NIP Rings

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October 20, 2025

It all starts with...

Shelah's Conjecture

NIP fields are either ACF, RCF or admits a nontrivial henselian valuation.

Proved by Will Johnson for dp-finite fields (2015-2020). For dp-minimal:

Theorem (Johnson, 2020)

$(K, +, \cdot)$ is dp-minimal iff finite or there is a henselian defectless valuation ring $\mathcal{O} \subseteq K$ such that

- $\Gamma := K^\times/\mathcal{O}^\times$ is trivial or dp-minimal OAG (class. in Jahnke-Simon-Walsberg 2017)

- $k := \mathcal{O}/\mathfrak{m}$ is ACF, RCF or p-CF

- if $k \vDash ACF_p$ with $p > 0$ then $[-v(p), v(p)]$ is p-divisible ($v(p) := \infty$ if $\mathrm{char}(K) = p$).

E.g. $\mathbb{R}((X))((Y))$, $\mathbb{Q}_p((t^{\mathbb{Z}_{(q)}}))$...

I. NIP division algebras

DA Conjecture

NIP division algebras are finite dimensional over their center.

The DA conjecture is known to hold for:

- (Milliet, 2020) NIP division algebra of positive characteristic (in fact NTP_2).
- (Hempel-Palacin, 2016) dp-finite division algebras (in fact finite burden).

It seems:

Still open in characteristic 0. *(Even for stable?)*

Recall the stable fields conjecture:

Stable fields Conjecture

Stable fields are separably closed.

DA Conjecture + Stable fields Conjecture implies:

Every stable division algebra is a field.

Characteristic 0: no central finite extension of an algebraically closed field
Characteristic $p > 0$: (Milliet) if $a \in D \setminus Z(D)$ then Millet proves that $ax - xa = 1$ has no solution hence $Z(D)(a)$ is a separable extension of $Z(D)$ by a result of Cohn, contradiction.

Under DA Conjecture, if D is a noncommutative division algebra and $[D, Z(D)] = n$ then writing the multiplication table of D in $Z(D)^n$, we have that D *is definable in* $Z(D)$. So

$$\text{Description of NIP fields} \implies \text{description of NIP DA}$$

Under Shelah's Conjecture (D noncommutative):

Z(D)	**D**
ACF	(no finite central extension)
RCF	Quaternions $\mathbb{H}$
Definable Henselian valuation	??

(It becomes an algebra problem.)

Except for the quaternions over the reals, the study of NIP DA reduces to study DA over a Henselian valued field, which is hard and has its own litterature (Morandi, Wadworth,...).

Property C_1 (quasi-algebraically closed): for all $d \in \mathbb{N}^{>0}$, an homogeneous polynomial in $> d$ variables has a nontrivial zero.

There are no nontrivial finite central extensions of a C_1 field (Artin, Tsen, 1933).

$\mathbb{C}((X))$ has no nontrivial finite central extension.
Another very important case:

Theorem

Every finite central extension of a p-adically closed field is cyclic.

Cyclic algebras. K/F be a cyclic field extension of degree n, X indeterminate, K-vector space

$$D = K \cdot 1 \oplus K \cdot X \oplus \ldots \oplus K \cdot X^{n-1}.$$

Let σ_0 a generator of $\mathrm{Gal}(K/F)$ and $\alpha \in F \setminus \{0\}$. Define a multiplication on D:

$$X^n = \alpha \qquad X \cdot b = \sigma_0(b)X \ \ (\forall b \in K)$$

extend to D linearly. This turns D into an algebra over F of dimension n^2, of center F. Denote $D := (K/F, \sigma_0, \alpha)$, the *cyclic algebra relative to* K/F, σ_0 *and* α.
Example, the quaternions $\mathbb{H}(\mathbb{R})$ is the cyclic algebra $(\mathbb{C}/\mathbb{R}, \sigma, -1)$ for σ the complex conjugation. The *norm* $N(a) = \prod_{\sigma \in \mathrm{Gal}(K/F)} \sigma(a)$, then

$$(K/F, \sigma_0, \alpha) \text{ is a division algebra if and only if } \alpha \notin N(K).$$

(Solvability of the norm equation.)

