Axiomatic Theory of Independence Relations in Model Theory

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Introduction

Those are the notes from a course [1] in neostability given at the University of Bonn in the summer semester 2023.

This course is an introduction to the fruitful links between model theory and a combinatoric of sets given by independence relations. An independence relation on a set is a ternary relation between subsets, usually denoted $\mathop{\smile\!\!\!\!\!\!\mid}$. Properties, or axioms, satisfied by such a relation –sometimes related with an ambient closure operator, or an ambient first-order theory– can be a witness of nice combinatorial behaviour, or tameness. A motivating example of this phenomenon is the Kim-Pillay theorem, which is a characterisation of simple theories by the existence of an independence relation satisfying a certain set of axioms. We distinguish between three sorts of axioms for independence relations, those that can be stated in a purely set-theoretic framework (monotonicity, base monotonicity, transitivity) those that necessitate an ambient closure operator (closure, anti-reflexivity) and those that necessitate an ambient theory to be stated (extension, the independence theorem). For clarity, those axioms will be stated separately, but a complete list of axioms is given on p. 6.

Chapter 1 should be considered as an introductory chapter. It does not mention first-order theories nor formulas and introduces independence relations in a naive set theory framework. Its main goal is to get the reader familiar with the basic axioms of independence relations (those that do not need an ambient theory to be stated) as well as introducing closure operators and pregeometries.

Chapter 2 introduces the model-theoretic context. The two main examples (algebraically closed fields and the random graph) are described as well as independence relations in those examples.

Chapter 3 gives the axioms of independence relations in a model-theoretic context and introduces what I believe is nowadays called “forking calculus” (although forking will be defined in Chapter 4). It introduces the general toolbox of the model-theorists (indiscernible sequences, Ramsey/Erdös-Rado and compactness) as well as the independence relations $\mathop{\smile\!\!\!\!\!\!\mid}^h$ and $\mathop{\smile\!\!\!\!\!\!\mid}^u$ of heirs/coheirs with two main applications: Adler's theorem of symmetry (how symmetry emerges from a weaker set of axioms, which is rooted in the work of Kim and Pillay) and a criterion for NSOP_4 using stationary independence relations in the style of Conant. Independence relations satisfying Adler's theorem of symmetry are here called *Adler independence relations* (or AIR).

Chapter 4 treats forking and dividing as abstract independence relations $\mathop{\smile\!\!\!\!\!\!\mid}^f$, $\mathop{\smile\!\!\!\!\!\!\mid}^d$. It is proved that $\mathop{\smile\!\!\!\!\!\!\mid}^d$ is always stronger than any AIR (even though it is not an AIR in general) as well as an abstract connection between the independence theorem and forking independence, which holds in all generality and is based on Kim-Pillay's approach. Then, simplicity is defined and the interesting direction of the Kim-Pillay theorem (namely that the existence of an Adler independence relation satisfying the independence theorem yields simplicity) is deduced from earlier results.

I believe the complete list of prerequisites in model theory for this course is the following: languages, sentences, theories, formulas, types, structures, definable sets, substructures, elementary substructures, models, elementary maps, elementary bijections and automorphism of models. As for set theory, I only assume the existence and basic properties of cardinals and ordinals.

[1] The course is called: V5A7 - Advanced Topics in Mathematical Logic. The link to basis is https://basis.uni-bonn.de/qisserver/rds?state=verpublish&status=init&vmfile=no&publishid=224952&moduleCall=webInfo&publishConfFile=webInfo&publishSubDir=veranstaltung.

Here is a complete list of the axioms of independence that will be used in this text.

Definition 1.1.1 (Axioms of independence relations, part 1).

(1) (finite character) If $a \mathop{\smile\!\!\!\!\!\!\mid}_C B$ for all finite $a \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$.
(2) (existence) $A \mathop{\smile\!\!\!\!\!\!\mid}_C C$ for any A and C.
(3) (symmetry) If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ then $B \mathop{\smile\!\!\!\!\!\!\mid}_C A$.
(4) (local character) For all A there is a cardinal $\kappa = \kappa(A)$ such that for all B there is $B_0 \subseteq B$ with $|B_0| < \kappa$ with $A \mathop{\smile\!\!\!\!\!\!\mid}_{B_0} B$.
(5) (right normality) If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ then $A \mathop{\smile\!\!\!\!\!\!\mid}_C BC$.
(6) (right monotonicity) If $A \mathop{\smile\!\!\!\!\!\!\mid}_C BD$ then $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$.
(7) (right base monotonicity) Given $C \subseteq B \subseteq D$ if $A \mathop{\smile\!\!\!\!\!\!\mid}_C D$ then $A \mathop{\smile\!\!\!\!\!\!\mid}_B D$.
(8) (right transitivity) Given $C \subseteq B \subseteq D$, if $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}_B D$ then $A \mathop{\smile\!\!\!\!\!\!\mid}_C D$.

Definition 1.2.10 (Axioms of independence relations, part 2). With an ambient closure operator cl.

(9) (anti-reflexivity) If $a \mathop{\smile\!\!\!\!\!\!\mid}_C a$ then $a \in \mathrm{cl}(C)$;
(10) (right closure) $A \mathop{\smile\!\!\!\!\!\!\mid}_C B \implies A \mathop{\smile\!\!\!\!\!\!\mid}_C \mathrm{cl}(B)$.
(11) (strong closure) $A \mathop{\smile\!\!\!\!\!\!\mid}_C B \iff \mathrm{cl}(AC) \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}(C)} \mathrm{cl}(BC)$.

Definition 3.1.1 (Axioms of independence relations, part 3). Let $\mathop{\smile\!\!\!\!\!\!\mid}$ be an invariant ternary relation on small subsets of $\mathbb{M}$.

(12) (extension) If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ then for any $D \supseteq B$ there is $A' \equiv_{BC} A$ with $A' \mathop{\smile\!\!\!\!\!\!\mid}_C D$.
(13) (full existence) For all A, B, C there exists $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!\mid}_C B$.
(14) (the independence theorem over models) Let M be a small model, and assume $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$, $C_1 \mathop{\smile\!\!\!\!\!\!\mid}_M A$, $C_2 \mathop{\smile\!\!\!\!\!\!\mid}_M B$, and $C_1 \equiv_M C_2$. Then there is a set C such that $C \mathop{\smile\!\!\!\!\!\!\mid}_M AB$, $C \equiv_{MA} C_1$, and $C \equiv_{MB} C_2$.
(15) (stationarity over models) Let M be a small model, and assume $C_1 \mathop{\smile\!\!\!\!\!\!\mid}_M A$, $C_2 \mathop{\smile\!\!\!\!\!\!\mid}_M A$, and $C_1 \equiv_M C_2$. Then $C_1 \equiv_{MA} C_2$.

CHAPTER 1

Independence relations in a set-theoretic setting

1.1. Independence relations and notations

In this course, the symbol $\mathop{\smile\!\!\!\!\!\!\downarrow}$ (sometimes indexed $\mathop{\smile\!\!\!\!\!\!\downarrow}^0$, $\mathop{\smile\!\!\!\!\!\!\downarrow}^i$, etc...) will always denote a ternary relation on the powerset of an ambient set. We will sometimes call $\mathop{\smile\!\!\!\!\!\!\downarrow}$ an *independence relation* [1]. We start with an easy set of axioms, where $A, B, C, \ldots$ are subsets of the ambiant set. We often denote by juxtaposition AB the union $A \cup B$.

Definition 1.1.1 (Axioms of independence relations, part 1).

(1) (finite character) If $a \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$ for all finite $a \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$.
(2) (existence) $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C C$ for any A and C.
(3) (symmetry) If $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$ then $B \mathop{\smile\!\!\!\!\!\!\downarrow}_C A$.
(4) (local character) For all A there is a cardinal $\kappa = \kappa(A)$ such that for all B there is $B_0 \subseteq B$ with $|B_0| < \kappa$ with $A \mathop{\smile\!\!\!\!\!\!\downarrow}_{B_0} B$.
(5) (right normality) If $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$ then $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C BC$.
(6) (right monotonicity) If $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C BD$ then $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$.
(7) (right base monotonicity) Given $C \subseteq B \subseteq D$ if $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C D$ then $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B D$.
(8) (right transitivity) Given $C \subseteq B \subseteq D$, if $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$ and $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B D$ then $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C D$.

Every property with a "right —" prefix has a symmetric counterpart "left —". For instance, left normality is $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B \implies AC \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$. We will often omit the left/right prefix when the context is clear, for instance if the relation is symmetric or if it is clear which side we are refering to.

Some terminology. If $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$, we say that A *is independent from* B *over* C, or A *and* B *are independent over* C. We call C the *base set* of the instance $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$. Given two independence relations $\mathop{\smile\!\!\!\!\!\!\downarrow}$, $\mathop{\smile\!\!\!\!\!\!\downarrow}^0$, we say that $\mathop{\smile\!\!\!\!\!\!\downarrow}$ is *stronger than* $\mathop{\smile\!\!\!\!\!\!\downarrow}^0$, or $\mathop{\smile\!\!\!\!\!\!\downarrow}^0$ is *weaker than* $\mathop{\smile\!\!\!\!\!\!\downarrow}$, denoted $\mathop{\smile\!\!\!\!\!\!\downarrow} \to \mathop{\smile\!\!\!\!\!\!\downarrow}^0$, if $\mathop{\smile\!\!\!\!\!\!\downarrow} \subseteq \mathop{\smile\!\!\!\!\!\!\downarrow}^0$, in other words, for all A, B, C we have $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B \implies A \mathop{\smile\!\!\!\!\!\!\downarrow}^0_C B$.

Exercise 1. Let S be any set and define the relation $\mathop{\smile\!\!\!\!\!\!\downarrow}$ by $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$ if and only if $A \cap B \subseteq C$. Prove that $\mathop{\smile\!\!\!\!\!\!\downarrow}$ satisfies all properties above.

