A short tour in Model Theory

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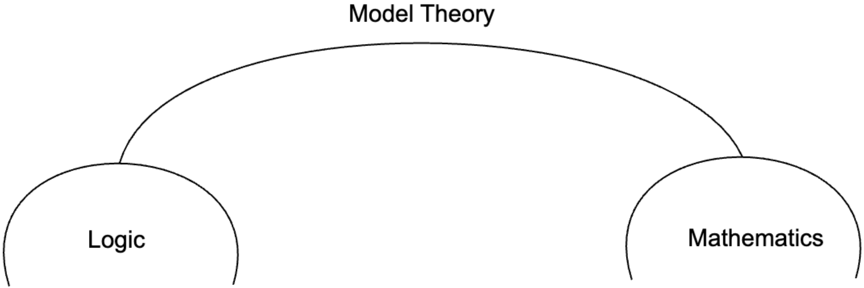
October 24, 2025


Model Theory
Logic
Mathematics

SEMANTIC

"*Give a meaning to the syntax*"

Naive set Theory

$X, Y, \ldots$

$x \in Y$, $X \cup Y$

Structure:

$(M, R, f, c, \ldots)$

SYNTAX

"*Formalize the semantic*"

Strings of character

$\forall x \exists y R(x, y), \ldots$

$\exists x \exists y \exists z x + y = z, \ldots$

Rules of Deduction

$A,\ A \to B \implies B \ldots$

$$\boxed{M \models \theta}$$

1st Order Logic:

$$\Phi \models \theta \quad \overset{\text{(Gödel, 1930)}}{\Longleftrightarrow} \quad \Phi \vdash \theta$$

Remark 1. (Completeness of 1st order logic.)
Gödel Completeness is a "$\exists \iff \forall$" statement.
Henkin's Proof: if Φ is consistent, then Φ has a model (consistent $=$ does not prove contradiction). *Constructing the semantic from the syntax.*

Remark 2. (Set Theory.)
Schizophrenic approach of set theory: a 1st order theory ZF of sets built on top of a naive set theory. Then the new theory of sets gives insight on what we can expect concerning the behaviour of sets (cardinals, ordinals, etc.). Some mathematicians stay in naive set theory, while others work in ZF. Model theorists use ZF.

High-school mathematical structures:

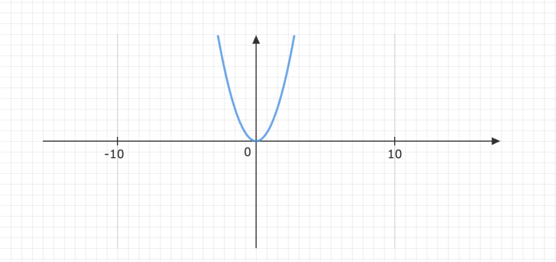
The natural numbers: $\mathbb{N}$, addition $+$, order $<$.

The line of real numbers $\mathbb{R}$, addition $2.7 + 3.2 = 5.9$ and multiplication $4 \cdot 2.5 = 10$, order: $2.5 < 7$:

The complex numbers: $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ will be a minor example, all nonconstant polynomials equations $ax^2 + bx + c = 0$ have solutions in $\mathbb{C}$.

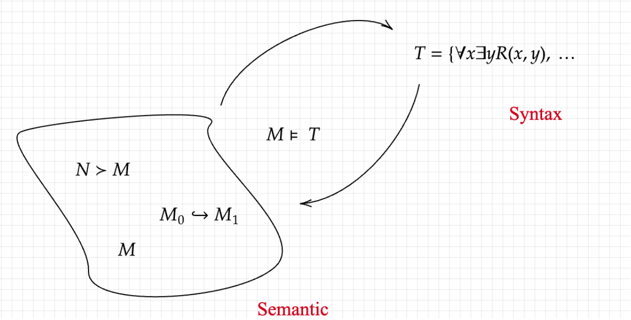
As for many discipline, people struggle finding an exact definition of what model theory is. One important reason is because of its capacity to interact with other branches of mathematics, and get transformed into something else, I'll come back later on this. (Another reason is because some people are very opinionated.) I would say that people agree on the following

(1) Model theorists looks at a structure $(M, R, f, c, \ldots)$ through the prism of its *definable sets*: each formula $\phi(x_1, \ldots, x_n)$ defines a subset of $M \times \ldots \times M$ given by all $(a_1, \ldots, a_n)$ such that $M \vDash \phi(a_1, \ldots, a_n)$. For instance, the graph Γ of the function $x \mapsto x^2$ is definable in the structure $(\mathbb{R}, +, \cdot)$.



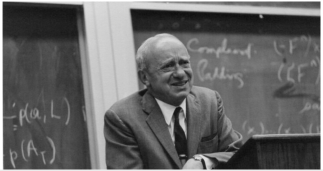
(2) Model theorists focus on the semantics rather than on the syntax. It does not matter if a set is defined by one or another formula, only the set is important: the formulas $y = x^2$ and $x_1^2 - x_2 = 0$ define the same sets. Model theorists look at the *geometry* of the definable sets.