Theorem (d'Elbée, Milliet)

Let k_0 be a dp-minimal field of characteristic $p \geq 0$ and q a prime number different from p. Then there exists a dp-minimal field extension F of k_0 and a cyclic central division algebra D over F of dp-rank q^2.

If $p > 0$, $\Gamma = \langle \{ \frac{1}{p^n} \mid n \in \mathbb{N} \} \rangle \subseteq \mathbb{Q}$, if $p = 0$, let $\Gamma = \mathbb{Z}$. Then Γ is p-divisible and not q-divisible, and dp-minimal.
Let $k = k_0((x^\Gamma))$ and $F = k((t^\Gamma))$, F is dp-minimal.
Let u be such that $u^q = t$, and let $K = F(u)$, K is a cyclic extension of F of order q. Consider σ_0 a generator of $\mathrm{Gal}(K/F)$, then

$$D = (K/F, \sigma_0, x)$$

is an F-algebra of dp-rank q^2. To prove that D is a division algebra, we prove that $x \notin N(K)$ for $N = N_{K/F}$ (by hand, not completely trivial).

Essentially there are a lot of cyclic extensions of NIP fields.

All of them cyclic?

Answer: NO!We revisit a construction of Albert (1933):

Theorem (d'Elbée)

There exists a non-cyclic division algebra of dp-rank 16 *over* $\mathbb{R}((X))((Y))$.

(Remark: one of the most important problem in noncommutative algebra is finding a non-cyclic division algebra of degree p over some field.)

Quaternion algebra. *char*$(F) \neq 2$, for $u, v \in F^\times$, we define the *quaternion algebra* $\left(\frac{u,v}{F}\right)$ to be the set of expressions of the form $a + bi + cj + dij$ for $a, b, c, d \in F$ and symbols i, j. Define a multiplication on $\left(\frac{u,v}{F}\right)$ based on the rules:

$$i^2 = u, \quad j^2 = v, \quad ij = -ji$$

with which $\left(\frac{u,v}{F}\right)$ is an F-algebra. (Example: $\mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$.)

For $F = \mathbb{R}((X))((Y))$, consider the *biquaternion algebra*

$$D := \left(\frac{X,-1}{F}\right) \otimes_F \left(\frac{-X,Y}{F}\right)$$

Then D is a non-cyclic DA of dp-rank 16.
(Inverse exists $\iff$ for any quadratic extension K of F, the quadratic form $\phi : K^6 \to K$ defined by

$$\phi(a_1, a_2, a_3, a_4, a_5, a_6) = Xa_1^2 + -a_2^2 + Xa_3^2 + Xa_4^2 - Ya_5^2 - XYa_6^2$$

is anisotropic, i.e. $\phi(\bar{a}) = 0$ only if $\bar{a} = 0$.)

II. NIP Commutative rings

(All rings are unital). Why is it hard?

In (any) commutative ring, prime ideals are externally definable (!).

$$R \text{ NIP} \implies R/\mathfrak{p} \text{ NIP}$$

Let $\Sigma(x) = \{a \mid x : a \in R \setminus \mathfrak{p}\} \cup \{b \nmid x : b \in \mathfrak{p}\}$. $\Sigma(x)$ is finitely satisfiable: $a_1 a_2 \cdots a_n$ satisfies

$$\{a_1 \mid x,\ a_2 \mid x,\ \ldots,\ a_n \mid x,\ b_1 \nmid x,\ b_2 \nmid x,\ \ldots,\ b_m \nmid x\} \subseteq \Sigma(x)$$

If $c \in R' \succ R$ with $c \models \Sigma(x)$, then $\{x \nmid c\} \cap R = \mathfrak{p}$.

Harder (Johnson):
- I radical, R NIP $\implies$ I ext. def.
- $S \subseteq R$ multiplicative, R NIP $\implies \mathrm{S}^{-1}R$ NIP.

R a commutative unital ring. $\mathrm{Spec}(R)$ is the lattice of prime ideals.

Basic Facts

If R is NIP then $\mathrm{Spec}(R)$ has finite width (=maximal size of antichain).
$\rightsquigarrow$ (Dilworth's Thm) $\mathrm{Spec}(R)$ is a union of chains.
$\rightsquigarrow$ R has finitely many maximal/minimal ideals.
$\rightsquigarrow$ $0 \neq \mathrm{Jac}(R)(= \bigcap \text{maximal ideals})$.