Exercise 2. Check that $\mathop{\smile\!\!\!\!\!\!\downarrow}$ satisfies right monotonicity if and only if for all A, C and $B \subseteq D$, we have $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C D \implies A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$.

Exercise 3. Prove that if $\mathop{\smile\!\!\!\!\!\!\downarrow}$ satisfies (right) normality, (right) monotonicity and (right) transitivity, then $\mathop{\smile\!\!\!\!\!\!\downarrow}$ satisfies the following stronger version of transitivity:

$$A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B \text{ and } A \mathop{\smile\!\!\!\!\!\!\downarrow}_{BC} D \implies A \mathop{\smile\!\!\!\!\!\!\downarrow}_C BD.$$

Exercise 4. Prove that if $\mathop{\smile\!\!\!\!\!\!\downarrow}$ satisfies (right) normality, (right) monotonicity and (right) base monotonicity, then $\mathop{\smile\!\!\!\!\!\!\downarrow}$ satisfies the following stronger version of base monotonicity:

$$A \mathop{\smile\!\!\!\!\!\!\downarrow}_C BD \implies A \mathop{\smile\!\!\!\!\!\!\downarrow}_{CD} B.$$

[1] Unlike other authors [1], we do not reserve the denomination "independence relation" for certain ternary relation satisfying a fixed set of axioms, we rather use it freely for ternary relations that will satisfy some axioms.

Exercise 5. Assume that $\mathop{\smile\!\!\!\!\!\!\perp}$ satisfies finite character, base monotonicity and the following weak version of locality: if a is finite, for all B there exists a finite $b \subseteq B$ with $a \mathop{\smile\!\!\!\!\!\!\perp}_b B$. Prove that $\mathop{\smile\!\!\!\!\!\!\perp}$ satisfies local character.

Exercise 6. If $\mathop{\smile\!\!\!\!\!\!\perp}$ satisfies local character and base monotonicity then $\mathop{\smile\!\!\!\!\!\!\perp}$ satisfies existence.

1.2. Independence relations in pregeometries

We now define a purely set-theoretic and combinatorial context where two independence relations appear very naturally.

Definition 1.2.1. A *(finitary) closure operator* cl on a set S is a function:

$$\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$$

which satisfies the following properties:

(1) (Reflexivity) $A \subseteq \mathrm{cl}(A)$;
(2) (Monotonicity) if $A \subseteq B$ then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;
(3) (Transitivity) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;
(4) (Finite Character) $\mathrm{cl}(A) = \bigcup_{\text{finite } a \subseteq A} \mathrm{cl}(a)$.

If cl also satisfies:

(5) (Exchange) if $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$ then $b \in \mathrm{cl}(Aa)$.

then (S, cl) is called a *pregeometry*.

Example 1.2.2. *Let $(G, +, 0)$ be an abelian group. For $A \subseteq G$, we denote by $\langle A \rangle$ the group span by A and $\langle A \rangle^{\mathrm{div}}$ the divisible closure of $\langle A \rangle$ in G, that is*

$$\langle A \rangle^{\mathrm{div}} = \{g \in G \mid ng \in \langle A \rangle \text{ for some } n \in \mathbb{N}\}$$

For any abelian group $(G, +, 0)$, the map $A \mapsto \langle A \rangle$ is a finitary closure operator. It does not satisfy exchange in general (think of 1 and 2 in $\mathbb{Z}$). If G is divisible, then $A \mapsto \langle A \rangle^{\mathrm{div}}$ is a pregeometry.

Exercise 7. Assume that cl is a finitary closure operator on a set S. Let $B \subseteq S$. We define:

- $\mathrm{cl}_B(A) := \mathrm{cl}(A \cup B)$ for all $A \subseteq S$ (*Relativisation*);
- $\mathrm{cl}_{\restriction B}(A) = \mathrm{cl}(A) \cap B$ for all $A \subseteq S$ (*Restriction*).

Prove that cl_B (resp. $\mathrm{cl}_{\restriction B}$) is a finitary closure operator on S (resp. on B). Similarly, if cl is a pregeometry, so are cl_B and $\mathrm{cl}_{\restriction B}$.

Definition 1.2.3. Given a closure operator cl on a set S we define $A \mathop{\smile\!\!\!\!\!\!\perp}^a_C B$ if $\mathrm{cl}(AC) \cap \mathrm{cl}(BC) = \mathrm{cl}(C)$.

Exercise 8. Check that if cl is a finitary closure operator, then $\mathop{\smile\!\!\!\!\!\!\perp}^a$ satisfies all properties of Definition 1.1.1 except base monotonicity.

Exercise 9. Assume that cl is a pregeometry on S. Prove that if $a \notin \mathrm{cl}(B)$, then $a \mathop{\smile\!\!\!\!\!\!\perp}^a_C B$ for all $C \subseteq B$.

Definition 1.2.4. Let cl be a closure operator on S. Let $A_0, A, B \subseteq S$.

- We say that A_0 is *independent* over B if $a \notin \mathrm{cl}(B \cup (A_0 \setminus \{a\}))$ for all $a \in A_0$.
- We say that A_0 *generates* A over B if $A \subseteq \mathrm{cl}(BA_0)$.

Exercise 10. Prove that A is independent over B for cl if and only if A is independent over $\emptyset$ for cl_B. Prove that A_0 generates A over B if and only if A_0 generates AB over $\emptyset$ for cl_B.

Proposition 1.2.5. *Let* cl *be a closure operator on S. Let $A_0 \subseteq A, B \subseteq S$ such that A_0 is independent over B. Then, there exists a set $A_1 \subseteq \mathrm{cl}(A)$ containing A_0 which is maximal among the subsets of A containing A_0 that are independent over B.*

Proof. First, by working with cl_B, we may assume that $B = \emptyset$. Let $\mathcal{F}$ be the set of independent (over $\emptyset$) subsets of A containing A_0. Let $(X_n)_n$ be a chain of elements in $\mathcal{F}$ and $X = \bigcup_n X_n$. We check that X is independent. If $x \in X$, then $x \in X_n$ for cofinitely many n. If $x \in \mathrm{cl}(X \setminus \{x\})$ then by finite character, $x \in \mathrm{cl}(X_n \setminus \{x\})$ for cofinitely many n, hence there is n such that $x \in X_n$ and $x \in \mathrm{cl}(X_n \setminus \{x\})$, which contradicts that X_n are independent. By Zorn's lemma, $\mathcal{F}$ admits a maximal element. □

Proposition 1.2.6 (and Definition). *Let* cl *be a pregeometry on S. Let $A_0, A, B \subseteq S$ with A_0 independent over B. Then, there exists a set $A_1 \subseteq \mathrm{cl}(A)$ containing A_0 such that A_1 is independent over B and A_1 generates A over B. Such A_1 is called a basis of A over B. For all bases A_1, A_2 of A over B we have $|A_1| = |A_2|$. We call $|A_1|$ the dimension of A over B, denoted* $\dim(A/B)$.

Proof. Similarly as above, if (S, cl) is a pregeometry, then so is the closure operator cl_B on S. This implies that we may assume that $B = \emptyset$. We start with a claim.

Claim 1. If X is independent and $a \notin \mathrm{cl}(X)$. Then $X \cup \{a\}$ is again independent

Proof of the claim. Otherwise, either $a \in \mathrm{cl}(X)$ which is excluded or $x \in \mathrm{cl}(Xa \setminus \{x\})$ for some $x \in X$. By assumption $x \notin \mathrm{cl}(X \setminus \{x\})$ hence

$$x \in \mathrm{cl}((X \setminus \{x\})a) \setminus \mathrm{cl}(X \setminus \{x\}).$$

By exchange, $a \in \mathrm{cl}(X \setminus \{x\} \cup \{x\}) = \mathrm{cl}(X)$, a contradiction. □

Let A_1 be a maximal independent subset of A extending A_0. If A_1 does not generates A, there exists $a \in A \setminus \mathrm{cl}(A_1)$. By Claim 1, $A_1 \cup \{a\}$ is again independent, which contradicts the maximality of A_1.

For the second part, it is enough to show that if X is independent in A and Y is generating in A, then $|X| \leq |Y|$.

Claim 2. If $x \in X \setminus Y$ there exists $y \in Y \setminus X$ such that $(X \setminus \{x\}) \cup \{y\}$ is independent.

Proof of the claim. As X is independent, $x \notin \mathrm{cl}(X \setminus \{x\})$ hence Y is not included in $\mathrm{cl}(X \setminus \{x\})$, otherwise $\mathrm{cl}(X \setminus \{x\})$ would be all A by transitivity and the fact that Y is generating. Then for any $y \in Y \setminus \mathrm{cl}(X \setminus \{x\})$ we have $(X \setminus \{x\}) \cup \{y\}$ is independent by Claim 1. □

Assume that X is finite. Then using iteratively Claim 2, we can replace elements of X by elements of Y to get an independent family, hence there is a subset of Y with the same cardinality as X which is independent, hence $|X| \leq |Y|$.

If X is infinite, let X' be a basis of A extending X. Then by finite character, for every $y \in Y$ there is a finite tuple x_y from X' such that $y \in \mathrm{cl}(x_y)$. We have $Y \subseteq \bigcup_{y \in Y} x_y$ hence $\bigcup_{y \in Y} x_y$ is a generating subset of X'. Assume that $\bigcup_{y \in Y} x_y \subsetneq X'$, then there exists $x \in X' \setminus \bigcup_{y \in Y} x_y$. Hence $\bigcup_{y \in Y} x_y \subseteq X' \setminus \{x\}$. As X' is independent, $X' \setminus \{x\}$ is not a generating set (since $x \notin \mathrm{cl}(X' \setminus \{x\})$). This contradicts that $\bigcup_{y \in Y} x_y$ is generating. It follows that $X' = \bigcup_{y \in Y} x_y$. As each x_y is finite, this implies that Y is infinite and hence $|\bigcup_{y \in Y} x_y| \leq |Y|$. We conclude that $|X| \leq |X'| = |\bigcup_{y \in Y} x_y| \leq |Y|$. □

Exercise 11. Let cl be a closure operator on S. Let A be a finite subset of S, prove that there exists a finite set $A_0 \subseteq \mathrm{cl}(A)$ which is a basis of A.