(3) Model theorists study 1st-order theories by looking at their models:



Early names in model theory: Skolem (completude 1920), Lowenheim, Presburger, Gödel, more importantly Alfred Tarski and Abraham Robinson. Tarski is considered the father of model theory.

One of the main classical tool of model theorists (still very much used today is called *quantifier elimination*

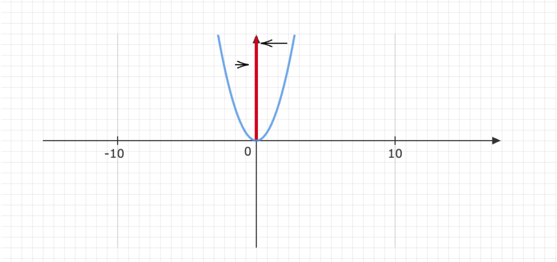


Definition

A theory T has *quantifier elimination* if for all formula ϕ there exists another formula ψ which does not use the quantifier $\forall, \exists$ and such that $T \vDash \phi \leftrightarrow \psi$.
A structure $M = (M, R, f, \ldots)$ has QE if the theory of M (its *axioms*) has quantifier elimination.

Remark 4. This phenomenon in fact witnesses a *default of expressability*, or a *collapse of semantic*: in a sense, there are no more definable sets than the ones defined without quantifiers.

Example. In $(\mathbb{R}, +, \cdot, >)$ the formulas $x > 0$ and $\exists y\ x = y^2$ are equivalent. One has quantifier while the other does not. This also has a geometric flavour, that model theorists would prefer:



(The class of quantifier-free definable sets is closed under projection.)

It is a model-theoretic property of M that may happen for various reasons:

- $\star$ the structure is very rich: in $(\mathbb{C}, +, \cdot)$, for a non-constant polynomial, $\exists x \; P(x) = 0$ is *equivalent* to $x = x$ since every non-constant polynomial has a root !
- $\star$ the structure is very poor: in $(\mathbb{N}, R)$, where $R(x, y)$ if and only if x and y are adjacent, then $\exists y_1 \neq y_2 R(x, y_1) \wedge R(x, y_2)$ if and only if $x \neq 0$.

Quantifier elimination (or variants of it) is still very much used today because it gives a nice description of definable sets. It is often the starting point for studying a new structure.

Compactness Theorem. (Follows from completeness.) For any set of formulas $\Phi \vDash \theta$ iff $\Phi_0 \vDash \theta$ for some finite $\Phi_0 \subseteq \Phi$.

Myriad of uses and applications. Some people would argue that its use defines MT. One example of application is to know whether certain concepts are axiomatizable or not.

Is the concept of "being of finite size" 1st order axiomatizable?

The answer is no: assume that T is an axiomatization of that concept, then:

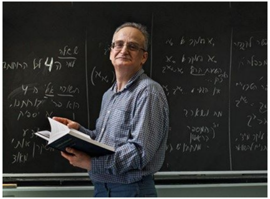
$$\Phi \cup \{\text{exists at least } n \text{ elements} \mid n \in \mathbb{N}\} \vDash \bot$$

$$\Phi \cup \{\text{exists at least 42 elements}\} \vDash \bot$$

$$\Phi \vDash \{\text{exists at most 42 elements}\}$$

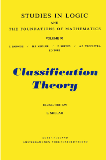
a contradiction.

Saharon Shelah, in the tracks of Michael Morley developped a completely new approach for model theory, based on infinite combinatorics, and more flexible constraints on structures.



Consider (again) the real numbers, then $(\mathbb{R}, +, \cdot, <)$ has quantifier elimination but forgetting about the order "$<$", the numbers $(\mathbb{R}, +, \cdot)$ *do not* have quantifier elimination[1]. The system $(\mathbb{R}, +, \cdot)$ which is a priory easier than $(\mathbb{R}, +, \cdot, <)$ does not enjoy the nice property of QE. In fancy terms: QE is not *preserved by interpretation*. Even worse, in fact $(\mathbb{R}, +, \cdot, <)$ and $(\mathbb{R}, +, \cdot)$ have the same definable sets !

Quantifier elimination is a property which is too syntax-dependent (it actually depends on the language). Shelah defined model-theoretic properties which do not depend on the language. Those properties have more *semantic* descriptions, via infinite combinatorics. Let us see an example.



[1] Precisely because "$<$" eliminates a quantifier via $x < y \leftrightarrow \exists z\ x = y + z^2$.