$\{\mathfrak{p}_i\}_{1\leq i\leq n}$ antichain of prime ideals.
- for $1 \leq i \leq n$ choose $a_i \in \mathfrak{p}_i \setminus \bigcup_{j\neq i} \mathfrak{p}_j$ (prime avoidance lemma),
- for $I \subseteq \{1, \ldots, n\}$, set $b_I = \prod_{i\in I} a_i$.

Then $a_i \mid b_I \iff i \in I$.

In any commutative ring: $\mathrm{Width}(\mathrm{Spec}(R)) \leq \text{VC-dim}(x \mid y)$.

Stable commutative rings, (Cherlin-Reineke, 1976)

Every stable commutative ring is isomorphic to a product $R_1 \times \ldots \times R_n$ where R_i is a local ring with stable residue field.

In fact we conjecture the same for NIP commutative rings (+Henselianity Conjecture)

Generalized Henselianity Conjecture

If R is an NIP commutative ring then R is a product of finitely many henselian local rings.

(*Henselian local ring*: $(R, \mathfrak{m})$ and $a_n x^n + \ldots a_0 \in R[x]$, $a_0 \in \mathfrak{m}$, $a_1 \notin \mathfrak{m} \implies$ has a root in $\mathfrak{m}$.)

Theorem (Johnson, 2022-2023)

The conjecture holds if:

- *R NIP commutative ring of prime characteristic ($\mathbb{F}_p$-algebra).*
- *R dp-finite commutative ring.*

(More cases by Johnson, W_n-rings, Noetherian...)

In the dp-minimal case we have refined results (d'Elbée, Halevi, Johnson, 2025):

Facts

- Every dp-minimal commutative ring is of the form $R \times S$ where R is a dp-minimal henselian local ring and S is a finite ring.
- If R is a dp-minimal local ring then the prime ideals are linearly ordered.
- If R is a dp-minimal local ring then every prime ideal containing the zero divisors is comparable to any other ideal.

Integral domains arise naturally as $R/\mathfrak{p}$...

III. Dp-minimal integral domains

(Joint with Yatir Halevi and Will Johnson)
Examples of dp-minimal integral domains: given a valued dp-minimal field (K, v), consider $\mathcal{O}$ the valuation ring of (K, v).
One should expect that dp-minimal domains are "close" to be valuation rings. This is **almost true** but not quite:

Example

$K = \mathbb{Q}_3(i)$, where $i = \sqrt{-1}$. Let $\mathcal{O}$ be the natural valuation ring on K, namely $\mathcal{O} = \mathbb{Z}_3[i]$. Then

$$R := \mathbb{Z} + 27\mathcal{O} = \{0, 1, 2, \ldots, 26\} + 27\mathcal{O}$$

is a dp-minimal subring of $\mathcal{O}$ which is not a valuation ring (27 and $27i$ are incomparable).
(Similar examples exist starting e.g. with $\mathbb{F}_p^{\mathrm{alg}}(t^{\mathbb{Q}})$.)

Why is it almost true?

Valuation ring	$I \subseteq J$ or $J \subseteq I$ for all ideals I, J
Divided domain	$I \subseteq \mathfrak{p}$ or $\mathfrak{p} \subseteq I$ for all ideal I and prime $\mathfrak{p}$
Local treed domain	$\mathfrak{q} \subseteq \mathfrak{p}$ or $\mathfrak{p} \subseteq \mathfrak{q}$ for all primes $\mathfrak{p}, \mathfrak{q}$

Theorem (d'Elbée Halevi, 2021)

Every dp-minimal integral domain is divided.

(Note: in dp-rank 2, dp-minimal rings are not even local treed.)