The following example shows that the previous exercise does not generalise to arbitrary sets A.

Example 1.2.7. *If (S, cl) is a closure operator which is not a pregeometry, then infinite subsets may not admit a basis. Here is an example cooked up by Frederick Gebert. Take $S = \mathbb{N}$ and* $\mathrm{cl}(A) = [0, \max A]$. *Then* cl *is a closure operator on* $\mathbb{N}$. *It is easy to check that* $(\mathbb{N}, \mathrm{cl})$ *is not a pregeometry and that $A \subseteq \mathbb{N}$ is an independent set if an only if $A = \{a\}$. Therefore $\mathbb{N}$ do not admit a basis. In this example, every finite set A_0 admit a basis:* $\max A_0$.

We denote $\dim(A) = \dim(A/\emptyset)$.

Lemma 1.2.8. *Let (S, cl) be a pregeometry and $A, B \subseteq S$. Then*

(1) $\dim(AB) = \dim(A/B) + \dim(B)$*;*
(2) If $C \subseteq B$ then $\dim(A/B) \leq \dim(A/C)$*;*
(3) $\dim(AB) + \dim(A \cap B) \leq \dim A + \dim B$*, for A, B closed.*

Proof. *(1)* Let B_0 be a basis of B over $\emptyset$ using Proposition 1.2.6. Using again Proposition 1.2.6, let A_0 be a basis of AB over B. We check that A_0B_0 is a basis of AB over $\emptyset$. As A_0 is independent over B_0, in particular, $A_0 \cap \mathrm{cl}(B) = \emptyset$ hence $A_0 \cap B_0 = \emptyset$. Let $x \in A_0 \cup B_0$. If $x \in A_0$, then $x \notin \mathrm{cl}(B \cup (A_0 \setminus \{x\})) = \mathrm{cl}(A_0B_0 \setminus \{x\})$ as $A_0 \cap B_0 = \emptyset$ hence we are done. If $x \in B_0$ and $x \in \mathrm{cl}(A_0 \cup (B_0 \setminus \{x\}))$. Let $(a_1, \ldots, a_n)$ be a tuple of elements of A_0 of minimal size such that $x \in \mathrm{cl}(\{a_1, \ldots, a_n\} \cup (B_0 \setminus \{x\}))$. Thus $x \notin \mathrm{cl}(\{a_2, \ldots, a_n\} \cup (B_0 \setminus \{x\}))$. By exchange, $a_1 \in \mathrm{cl}(\{a_2, \ldots, a_n\} \cup B_0)$, which contradicts the fact that a_i are elements of A_0. It follows that $\dim(AB) = |A_0| + |B_0|$. By definition, $|B_0| = \dim B$ and $\dim(A/B) = |A_0|$ so we are done. (2) Easily follows from (1), but it is also clear by definition. (3) Let $C = A \cap B$ and C_0 a basis of C. Let A_0 a basis of A over C and B_0 a basis of B over C. Then certainly $A_0B_0C_0$ is a generating set for AB. $\dim(AB) \leq |A_0| + |B_0| + |C_0|$. As $|C_0| = \dim C$, $|A_0| + |C_0| = \dim A$, $|B_0| + |C_0| = \dim B$, we get $\dim(AB) \leq \dim A + \dim B - \dim C$, as requested. □

Definition 1.2.9. In a pregeometry, we denote $A \mathop{\perp}^{\mathrm{cl}}_C B$ if for all finite subsets $A_0 \subseteq A$ we have $\dim(A_0/BC) = \dim(A_0/C)$.

Equivalently, $A \mathop{\perp}^{\mathrm{cl}}_C B$ if and only if every finite $A_0 \subseteq A$ which is independent over C stays independent over BC.

Exercise 12. Prove that for all A, B, C $A \mathop{\perp}^{\mathrm{cl}}_C B$ if and only if $A \mathop{\perp}^{\mathrm{cl}_C}_{\emptyset} B$.

Definition 1.2.10 (Axioms of independence relations, part 2). With an ambient closure operator cl.

(9) (anti-reflexivity) If $a \mathop{\perp}_C a$ then $a \in \mathrm{cl}(C)$;
(10) (right closure) $A \mathop{\perp}_C B \implies A \mathop{\perp}_C \mathrm{cl}(B)$.
(11) (strong closure) $A \mathop{\perp}_C B \iff \mathrm{cl}(AC) \mathop{\perp}_{\mathrm{cl}(C)} \mathrm{cl}(BC)$.

Remark 1.2.11. Note that if $\mathop{\perp}$ satisfies two-sided monotonicity, anti-reflexivity and strong closure then $\mathop{\perp} \to \mathop{\perp}^a$.

Exercise 13. Prove that if $\mathop{\perp}$ satisfies symmetry, normality, base monotonicity and closure, then $\mathop{\perp}$ satisfies the "⇒" direction of strong closure. What about the "⇐" direction?

Exercise 14. Check that if cl is a finitary closure operator, then $\mathop{\perp}^a$ satisfies further anti-reflexivity and strong closure.

Theorem 1.2.12. *Let (S, cl) be a pregeometry. Then, $\mathop{\perp}^{\mathrm{cl}}$ satisfies symmetry, finite character, existence, normality, monotonicity, base monotonicity, transitivity, anti-reflexivity, closure, strong closure, local character.*

Proof. (finite character left and right). Left finite character is clear by definition. For right finite character, assume that $A \mathop{\not\perp}^{\mathrm{cl}}_C B$ then there is a finite subset of A which is independent over C and which is not independent over BC. In particular, it is not independent over B_0C for a finite subset $B_0 \subseteq B$, by finite character, hence $A \mathop{\not\perp}^{\mathrm{cl}}_C B_0$. (monotonicity left and right) Left monotonicity is clear by definition. For right monotonicity, observe that for any finite A we have $\dim(A/C) \geq \dim(A/BC) \geq \dim(A/BDC)$ hence if $\dim(A/C) = \dim(A/BDC)$ then $\dim(A/C) = \dim(A/BC)$. It follows that $A \mathop{\perp}^{\mathrm{cl}}_C BD \implies A \mathop{\perp}^{\mathrm{cl}}_C B$. (symmetry) By Exercise 12, we may assume that $C = \emptyset$ and by two-sided finite character and monotonicity that A, B are finite. By Lemma 1.2.8, we have

$$\begin{aligned}\dim(AB) &= \dim(A/B) + \dim(B)\\ &= \dim(B/A) + \dim(A)\end{aligned}$$

Hence, if $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}} B$, then $\dim(A/B) = \dim(A)$, so $\dim(B/A) = \dim(B)$ and $B \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}} A$. (existence, normality) are trivial. Note that by left finite character we may assume that A is finite in what follows. (base monotonicity) Similarly as for monotonicity, $\dim(A/C) \geq \dim(A/CD) \geq \dim(A/BDC)$ hence if $\dim(A/C) = \dim(A/BDC)$ then $\dim(A/CD) = \dim(A/BCD)$. It follows that $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_C BD \implies A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{CD} B$. (transitivity) Assume that $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_B D$, with $C \subseteq B \subseteq D$. Then $\dim(A/C) = \dim(A/B)$ and $\dim(A/B) = \dim(A/D)$, hence $\dim(A/C) = \dim(A/D)$ so $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_C D$. (anti-reflexivity) If $\dim(a/Ca) = \dim(a/C)$, then $\dim(a/C) = 0$, so $\mathrm{cl}(C) = \mathrm{cl}(Ca)$ i.e. $a \in \mathrm{cl}(C)$. (closure, strong closure) We prove only strong closure as closure is proved similarly. By definition, for any A, D, $\dim(A/D) = \dim(\mathrm{cl}(AD)/\mathrm{cl}(D))$ hence $\dim(A/C) = \dim(A/BC)$ if and only if $\dim(\mathrm{cl}(AC)/\mathrm{cl}(C)) = \dim(\mathrm{cl}(ABC)/\mathrm{cl}(BC)) = \dim(\mathrm{cl}(AC)/\mathrm{cl}(BC))$, so

$$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_C B \iff \mathrm{cl}(AC) \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{\mathrm{cl}(C)} \mathrm{cl}(BC).$$

(local character) Assume that A is finite and B is any set. Let X be a basis of A over B. For each singleton $a \in A$, there exists a finite set $b_a \subseteq B$ such that $a \in \mathrm{cl}(Xb_a)$. Let $B_0 = \bigcup_{a \in A} b_a$. Then X is a basis of A over B_0, hence $\dim(A/B) = \dim(A/B_0)$, so $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{B_0} B$. We proved that: for all finite A and any B, there exists a finite $B_0 \subseteq B$ such that $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{B_0} B$. We conclude by Exercise 5. □

Exercise 15. Prove that $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_C B$ if and only if all subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ which are both independent over C are disjoint and their union is again independent over C.

Of course, we have $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}} \to \mathop{\smile\!\!\!\!\!\!|}^{a}$. In terms of properties they satisfy, the two relations $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ and $\mathop{\smile\!\!\!\!\!\!|}^{a}$ only differ by the property base monotonicity. Recent results in model theory make apparent a tension associated with the presence or the absence of the property base monotonicity for different notions of forking. As we will see later it is the turning point between simple and NSOP_1 theories when considering the independence relation associated to Kim-forking. In a pregeometry, $\mathop{\smile\!\!\!\!\!\!|}^{a}$ and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ are "close" to each other, as we will see now, and forcing the base monotonicity axiom on $\mathop{\smile\!\!\!\!\!\!|}^{a}$ preserves most of the axioms mentioned above. This is not true in general, as we will see later.