Definition (Stability)

A theory T is *stable* if

for all model M of T

for all formula $\phi(\vec{x}, \vec{y})$

there is not infinite sequence $(\vec{a}_0, \vec{a}_1, \ldots, \vec{a}_n, \ldots)$ of elements of M such that the following holds:

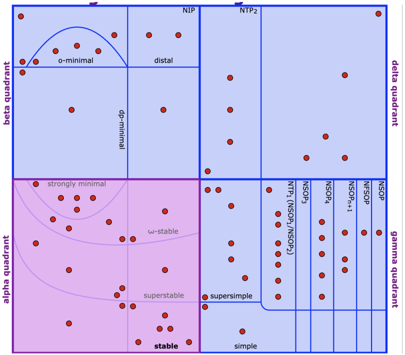
$$i \leq j \iff M \vDash \phi(\vec{a}_i, \vec{a}_j)$$

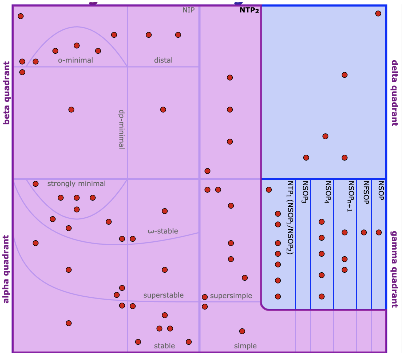
(Semantic, in $(\mathbb{N}, \leq)$) (Syntax, in $(M, R, f, \ldots)$)

Informally: no formula interprets an order that behaves like $(\mathbb{N}, \leq)$. There is a persistence in those types of properties, anything "constructed" out of a stable theory will again be stable. There is an intuitive reason why it is a good idea to prevent infinite orders to exist.

Example. $(\mathbb{R}, +, \cdot, <)$ is not stable, while $(\mathbb{C}, +, \cdot)$ is stable.

Stability is an extremely successful notion in model theory, it allows to generalize phenomena that occur in $(\mathbb{C}, +, \cdot)$ to many structures, only based on the fact that they are stable (structures in MT can be anything, graphs, groups, orders, fields, vectors, etc) and make apparent links between a priori different structures (algebraic geometry, Mordell-Lang,etc.).
The point is that vagues intuitive notions in mathematics such as "there is no underlying linear order" could be made precise and formal using the formalism of logic. This approach is at the core of model theory: either look at mathematical problems from the point of view of a logician, or treating logical object with the eyes of a mathematician.
There are many other such tameness properties (with barbaric names): NIP, NTP_2, NSOP_1, NSOP_4, etc. They constitute dividing lines in the spectrum of all mathematical structures, as can be seen https://www.forkinganddividing.com/. The quest for a complete map of the universe (i.e. putting any given mathematical structure in is correct class of tameness) is called *classification theory*, a very active trend of research in model theory.


NIP
NTP$_2$
o-minimal
distal
dp-minimal
beta quadrant
delta quadrant
strongly minimal
ω-stable
superstable
stable
supersimple
simple
NTP$_1$ (NSOP$_1$/NSOP$_2$)
NSOP$_3$
NSOP$_4$
NSOP$_{n+1}$
NFSOP
NSOP
alpha quadrant
gamma quadrant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Most mathematicians work with very powerful tools developed and adapted to answer precise questions, and using a certain amount of different structures. Model theorists have tools to deal with very wide range of structures at the same time, since everything is often stated in terms of formulas in an arbitrary language. The high level of generality associated to model theory is at the same time a strength and a weakness:

(Strength.) Results are often very general and often allows to uncover new connections between mathematical objects.

(Weakness.) General tools are weak when applied in particular examples. Often, this means that the tools are to be refined in order to yield stronger results.

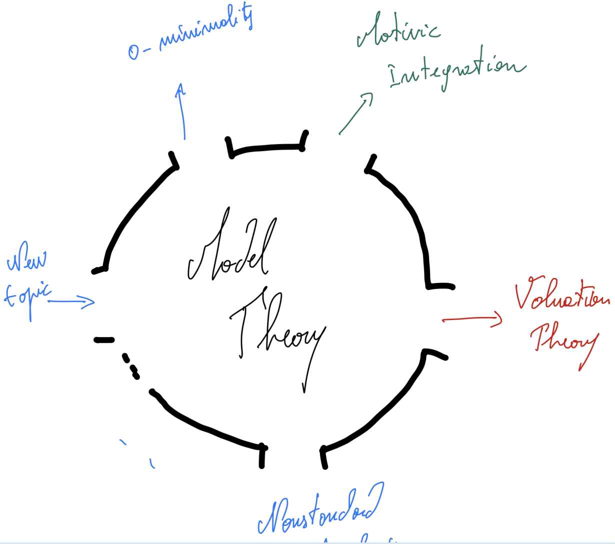
This has a perverse effect resulting in the partitioning of MT in various areas, becoming more and more distant from each other. A reccurent pattern is the following:

(*i*) a new area emerges, a new connection between a mathematical question and MT,

(*ii*) the tools of MT are applied to this new connection,

(*iii*) the tools get more and more specific and adapted to this new area, which then does not rely on model theory anymore.

There is sometimes a retroactive loop: specific tools can then be generalized to the model-theoretic framework.


o-minimality
Motivic Integration
New topic
Model Theory
Valuation Theory
Nonstandard

In conclusion, MT is a branch of logic and a branch of mathematics. It focuses on mathematical questions with the point of view of logicians (theories, definable sets, etc.), and on logical questions with the approach of mathematicians (semantic, geometry, etc.).

Thanks for your attention!