Let's get back to our example:

Example

$K = \mathbb{Q}_3(i)$, where $i = \sqrt{-1}$. Let $\mathcal{O}$ be the natural valuation ring on K, namely $\mathcal{O} =_3 [i]$. Then

$$R := \mathbb{Z} + 27\mathcal{O} = \{0, 1, 2, \ldots, 26\} + 27\mathcal{O}$$

is a dp-minimal subring of $\mathcal{O}$ which is not a valuation ring.
Taking $R_0 = \mathbb{Z}/27\mathbb{Z} \subseteq \mathcal{O}/27\mathcal{O}$, and consider the map $\pi : \mathcal{O} \to \mathcal{O}/27\mathcal{O}$ we have:

$$R = \pi^{-1}(R_0).$$

$\rightsquigarrow$ All dp-minimal integral domains which are not valuation arise that way!

Theorem

Let R be an integral domain. Then R is dp-minimal if and only if one of the following holds:

(1) *R is a dp-minimal field.*

(2) *R is a dp-minimal valuation ring.*

(3) *There is a valuation subring $\mathcal{O}$ of $K = \mathrm{Frac}(R)$,a proper ideal $I \lhd \mathcal{O}$,and a finite subring R_0 of $\mathcal{O}/I$ such that R is the preimage of R_0 under the quotient map $\mathcal{O} \to \mathcal{O}/I$, and the valuation ring $(\mathcal{O}, +, \cdot)$ is dp-minimal.*

Moreover, in (3) the ring $\mathcal{O}$, ideal I, and subring R_0 can be chosen to be definable in R.

Remarks:

(a) dp-rk$(M^{\mathrm{sh}}) =$ dp-rk(M), dp-rk$(R) =$ dp-rk$(\mathrm{Frac}(R), R) =$ dp-rk$(R_{\mathfrak{p}})$

(b) (2) tells the whole story by (a) since dp-minimal valued fields are classified (Johnson 2023).

(c) The construction in (3) indeed yields a dp-minimal integral domain: every ideal of a dp-minimal valuation ring is externally definable.

(d) A nontrivial consequence of the proof and of the result: Let K be a dp-minimal field, then

$$K \textit{ defines a non-trivial valuation } \iff K \textit{ defines an infinite proper subgroup of } (K,+).$$

(e) Let R be NIP ring, then for the logic topology, R/R^{00} is a compact Hausdorff ring hence it is a profinite ring (litterature) hence $R^0 = R^{00}$. If every principal ideal in R has finite index, this further implies that $\mathrm{Frac}(R/R^{00})$ is isomorphic to a finite extension of $\mathbb{Q}_p$ (using classification of locally compact fields and KSW).

Let R be an *exceptional* dp-minimal integral domain (i.e. neither a field nor a valuation ring) and $K = \mathrm{Frac}(R)$, dp-minimal.

Step 1: K is either

(i) ACVF-like: there is a henselian valuation on K with ACF residue field

(ii) pCF-like: there is a henselian valuation on K with finite residue field

Step 2: Accordingly, R satisfies either

(i) (ACVF-like) R/aR is infinite for every $a \in \mathfrak{m}$ and $\sqrt{\mathfrak{m}^{00}} = \mathfrak{m}$ ($\mathfrak{m} \sim \mathfrak{m}^{00}$).

(ii) (pCF-like) R/aR is finite for some $a \in \mathfrak{m}$ and $\mathfrak{m}^{00}$ is the second largest prime ideal in R ($\mathfrak{m} >> \mathfrak{m}^{00}$). We show how to conclude the pCF-like case.

Let $\mathcal{O}_{\mathrm{can}}$ be the intersection of all definable valuation rings on K. By Johnson, $\mathcal{O}_{\mathrm{can}}$ is henselian, definable and as K is pCF-like, it has finite residue field.
By dp-minimality, $[-v(p), v(p)]$ is finite, so value group discretely ordered and so $\mathfrak{m}_{\mathrm{can}}$ is principal and with a bit more work one gets that $\mathcal{O}_{\mathrm{can}}^{00}$ is the second largest prime in $\mathcal{O}_{\mathrm{can}}$.
Then $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$ is an integral domain but not a field, in particular it is infinite.

Claim: $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$ is of characteristic 0.