Definition 1.2.13. Let $\mathop{\smile\!\!\!\!\!\!|}$ be a ternary relation. We associate the *monotonisation* $\mathop{\smile\!\!\!\!\!\!|}^{m}$ *of* $\mathop{\smile\!\!\!\!\!\!|}$ which is defined as the following:

$$A \mathop{\smile\!\!\!\!\!\!|}^{m}_C B \iff A \mathop{\smile\!\!\!\!\!\!|}_X B \text{ for all } X \text{ with } C \subseteq X \subseteq \mathrm{cl}(BC).$$

Proposition 1.2.14. *The relation* $\mathop{\smile\!\!\!\!\!\!|}^{m}$ *satisfies right base monotonicity.*

- *If* $\mathop{\smile\!\!\!\!\!\!|}$ *satisfies left or right monotonicity, left or right closure, left normality, so does* $\mathop{\smile\!\!\!\!\!\!|}^{m}$*. If* $\mathop{\smile\!\!\!\!\!\!|}$ *further satisfies right normality or left transitivity, then so does* $\mathop{\smile\!\!\!\!\!\!|}^{m}$*.*
- *If* $\mathop{\smile\!\!\!\!\!\!|}$ *satisfies anti-reflexivity or (left or right) closure then so does* $\mathop{\smile\!\!\!\!\!\!|}^{m}$*.*
- *If* $\mathop{\smile\!\!\!\!\!\!|}^{0} \to \mathop{\smile\!\!\!\!\!\!|}$ *and* $\mathop{\smile\!\!\!\!\!\!|}^{0}$ *satisfies the right-sided instance of: normality, monotonicity, closure and base monotonicity, then* $\mathop{\smile\!\!\!\!\!\!|}^{0} \to \mathop{\smile\!\!\!\!\!\!|}^{m}$*.*

Proof. Let $A, C \subseteq B \subseteq D$ be such that $A \mathop{\smile\!\!\!\!\!\!|}^{m}_C D$. Then for all X such that $C \subseteq X \subseteq \mathrm{cl}(D)$ we have $A \mathop{\smile\!\!\!\!\!\!|}_X D$. In particular, for all Y such that $B \subseteq Y \subseteq \mathrm{cl}(BD) = \mathrm{cl}(D)$ we have $A \mathop{\smile\!\!\!\!\!\!|}_Y D$ so $A \mathop{\smile\!\!\!\!\!\!|}^{m}_B D$.

We turn to the first item. Left monotonicity, closure and normality are clearly preserved. For right monotonicity: if $A \mathop{\smile\!\!\!\!\!\!|}^{m}_C BD$ then in particular for all X with $C \subseteq X \subseteq \mathrm{cl}(CB)$ we have $A \mathop{\smile\!\!\!\!\!\!|}_X BD$ hence $A \mathop{\smile\!\!\!\!\!\!|}_X B$ by right monotonicity, hence $A \mathop{\smile\!\!\!\!\!\!|}^{m}_C B$. For right normality, assume that $A \mathop{\smile\!\!\!\!\!\!|}^{m}_C B$. Then for any X with $C \subseteq X \subseteq \mathrm{cl}(BC)$ we have $A \mathop{\smile\!\!\!\!\!\!|}_X B$ hence $A \mathop{\smile\!\!\!\!\!\!|}_X BX$ by right normality of $\mathop{\smile\!\!\!\!\!\!|}$ and hence $A \mathop{\smile\!\!\!\!\!\!|}_X BC$ by right monotonicity of $\mathop{\smile\!\!\!\!\!\!|}$. We conclude that $A \mathop{\smile\!\!\!\!\!\!|}^{m}_C BC$. We assume left transitivity and assume that $B \mathop{\smile\!\!\!\!\!\!|}^{m}_C A$ and $D \mathop{\smile\!\!\!\!\!\!|}^{m}_B A$ for $C \subseteq B \subseteq A$. To get $D \mathop{\smile\!\!\!\!\!\!|}^{m}_C A$, let X be such that $C \subseteq X \subseteq \mathrm{cl}(AC)$. Using $B \mathop{\smile\!\!\!\!\!\!|}^{m}_C A$ we get $B \mathop{\smile\!\!\!\!\!\!|}_X A$ and by left normality, we get $BX \mathop{\smile\!\!\!\!\!\!|}_X A$ $(*)$. Also, $B \subseteq XB \subseteq \mathrm{cl}(AB)$ hence using $D \mathop{\smile\!\!\!\!\!\!|}^{m}_B A$ we get

$D \mathop{\perp\!\!\!\perp}_{XB} A$ and by left normality we have $DXB \mathop{\perp\!\!\!\perp}_{XB} A$ $(**)$. By putting $(*)$ and $(**)$ together with left transitivity for $\mathop{\perp\!\!\!\perp}$ we get $DXB \mathop{\perp\!\!\!\perp}_X A$. By left monotonicity, we get $D \mathop{\perp\!\!\!\perp}_X A$.

We prove the second item. Preservation of anti-reflexivity is clear since $\mathop{\perp\!\!\!\perp}^m$ is stronger than $\mathop{\perp\!\!\!\perp}$. Preservation of left closure is clear. Assume that $\mathop{\perp\!\!\!\perp}$ satisfies right closure, and $A \mathop{\perp\!\!\!\perp}^m_C B$. Then $\mathrm{cl}(CB) = \mathrm{cl}(C\mathrm{cl}(B))$ hence for any $C \subseteq X \subseteq \mathrm{cl}(C\mathrm{cl}(B))$ we have $A \mathop{\perp\!\!\!\perp}_X B$ hence $A \mathop{\perp\!\!\!\perp}^m_C \mathrm{cl}(B)$.

We prove the last item. If $A \mathop{\perp\!\!\!\perp}^0_C B$, then by normality and closure, we have $A \mathop{\perp\!\!\!\perp}^0_C \mathrm{cl}(BC)$. Then, by base monotonicity, for all D with $C \subseteq D \subseteq \mathrm{cl}(BC)$ we have $A \mathop{\perp\!\!\!\perp}^0_D \mathrm{cl}(BC)$ so $A \mathop{\perp\!\!\!\perp}^0_D B$ by monotonicity. As $\mathop{\perp\!\!\!\perp}^0 \to \mathop{\perp\!\!\!\perp}$, we get $A \mathop{\perp\!\!\!\perp}_D B$. We conclude that $A \mathop{\perp\!\!\!\perp}^m_C B$. $\square$

Exercise 16. If $\mathop{\perp\!\!\!\perp}$ satisfies left base monotonicity does $\mathop{\perp\!\!\!\perp}^m$ satisfies left base monotonicity?

Theorem 1.2.15. *Let (S, cl) be a pregeometry then $\mathop{\perp\!\!\!\perp}^{a\, m} = \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}$. In particular, $\mathop{\perp\!\!\!\perp}^{\mathrm{cl}}$ is minimal (for $\to$) among the relations stronger than $\mathop{\perp\!\!\!\perp}^a$ which satisfy the right version of: normality, monotonicity, closure and base monotonicity.*

Proof. The direction $\mathop{\perp\!\!\!\perp}^{\mathrm{cl}} \to \mathop{\perp\!\!\!\perp}^{a\, m}$ always holds since $\mathop{\perp\!\!\!\perp}^{\mathrm{cl}} \to \mathop{\perp\!\!\!\perp}^a$ and $\mathop{\perp\!\!\!\perp}^{\mathrm{cl}}$ satisfies base monotonicity by Theorem 1.2.12. We prove the contrapositive of the converse. Assume that $A \mathop{\not\perp\!\!\!\perp}^{\mathrm{cl}}_C B$ for some A, B, C. Using strong closure we may assume that $C \subseteq A \cap B$ and A, B, C are cl-closed. By symmetry, we have $B \mathop{\not\perp\!\!\!\perp}^{\mathrm{cl}}_C A$, hence there exists a finite set $B_0 \subseteq B$ independent over C such that B_0 is not independent over A. Hence there exists $b \in B_0$ such that $b \in \mathrm{cl}(A \cup (B_0 \setminus \{b\}))$. Let $D = B_0 \setminus \{b\}$. As B_0 is independent over C, we have $b \notin \mathrm{cl}(CD)$. However $b \in \mathrm{cl}(AD) \cap \mathrm{cl}(B)$, hence $A \mathop{\not\perp\!\!\!\perp}^a_{CD} B$. In particular $A \mathop{\not\perp\!\!\!\perp}^{am}_C B$. $\square$

This gives a general definition of $\mathop{\perp\!\!\!\perp}$ cl even if there is no notion of dimension (i.e. if cl does not satisfy exchange).

Example 1.2.16. *Assume that* cl *is a closure operator on a set S. If* cl *satisfies exchange, then $\mathop{\perp\!\!\!\perp}^{a\, m}$ is symmetric. Is the converse true? The answer is no, we can see this in Example 1.2.7. One checks that $A \mathop{\perp\!\!\!\perp}^a_C B$ if and only if $\mathrm{cl}(A) \subseteq \mathrm{cl}(C)$ or $\mathrm{cl}(B) \subseteq \mathrm{cl}(C)$ if and only if* $\sup A \leq \sup C$ *or* $\sup B \leq \sup C$. *It is easy ti check that this relation satisfies base monotonicity.*

Proposition 1.2.17. *The following are equivalent:*

(1) If $x \in \mathrm{cl}(AB)$, then there exist singletons $a \in \mathrm{cl}(A)$ and $b \in \mathrm{cl}(B)$ such that $x \in \mathrm{cl}(ab)$;
(2) $\mathop{\perp\!\!\!\perp}^a$ satisfies base monotonicity;
(3) $\mathop{\perp\!\!\!\perp}^a = \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}$;
(4) For all A, B, $A \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{\mathrm{cl}(A)\cap\mathrm{cl}(B)} B$;
(5) (Modular law) $\dim(AB) + \dim(A \cap B) = \dim A + \dim B$, for all cl*-closed A, B.*

We say that cl *is modular if one of those statements hold.*

Proof. ($1 \implies 2$) Assume that $A \mathop{\perp\!\!\!\perp}^a_C BD$, we show $A \mathop{\perp\!\!\!\perp}^a_{CD} B$. Using (1), if $x \in \mathrm{cl}(ACD) \cap \mathrm{cl}(BCD)$, there exists $a \in \mathrm{cl}(AC)$ and $d \in \mathrm{cl}(D)$ such that $x \in \mathrm{cl}(ad)$. If $x \in \mathrm{cl}(D)$ we are done hence we may assume otherwise. By exchange, this implies $a \in \mathrm{cl}(xd) \subseteq \mathrm{cl}(BCD)$. Also, $a \in \mathrm{cl}(AC)$ hence by $A \mathop{\perp\!\!\!\perp}^a_C BD$ we get $a \in \mathrm{cl}(C)$ so $x \in \mathrm{cl}(ad) \subseteq \mathrm{cl}(CD)$. We have proved that $\mathrm{cl}(ACD) \cap \mathrm{cl}(BCD) = \mathrm{cl}(CD)$ hence $A \mathop{\perp\!\!\!\perp}^a_{CD} B$.

($2 \implies 3$) By Theorem 1.2.15 and Proposition 1.2.14.

($3 \implies 4$) By definition, $\mathrm{cl}(A) \mathop{\perp\!\!\!\perp}^a_{\mathrm{cl}(A)\cap\mathrm{cl}(B)} \mathrm{cl}(B)$ for all A, B. By two-sided monotonicity, we get $A \mathop{\perp\!\!\!\perp}^a_{\mathrm{cl}(A)\cap\mathrm{cl}(B)} B$ hence $A \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{\mathrm{cl}(A)\cap\mathrm{cl}(B)} B$ by hypothesis.

($4 \implies 5$) Let A, B be closed sets. By Lemma 1.2.8 *(1)*, we have $\dim(AB) = \dim(A/B) + \dim(B)$. As $A \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{A\cap B} B$, we have $\dim(A/B) = \dim(A/A \cap B)$. Using again Lemma 1.2.8 *(1)*, we get $\dim(A) = \dim(A/A \cap B) + \dim A \cap B$. We conclude by putting together the two equations.

($5 \implies 1$) We start with a claim.

Claim 3. Let A be closed and b be a singleton. If $x \in \mathrm{cl}(Ab)$ then there exists $a \in A$ such that $x \in \mathrm{cl}(ab)$.

Proof of the claim. We may assume that $x, b \notin A$. As $x \in \mathrm{cl}(Ab)$ (hence $b \in \mathrm{cl}(Ax)$) we have $\dim(xb/A) = 1$. Also we may assume $\dim(xb) = 2$. By the modular law, we have

$$\dim(Abx) + \dim(A \cap \mathrm{cl}(bx)) = \dim(A) + \dim(bx). \quad (*)$$

By Lemma 1.2.8 *(1)*, we also have

$$\dim(Abx) = \dim(bx/A) + \dim A \quad (**)$$

By putting together $(*)$ and $(**)$ with the above we conclude $\dim(A \cap \mathrm{cl}(bx)) = 1$. Let a be a basis of $A \cap \mathrm{cl}(bx)$. We have $a, b \notin \mathrm{cl}(\emptyset)$ hence as $b \notin \mathrm{cl}(a)$ we have $a \notin \mathrm{cl}(b)$ by exchange. As $a \in \mathrm{cl}(xb)$ we conclude again by exchange that $x \in \mathrm{cl}(ab)$. $\square$

We may assume that $A = \mathrm{cl}(A_0)$ and $B = \mathrm{cl}(B_0)$ for finite bases A_0, B_0 of A, B (respectively). We prove it by induction on $\dim(A_0B_0)$. Let $b \in B_0$ be such that $x \notin \mathrm{cl}(A_0B_0 \setminus \{b\})$ and $b \notin \mathrm{cl}(A_0B_0 \setminus \{b\})$. Then by the claim, there exists $y \in \mathrm{cl}(A_0B_0 \setminus \{b\})$ such that $x \in \mathrm{cl}(yb)$. As $\dim(A_0(B_0 \setminus \{b\})) < \dim(A_0B_0)$, we get by induction hypothesis that there exists $a \in A$ and $b' \in \mathrm{cl}(B_0 \setminus \{b\})$ such that $y \in \mathrm{cl}(ab')$, hence $x \in \mathrm{cl}(abb')$. Using again the claim (as a is a singleton), there exist $b'' \in \mathrm{cl}(bb') \subseteq B$ such that $x \in \mathrm{cl}(ab'')$. $\square$

Exercise 17. Let $(V, +, 0)$ be a vector space over some field K. For $A \subseteq V$ define $\langle A \rangle$ to be the vector span of A.

(1) Prove that $(V, \langle\cdot\rangle)$ is a pregeometry.
(2) Prove that $\mathop{\smile\!\!\!\!\!\mid}^{a} = \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}$, hence the pregeometry is modular.

Example 1.2.18. *Let $(K, +, \cdot, 0, 1)$ be an algebraically closed field of infinite transcendence degree. Define for any subset $A \subseteq K$ the closure operator A^{alg} to be the relative algebraic closure in K of the field generated by A. By Steinitz exchange principle, $A \mapsto A^{\mathrm{alg}}$ defines a pregeometry on subsets of K. This pregeometry is not modular, as $\mathop{\smile\!\!\!\!\!\mid}^{a}$ does not satisfy base monotonicity. To see this, let F be an algebraically closed subfield of K, such that $\mathrm{Trdeg}(K/F)$ is large enough. Let x, y, z be algebraically independent over F and $t = xz + y$. Then consider $A = F(x, y)^{\mathrm{alg}}$ and $B = F(z, t)^{\mathrm{alg}}$. We have $A \cap B = F$ hence $xy \mathop{\smile\!\!\!\!\!\mid}^{a}_{F} zt$ (this necessitates some checking, left as an exercise). However we don't have $xy \mathop{\smile\!\!\!\!\!\mid}^{a}_{Fz} t$ since $t \notin F(z)^{\mathrm{alg}}$ (again left as an exercise).*

CHAPTER 2

Independence relations in theories

2.1. Model-theoretic setting.

2.1.1. Special models. We assume known the notions of languages, sentences, theories, formulas, types, structures, definable sets, substructures, elementary substructures, models, elementary maps, elementary bijections and automorphism of models. We are given a complete theory T in a language $\mathscr{L}$ which is at most countable. By convenience, we assume that T has only infinite models.

Definition 2.1.1. We say that a model M of T is:

- (κ-universal): every model N of T with $|N| < \kappa$ elementary embeds in M;
- (κ-saturated): for all $A \subseteq M$ with $|A| < \kappa$, every type over A is realized in M;
- (strongly κ-homogeneous): for all (partial) elementary bijection $f : A \to B$, for $A, B \subseteq M$ with $|A| = |B| < \kappa$, there is an automorphism σ of M extending f.

The following is classical, we will not cover it in the course. Proofs of such results can be found in e.g. [**3**, Chapter 10]. A more set-theoretic approach is given in [**4**, Chapter 6].

Fact 2.1.2. *For any uncountable cardinal κ there exists a model of T which is κ^+-universal, κ-saturated and strongly κ-homogeneous.*

A *saturated* model is a model which is saturated in its own cardinality. Under some set-theoretic assumption, such models always exist, but we will not worry about those for now. If a theory is stable, then saturated models exist. The interested reader might consult [**2**] for an interesting note on those matters.

2.1.2. Monster models. We work in a so-called *monster model*[1] $\mathbb{M}$ of T, that is a model $\mathbb{M} \vDash T$ which is $\bar{\kappa}^+$-universal, $\bar{\kappa}$-saturated and strongly $\bar{\kappa}$-homogeneous **for some big enough** $\bar{\kappa}$. We fix once and for all a cardinal $\bar{\kappa}$ and a monster model $\mathbb{M}$ of T. Subsets $A, B, C, \dots$ and tuples $a, b, c, \dots$ of $\mathbb{M}$ are *small* if their cardinality is strictly smaller than $\bar{\kappa}$. For instance, a small model M of T has an isomorphic copy in $\mathbb{M}$ by $\bar{\kappa}$-universality, hence we will just consider that $M \prec \mathbb{M}$. We will often omit the mention of $\bar{\kappa}$, or $\mathbb{M}$ by just considering small models, sets and tuples.

2.1.3. Types. For a small set $C \subseteq \mathbb{M}$, we denote by $\mathscr{L}(C)$ the set of $\mathscr{L}$-formulas with parameters in C. If $\phi \in \mathscr{L}(C)$ this implicitely means that there is a tuple $x = (x_1, \dots, x_n)$ of variables which are the free variables of ϕ. We write $S_n(C)$ for the set of complete n-types over C, i.e. maximal consistent sets of $\mathscr{L}(C)$-formulas in variables $x_1, \dots, x_n$. For us, a type is always complete (i.e. maximal), and we will call *partial type* a consistent set of formula that may not be complete. Note that a single formula may be considered as a partial type. By $\bar{\kappa}$-saturation, being a consistent type over a small set is equivalent to having a realisation in the monster. Types may be in an infinite number of variables: fix an enumeration $(x_i)_{i<\alpha}$ of variables along some ordinal $\alpha < \bar{\kappa}$. A type $p(x_i)_{i<\alpha}$ is just a maximal consistent set of formulas $\phi((x_i)_{i \in S})$ where S is a finite

[1]The concept of a monster model is a rhetorical subterfuge invented by model theorists to avoid statements of theorems that starts by "Let $A \subseteq M$ where M is $\beth_{2^{|A|^+}}$-saturated...". It allows to avoid taking successive extensions of models to realize types or have automorphisms. This strategy is very similar to the strategy taken in classical algebraic geometry (not the scheme language, before that!) where one would work in a big algebraically closed field of infinite transcendence degree, in order to have the existence of generics of a variety, and automorphisms by infinite Galois theory.

subset of α. In this case, we write $p \in S_\alpha(C)$. Often the number of variables in the type will not matter, hence we will just write $S(C)$ for the set of types over C in a small number of variables. For any small tuple a we write $\mathrm{tp}(a/C)$ for the complete type $\{\phi(x) \in \mathscr{L}(C) \mid \mathbb{M} \vDash \phi(a)\}$. This defines a map $\mathbb{M}^\alpha \to S_\alpha(C)$ via $a \mapsto \mathrm{tp}(a/C)$. This map is surjective by saturation. This implies that we may always consider a type (over a small set) via its realisations, and that proving that a type is consistent is equivalent to find a realisation of it.