As $\mathcal{O}_{\mathrm{can}}^{00}$ is the maximal ideal of $\mathcal{O}_{\mathrm{can}\,\mathcal{O}_{\mathrm{can}}^{00}}$ (valuation rings are divided) this implies in particular $a\mathcal{O}_{\mathrm{can}}^{00} = \mathcal{O}_{\mathrm{can}}^{00}$ for all $a \in \mathcal{O}_{\mathrm{can}} \setminus \mathcal{O}_{\mathrm{can}}^{00}$. Further, $\mathcal{O}_{\mathrm{can}}^{00} = a\mathcal{O}_{\mathrm{can}}^{00} = (a\mathcal{O}_{\mathrm{can}})^{00} \subseteq a\mathcal{O}_{\mathrm{can}}$ so $a\mathcal{O}_{\mathrm{can}}$ has finite index in $\mathcal{O}_{\mathrm{can}}$, so every nonzero principal ideal in $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$ has finite index. Using the classification of locally compact fields, one shows that: *If A is a compact Hausdorff integral domain such that every nontrivial principal ideal has finite index, then $\mathrm{Frac}(A)$ is either isomorphic to a finite extension of $\mathbb{Q}_p$ or to $\mathbb{F}_p((t))$.* Using Kaplan-Scanlon-Wagner, we exclude the latter hence we conclude that $\mathrm{Frac}(\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00})$ is a finite extension of $\mathbb{Q}_p$.

By d'Elbée-Halevi, $R \subseteq \mathcal{O}$ for all externally definable valuation overrings.

Claim: R has finite index in $\mathcal{O}_{\mathrm{can}}$.

It is enough to show that $R^{00} = \mathcal{O}_{\mathrm{can}}^{00}$, so assume not. Then $R^{00} \subsetneq \mathcal{O}_{\mathrm{can}}^{00} \subsetneq \mathfrak{m}_{\mathrm{can}}$. Let $\mathfrak{p} = R^{00} = \mathfrak{m}^{00}$. As R is divided, $\mathfrak{p}$ is the maximal ideal of $R_{\mathfrak{p}}$, but as $\mathfrak{p} = R^{00} \subseteq \mathfrak{m}_{\mathrm{can}}$, we have the reverse inclusion $\mathcal{O}_{\mathrm{can}} \subseteq R_{\mathfrak{p}}$ hence $\mathfrak{p}$ is an ideal of $\mathcal{O}_{\mathrm{can}}$. Both $\mathfrak{p}$ and $\mathcal{O}_{\mathrm{can}}^{00}$ are ideals of $\mathcal{O}_{\mathrm{can}}$ hence they are defined by cuts, and there is some closed ball $B_{\geq\gamma}(0)$ which separates them:

$$\mathfrak{p} \subseteq B_{\geq\gamma}(0) \subseteq \mathcal{O}_{\mathrm{can}}^{00}$$

As R/R^{00} is profinite, $R^{00} = R^0$ is a filtered intersection of definable ideals $I \subseteq R$ of finite index. As $B_{\geq\gamma}(0)$ is definable, saturation yields that there is a definable ideal J of finite index such that $J \subseteq B_{\geq\gamma}(0) \subseteq \mathcal{O}_{\mathrm{can}}^{00}$. Then there is a ring homomorphism

$$R/J \to \mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$$

but R/J is finite and $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$ has characteristic 0, contradiction.

Let $m = m_0 p^n$ be the index of R in $\mathcal{O}_{\mathrm{can}}$, then m_0 is invertible in $\mathcal{O}_{\mathrm{can}}$ so $m\mathcal{O}_{\mathrm{can}} = p^n\mathcal{O}_{\mathrm{can}}$ and we conclude:

$$p^n\mathcal{O}_{\mathrm{can}} \subseteq R \subseteq \mathcal{O}_{\mathrm{can}}$$

$p^n\mathcal{O}_{\mathrm{can}}$ has finite index in $\mathcal{O}_{\mathrm{can}}$ because $p^n\mathcal{O}_{\mathrm{can}}$ has finite residue field and $v(p)$ is a multiple of the minimal positive valuation element.

To conclude: take $I = p^n\mathcal{O}_{\mathrm{can}}$ and the projection $\pi : \mathcal{O}_{\mathrm{can}} \to \mathcal{O}_{\mathrm{can}}/I$ and $R_0 = \pi(R) = R/p^n\mathcal{O}_{\mathrm{can}}$.

Grazie a tutti!