2.1.4. Topology on the space of types. The "S" in $S(C)$ stands for Stone, as $S(C)$ is the Stone space of the boolean algebra $\mathscr{L}(C)$ of formulas, identified with definable subsets of $\mathbb{M}$. It is the set of ultrafilters on $\mathscr{L}(C)$ and is a topological space. The topology on $S(C)$ is given by basic clopen sets (=closed and open) of the form

$$[\phi] = \{p \in S(C) \mid p \vDash \phi\}$$

for $\phi \in \mathscr{L}(C)$.

Exercise 18. Check that the set $\{[\phi] \mid \phi \in \mathscr{L}(C)\}$ is a basis of topology. Check that $[\phi]$ are clopen for this topology and that the topology is Hausdorff.

The cornerstone of model theory is the so-called compactness theorem:

Fact 2.1.3 (Compactness). *$S(C)$ is a compact topological space.*

A more practical statement of the compactness theorem is the following.

Let $\pi(x)$ be a partial type over C, then $\pi(x)$ is consistent if and only if it is finitely satisfiable (π_0 admits a realisation for all finite subsets $\pi_0 \subseteq \pi$).

In our setting, this is equivalent to:

$\pi(x)$ admits a realisation $a \vDash \pi$ if and only if for all finite $\pi_0 \subseteq \pi$, there exist $a_0 \vDash \pi_0$.

To see that this statement is equivalent to the compactness of $S(C)$: observe that finite satisfiability of π is equivalent the family $\{[\phi] \mid \phi \in \pi\}$ having the finite intersection property: $a \vDash \phi_1 \wedge \ldots \wedge \phi_n$ if and only if $\mathrm{tp}(a/C) \in [\phi_1] \cap \ldots \cap [\phi_n]$. Compactness of $S(C)$ is then equivalent to saying that there exist $p \in \cap_{\phi\in\pi}[\phi]$ hence any realisation a of p will satisfy π.

2.1.5. Automorphisms and the Galois approach. The set of automorphisms of $\mathbb{M}$ is denoted $\mathrm{Aut}(\mathbb{M})$. Those are the bijections of $\mathbb{M}$ which satisfy $\mathbb{M} \vDash \phi(a_1, \ldots, a_n)$ if and only if $\mathbb{M} \vDash \phi(\sigma(a_1), \ldots, \sigma(a_n))$ for all $\mathscr{L}$-formula $\phi(x_1, \ldots, x_n)$, $a_1, \ldots, a_n \in \mathbb{M}$. Of course, $\mathrm{Aut}(\mathbb{M})$ is a group for the composition of maps. For a small set $C \subseteq \mathbb{M}$, an automorphism *over C* is an automorphism σ that fixes the set C pointwise ($\sigma(c) = c\ \forall c \in C$). We write $\mathrm{Aut}(\mathbb{M}/C)$ for the subgroup of $\mathrm{Aut}(\mathbb{M})$ consisting of all automorphisms over C. The group $\mathrm{Aut}(\mathbb{M}/C)$ acts on $\mathbb{M}$ and fixes the set $p(\mathbb{M}) = \{a \in \mathbb{M} \mid a \vDash p\}$ setwise, for any $p \in S(C)$. Conversely, by strong $\bar{\kappa}$-homogeneity, if $a, b \vDash p$ and $p \in S(C)$, then there is an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ such that $\sigma(a) = b$.

Exercise 19. Check that $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$ if and only if there is $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ such that $\sigma(a) = b$.

We denote $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$ by $a \equiv_C b$. We then have

$$a \equiv_C b \iff \sigma(a) = b \text{ for some } \sigma \in \mathrm{Aut}(\mathbb{M}/C).$$

Thus we may and will identify a types over a small sets $C \subseteq \mathbb{M}$ with an orbit under the automorphism group $\mathrm{Aut}(\mathbb{M}/C)$. The group $\mathrm{Aut}(\mathbb{M})$ also acts on the set $S(C)$ via the mapping $p \mapsto p^\sigma$ where $p^\sigma = \{\phi(x, \sigma(c)) \mid \phi(x, c) \in p\}$. For this action, $\mathrm{Aut}(\mathbb{M}/C)$ fixes $S(C)$ pointwise.

Before we continue, we recall notations and definitions that we just introduces and that we will use until the end of this text.

- $A, B, C, \ldots$ are small subsets of $\mathbb{M}$, $a, b, c, \ldots$ are small tuples from $\mathbb{M}$ maybe of infinite length, for instance indexed by some ordinal α;

- For a formula $\phi(x)$ and a a tuple from $\mathbb{M}$ such that $|a| = |x|$ (a and x are indexed by the same ordinal), then $a \models \phi(x)$ stands for $\mathbb{M} \models \phi(a)$, sometimes simply denoted $\models \phi(a)$. $\phi(\mathbb{M})$ is the set of realisations of ϕ in $\mathbb{M}$. Those notations extends to partial types: $a \models \pi(x)$, $\models \pi(a)$, $\pi(\mathbb{M})$.
- For two partial types in the same variable $\pi(x), \Sigma(x)$, we denote $T \models \forall x(\Sigma(x) \to \pi(x))$ or equivalently $\Sigma(\mathbb{M}) \subseteq \pi(\mathbb{M}))$ by $\Sigma \models \pi$. If Σ is closed under finite conjunctions, this is equivalent to $\pi(x) \subseteq \Sigma(x)$ (as sets of formulas). Also if $\phi \in p$ we denote $p \models \phi$;
- $\mathscr{L}$ is the language of T, $\mathscr{L}(C)$ denotes formulas with parameters in a set C, sometimes $\mathscr{L}_n(C)$ denotes formulas in variables $x_1, \ldots, x_n$;
- $S_\alpha(C)$ the space of (complete) types over C in variables indexed by an ordinal α, $S(C)$ stands for $S_\alpha(C)$ for some α;
- $\mathrm{Aut}(\mathbb{M}/C)$ is the group of automorphism of $\mathbb{M}$ fixing C pointwise;
- $a \equiv_C b$ iff $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$, where $\mathrm{tp}(a/C) = \{\phi(x) \mid\models \phi(a) \text{ for } \phi(x) \in \mathscr{L}(C)\}$;
- For small sets, $A \equiv_C B$ stands for "there exists an elementary bijection $f : A \to B$ fixing C pointwise". By strong $\bar{\kappa}$-homogeneity, this is equivalent to "there exists an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ such that $\sigma(A) = B$ (as sets)". This is equivalent to saying that for any enumeration $a = (a_i)_{i<\alpha}$ of A, there exists and enumeration $b = (b_i)_{i<\alpha}$ of B such that $a \equiv_C b$.

2.1.6. The algebraic closure and the definable closure.

Lemma 2.1.4. *Let X be a topological space, $Y_1, Y_2 \subseteq X$ be two compact subsets. Let $\mathcal{H}$ be a set of clopen in X such which is closed by finite union and intersections. The following are equivalent:*

(1) There exists $B \in \mathcal{H}$ such that $Y_1 \subseteq B$ and $Y_2 \cap B = \emptyset$.
(2) for all $y_1 \in Y_1$, $y_2 \in Y_2$, there exists $B \in \mathcal{H}$ such that $y_1 \in Y_1$ and $y_2 \notin Y_2$.

Proof. ($1 \implies 2$) is trivial. We assume that (2) holds.

Claim 4. For each $y_1 \in Y_1$ there exist $B(y_1) \in \mathcal{H}$ such that $y_1 \in B(y_1)$ and $Y_2 \cap B(y_1) = \emptyset$.

Proof of the claim. Let $y_1 \in Y_1$ and consider the set $\mathcal{H}(y_1)$ consisting of those $B \in \mathcal{H}$ such that $y_1 \in B$. By (2), for all $y_2 \in Y_2$ there exists $B \in \mathcal{H}(y_1)$ such that $y_2 \in X \setminus B$. It follows that

$$Y_2 \subseteq \bigcup_{B \in \mathcal{H}(y_1)} X \setminus B$$

As $\mathcal{H}$ is a set of clopen, $X \setminus B$ are open and hence by compactness of Y_2 there exists $B_1, \ldots, B_n \in \mathcal{H}(y_1)$ such that $Y_2 \subseteq (X \setminus B_1) \cup \ldots \cup (X \setminus B_n) = X \setminus (B_1 \cap \ldots \cap B_n)$. As $\mathcal{H}$ is closed under positive boolean combination we have $B(y_1) := B_1 \cap \ldots \cap B_n \in \mathcal{H}$. Also clearly $y_1 \in B(y_1)$. □

By the claim, we have

$$Y_1 \subseteq \bigcup_{y_1 \in Y_1} B(y_1)$$

As $B(y_1)$ is open, again by compactness, there exists $B^1, \ldots, B^n \in \{B(y_1) \mid y_1 \in Y_1\}$ such that $Y_1 \subseteq B^1 \cup \ldots \cup B^n$. Let $B = B^1 \cup \ldots \cup B^n$. As $\mathcal{H}$ is closed under unions, $B \in \mathcal{H}$. As $Y_2 \cap B^i = \emptyset$ for each i we have $Y_2 \cap B = \emptyset$. □

Proposition 2.1.5. *Let X be a clopen of $S(C)$, then $X = [\phi]$ for some $\phi(x) \in \mathscr{L}(C)$.*

Proof. Let $\mathcal{H}$ be the set of all $[\phi]$ for $\phi \in \mathscr{L}(C)$. Clearly, $\mathcal{H}$ is closed by positive boolean combinations (even by complement). If X is clopen, then X and $(S(C) \setminus X)$ are closed hence compact. For each $p \in X$ and $q \in S(C) \setminus X$ we have $p \neq q$ hence there exists $\phi \in p$ such that $\phi \notin q$. Then $p \in [\phi]$ and $q \notin [\phi]$. By Lemma 2.1.4 there exists $\phi \in \mathscr{L}(C)$ such that $X \subseteq [\phi]$ and $(S(C) \setminus X) \subseteq S(C) \setminus [\phi]$ hence $[\phi] \subseteq X$, so $X = [\phi]$. □

Exercise 20. Let $C \subseteq B$ and consider the restriction map $\pi : S(B) \to S(C)$ defined by

$$\pi(p) = p \restriction C = \{\theta \in \mathscr{L}(C) \mid \theta \in p\}.$$

Prove that π is continuous and onto.

Lemma 2.1.6. *Let X be a definable set and C a small set. The following are equivalent:*

(1) X is definable over C;

(2) $\sigma(X) = X$ for all $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$.

Proof. ($1 \implies 2$) is clear: if $a \in X = \phi(\mathbb{M}, c)$ for $\phi(x, c) \in \mathscr{L}(C)$. Then $\vDash \phi(\sigma(a), c)$ since σ fixes C pointwise, hence $\sigma(a) \in X$.

($2 \implies 1$) Assume that X is defined by a formula ϕ over B and that $C \subseteq B$ (you can change B to $B \cup C$). Consider the restriction map $\pi : S(B) \to S(C)$ such that $\pi(p) = p \upharpoonright C$. By Exercise 20 π is continuous and onto. Let $Y \subseteq S(C)$ be the image of $[\phi]$ under π. Then Y is closed as the direct image of a compact by a continuous function.

Claim 5. $[\phi] = \pi^{-1}(Y)$.

Proof. As $Y = \pi([\phi])$, we get $[\phi] \subseteq \pi^{-1}(Y)$. Assume that $p \in \pi^{-1}(Y)$, so that $p \upharpoonright C =: q \in Y$. As $q \in Y$ there exist $p' \in [\phi]$ such that $p' \upharpoonright C = q = p \upharpoonright C$. Consider $a \vDash p, a' \vDash p'$. As $X = \phi(\mathbb{M})$, $a' \in X$. Also, $a \equiv_C a'$ hence by invariance, $a \in X$. It follows that $\phi \in p$ hence $p \in [\phi]$. $\square$

As π is onto, we get from the claim that $S(C) \setminus Y = \pi(S(B) \setminus [\phi]) = \pi([\neg\phi])$ hence Y is open. By Proposition 2.1.5 we conclude that $Y = [\psi]$ for some $\psi \in \mathscr{L}(A)$. We conclude: $a \in X \iff \mathrm{tp}(a/B) \in [\phi] \iff \pi(\mathrm{tp}(a/B)) = \mathrm{tp}(a/C) \in [\psi] \iff a \vDash \psi$ hence $X = \psi(\mathbb{M})$. $\square$

Definition 2.1.7. Let A be a small set and b a singleton.

- (Algebraic closure) We say that b is *algebraic over A* if there exists an $\mathscr{L}(A)$-formula $\phi(x)$ such that $b \vDash \phi$ and $\phi(\mathbb{M})$ is finite. We denote by $\mathrm{acl}(A)$ the set of all $b \in \mathbb{M}$ which are algebraic over A.
- (Definable closure) We say that b is *definable over A* if there exists an $\mathscr{L}(A)$-formula $\phi(x)$ such that $\phi(\mathbb{M}) = \{b\}$. We denote by $\mathrm{dcl}(A)$ the set of all $b \in \mathbb{M}$ which are definable over A.

A partial type $\pi(x)$ with finitely many realisations ($\pi(\mathbb{M})$ is finite) is called *algebraic*. If $\pi = \{\phi\}$, the formula $\phi(x)$ is called *algebraic*.

Exercise 21. Prove that for elementary extensions $M \prec N$, $\phi(M)$ is finite if and only if $\phi(N)$ is finite.

Exercise 22 (*). Prove that acl and dcl are closure operators on $\mathbb{M}$.

Proposition 2.1.8. *Let C be a small set and a a singleton.*

(1) $a \in \mathrm{acl}(C)$ if and only if the orbit of a under $\mathrm{Aut}(\mathbb{M}/C)$ is finite.

(2) $a \in \mathrm{dcl}(C)$ if and only if $\sigma(a) = a$ for all $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$.

Proof. *(1)* If $a \in \mathrm{acl}(C)$, let $\phi(x)$ be a formula over C such that $\phi(x)$ has only finitely many realisations: $a_1, a_2, \ldots, a_n$ with $a = a_1$. If $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$, then for all $b \vDash \phi$ we have $\sigma(b) \vDash \phi$ hence the orbit of a under $\mathrm{Aut}(\mathbb{M}/C)$ is contained in $\{a_1, \ldots, a_n\}$. Conversely, let $a_1, \ldots, a_n$ be the set of all conjugates of a over C, say with $a_1 = a$. Then the set $\{a_1, \ldots, a_n\}$ is invariant under $\mathrm{Aut}(\mathbb{M}/C)$ hence this set is C-definable using Lemma 2.1.6. *(2)* Similarly as above, $a \in \mathrm{dcl}(C)$ if and only if the set $\{a\}$ is definable over C which is equivalent to a being invariant under $\mathrm{Aut}(\mathbb{M}/C)$ by Lemma 2.1.6. $\square$

Proposition 2.1.9. *Let $f : A \to B$ be an elementary bijection, for some small subsets A, B. Then f extends to an elementary bijection $f' : \mathrm{acl}(A) \to \mathrm{acl}(B)$.*

Proof. By strong $\bar{\kappa}$-homogeneity, there exists an automorphism σ extending f. For all $a \in \mathrm{acl}(A)$ we have $\sigma(a) \in \mathrm{acl}(B)$ hence $f' = \sigma \upharpoonright \mathrm{acl}(A)$ extends f and is an elementary bijection. $\square$

Exercise 23. Let $a \in \mathbb{M}$ be algebraic over C. Prove that there exists a formula $\phi(x) \in \mathscr{L}(C)$ such that $\phi(\mathbb{M})$ is the orbit of a over $\mathrm{Aut}(\mathbb{M}/C)$. Prove that $\phi(\mathbb{M}) = p(\mathbb{M})$ where $p(x) = \mathrm{tp}(a/C)$. Such a formula $\phi(x)$ is said to *isolates* the type of a over C.

Exercise 24 (*). Prove that if $a \equiv_C b$ then there exists $a' \equiv_C a$ such that $ab \equiv_C ba'$.

2.2. Theories: working examples

2.2.1. A first example: algebraically closed fields. Let $\mathscr{L}_{\text{ring}}$ be the language of rings, i.e. $\mathscr{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$ and let T be the (incomplete) theory of fields, that is $(K, +, -, \cdot, 0, 1) \vDash T$ if an only if T is a field. Note that, somehow traditionally, we do not include the function $x \mapsto x^{-1}$, hence an $\mathscr{L}$-substructure of a model of T is an integral domain. We consider the expansion ACF of T defined by adding to T the following axiom-scheme:

$$\forall x_0 \dots \forall x_n \exists y (x_n \neq 0 \to x_0 + x_1 y + \dots + x_n y^n = 0)$$

for all $n \in \mathbb{N}$, so that $K \vDash \text{ACF}$ if and only if K is an algebraically closed field.

We recall the following classical criterion for proving quantifier elimination:

FACT 2.2.1. *An $\mathscr{L}$-theory T has quantifier elimination if and only if for all $M, N \vDash T$ with a common substructure A and for all primitive existential formula $\exists x \phi(x)$ where $\phi \in \mathscr{L}(A)$ is a conjunction of atomic and negatomic,*

$$M \vDash \exists x \phi(x) \implies N \vDash \phi(x).$$

We may assume that N is $|A|$-saturated.

THEOREM 2.2.2 (Tarski). *ACF has quantifier elimination.*

PROOF. Let K_1, K_2 be two algebraically closed fields, with K_2 $|A|$-saturated. In particular the transcendence degree of K_2 over A is infinite. Let $\phi(x)$ be a conjunction of atomic and negatomic formulas with parameters in A. Then $\phi(x)$ is a conjunction of polynomial equations and inequation with parameters in A. Assume that $b \in K_1$ is such that $K_1 \vDash \phi(b)$. It is enough to find $b' \in K_2$ such that $K_2 \vDash \phi(b')$. The fraction field A_1 of A in K_1 and A_2 of A in K_2 are isomorphic via a function $f : A_1 \to A_2$, hence if $b \in A_1$ then we can choose $b' = f(b)$. We assume $b \notin A_1$. Let F_i be the algebraic closure of A_i in K_i. By Steinitz Theorem (unicity of algebraic closure) there exists a field isomorphism $g : F_1 \to F_2$ extending f, hence similarly, we may assume that $b \notin F_1$. It follows that $\phi(x)$ is a conjunction of polynomial inequations, hence we conclude by taking b' transcendental over A. $\square$

Recall that a theory T is *model-complete* if for all $M, N \vDash T$ we have

$$M \subseteq N \implies M \prec N.$$

Every theory with quantifier elimination is model-complete.

COROLLARY 2.2.3. *ACF is model-complete.*

EXERCISE 25. For p a prime number or 0, we denote ACF_p the expansion of ACF expressing that the characteristic is p. Check that $\mathbb{Q}^{\text{alg}}$ and $\mathbb{F}_p^{\text{alg}}$ are prime models and deduce that any completion of ACF is of the form ACF_p for some p prime or 0.

Let $\mathbb{K}$ be a monster model for ACF_p, for $p \in \mathbb{P} \cup \{0\}$.

COROLLARY 2.2.4. *Let A be a small subset of $\mathbb{K}$.*

(1) Every definable subset of $\mathbb{K}$ is either finite or cofinite;
(2) $\text{acl}(A)$ *is the field theoretic algebraic closure of the field generated by A, denoted A^{alg};*
(3) (p=0) $\text{dcl}(A)$ *is the field generated by A, $\mathbb{Q}(A)$;*
(4) (p>0) $\text{dcl}(A)$ *is the perfect hull of $\mathbb{F}_p(A)$, i.e.* $\text{dcl}(A) = \left\{ b \in \mathbb{K} \mid b^{p^n} \in \mathbb{F}_p(A),\ n \in \mathbb{N}^{>0} \right\}$.

PROOF. *(1)* By quantifier elimination, every definable set is a boolean combination of polynomial equations. Every polynomial equation defines a finite set and a boolean combination of finite sets is finite or cofinite.
(2) Clear by quantifier elimination.
(3) and (4) Let F be $\mathbb{Q}(A)$ or $\mathbb{F}_p(A)$ and $b \in \text{dcl}(F)$. If b is transcendental over F, then it has many conjugates over F by Proposition 2.1.8 so we may assume that $b \in F^{\text{alg}}$. Let K be the normal closure of $F(b)$. Any automorphism of F^{alg} over F extends to a global automorphism of

$\mathbb{K}$, which, by Proposition 2.1.8 fixes b. In particular, by Galois theory, b is the only root of its minimal polynomial over F hence $K = F(b)$ and b is purely inseparable over F. $\square$

Exercise 26. Let a be an element algebraic over a field C. Prove that the equation $m(x) = 0$ isolates $\mathrm{tp}(a/C)$, where $m(X)$ is the minimal polynomial of a over C.

Exercise 27 (*). Prove that for small subfields $A, B \subseteq \mathbb{K}$, we have that any field isomorphism $f : A \to B$ is a partial elementary map in $\mathbb{K}$. In particular, $A \equiv B$ if and only if A and B are isomorphic as fields.

2.2.2. Digression: strongly minimal theories.

Definition 2.2.5. A theory T is *strongly minimal* if $\phi(\mathbb{M})$ is finite or cofinite, for all $\phi(x) \in \mathscr{L}_1(C)$, for all (small) C.

Remark 2.2.6. In the definition above, it is important to look at realisations of formulas in a saturated model: consider $(\mathbb{N}, <)$ then every definable set is finite or cofinite however in elementary extensions of $(\mathbb{N}, <)$ there exists infinite co-infinite definable sets, so $\mathrm{Th}(\mathbb{N}, <)$ is not strongly minimal.

Lemma 2.2.7. *Let T be strongly minimal and $p \in S_1(C)$ be a non-algebraic type. Then for all $C \subseteq B$ there exists a unique non-algebraic type $q \in S_1(B)$ extending p.*

Proof. If $p \in S_1(C)$ is non algebraic then for every formula $\phi(x) \in \mathscr{L}_1(C)$ we have $\phi \in p$ if and only if $\phi(\mathbb{M})$ is cofinite. This gives an intuition of what q should be: we define

$$q := \{\psi \in \mathscr{L}(B) \mid \psi(\mathbb{M}) \text{ is cofinite}\}.$$

q is clearly consistent by compactness: any finite number of cofinite sets always intersect. q is unique by definition. $\square$

Proposition 2.2.8. *Let T be strongly minimal. Then we have the following:*

- *(full existence) for all $a, C \subseteq B$ where $|a| = 1$ there exists $a' \equiv_C a$ with $a' \notin \mathrm{acl}(B)$;*
- *(stationarity) for all $a, a', C \subseteq B$ with $C = \mathrm{acl}(C)$ and $|a| = |a'| = 1$, if $a \equiv_C a'$, $a \notin \mathrm{acl}(B)$ and $a' \notin \mathrm{acl}(B)$ then $a \equiv_B a'$.*

Further, for every $a \notin \mathrm{acl}(C)$ and ordinal α, there exists an infinite sequence $(a_i)_{i<\alpha}$ such that $a_i \equiv_C a$, $a_0 = a$ and $a_i \notin \mathrm{acl}(Ca_{<i})$ for all $i < \alpha$. For any other such sequence $(b_i)_{i<\beta}$ starting with some $b \notin \mathrm{acl}(C)$ we have

$$a_{i_1} \dots a_{i_n} \equiv_C b_{j_1} \dots b_{j_n}$$

for all $(i_1 < \ldots < i_n), (j_1 < \ldots < j_n)$ tuples of distinct elements in α, β respectively.

Proof. The two items are a rewriting of Lemma 2.2.7. If $a \in \mathrm{acl}(C)$ then take $a' = a$. Otherwise, take a' to satisfy the (unique) extension q to B of $\mathrm{tp}(a/C)$. We have $a' \equiv_C a$ and $a' \notin \mathrm{cl}(B)$. Assume the hypotheses of the second item. If $a \in \mathrm{acl}(C) = C$ then $a' = a$ hence $a \equiv_B a'$. Otherwise, $\mathrm{tp}(a/C)$ is non-algebraic and has a unique extension q over B. As $a' \equiv_C a$, we also have $a' \notin C$. By hypotheses, we also have $a \notin \mathrm{acl}(B)$ and $a' \notin \mathrm{acl}(B)$ hence a and a' both satisfy q, so $a \equiv_B a'$. For the rest, take $a_0 = a \notin \mathrm{acl}(C)$. By full existence, take $a_1 \equiv_C a_0$ with $a_1 \notin \mathrm{acl}(Ca_0)$. Then $a_2 \equiv_C a_0$ with $a_2 \notin \mathrm{acl}(Ca_0a_1)$ and by induction $a_i \equiv_C a$ with $a_i \notin \mathrm{acl}(Ca_{<i})$. To finish the proof it is enough to prove that $a_0 \dots a_n \equiv_C b_{i_0} \dots b_{i_n}$ for all $i_0 < \ldots < i_n < \beta$. For $n = 0$ this is by Lemma 2.2.7 (for $B = C$). Assume by induction that $a_0 \dots a_{n-1} \equiv_C b_{i_0} \dots b_{i_{n-1}}$ with $i_n > i_{n-1} > \ldots > i_0$. Let σ be an automorphism over C which maps $b_{i_0} \dots b_{i_{n-1}}$ to $a_0 \dots a_{n-1}$ and let $b' = \sigma(b_{i_n})$. As $b_{i_n} \notin \mathrm{acl}(Cb_{<i_n})$, we have $b_{i_n} \notin \mathrm{acl}(Cb_{i_0} \dots b_{i_{n-1}})$. Applying σ, we have $b' \notin \mathrm{acl}(Ca_0 \dots a_{n-1})$. Also, $a_n \notin \mathrm{acl}(Ca_0 \dots a_{n-1})$. Using the property (stationarity) we have $b' \equiv_{Ca_0 \dots a_{n-1}} a_n$ hence $a_0 \dots a_{n-1}b' \equiv_C a_0 \dots a_n$ so by applying σ^{-1} we get $b_{i_0} \dots b_{i_n} \equiv_C a_0 \dots a_n$. $\square$

Remark 2.2.9. The second item is false in general if C is not algebraically closed. For a counterexample, take a, a' two distinct conjugates over C and $B = \mathrm{acl}(C)$.

Corollary 2.2.10. *Let T be strongly minimal, then $A \mapsto \operatorname{acl}(A)$ defines a pregeometry.*

Proof. By considering the localised closure operator, it is enough to prove that if $a, b \notin \operatorname{acl}(\emptyset)$ and $b \in \operatorname{acl}(a)$ then $a \in \operatorname{acl}(b)$. Equivalently, we assume that $a \notin \operatorname{acl}(b)$ and we prove that $b \notin \operatorname{acl}(a)$. Let $a_0 = b$, $a_1 = a$. Proposition 2.2.8 let $a_2 \notin \operatorname{acl}(a_0a_1)$ with $a_2 \equiv a_0$, and iteratively $a_n \notin \operatorname{acl}(a_{<n})$ with $a_n \equiv a_0$, for all $n \leq \omega$. As $a \notin \operatorname{acl}(b)$, the sequence satisfies $a_i \notin \operatorname{acl}(a_{<i})$ for all $i < \omega$. By Proposition 2.2.8 we have $a_0a_1 \equiv a_ia_j$ for all $i < j \leq \omega$. On one side we have $a_0a_\omega \equiv a_ia_\omega$ hence $\operatorname{tp}(a_0/a_\omega) = \operatorname{tp}(a_i/a_\omega)$ for all $i < \omega$. As the sequence is infinite, this implies that $\operatorname{tp}(a_0/a_\omega)$ is non-algebraic. On the other side, as $a_0a_1 \equiv a_0a_\omega$, we have $a_0 \notin \operatorname{acl}(a_1)$ i.e. $b \notin \operatorname{acl}(a)$. $\square$

Remark 2.2.11. Consider $\mathop{\smile\!\!\!\!\!\mid}^{a}$ in T for $\operatorname{cl} = \operatorname{acl}$ i.e.

$$A \mathop{\smile\!\!\!\!\!\mid}^{a}_{C} B \iff \operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl}(C).$$

As acl is a pregeometry, $a \notin \operatorname{acl}(B)$ is equivalent to $a \mathop{\smile\!\!\!\!\!\mid}^{a\,m} B$ hence $a \mathop{\smile\!\!\!\!\!\mid}^{a}_{C} B$ for all $C \subseteq B$. By Proposition 2.2.8, $\mathop{\smile\!\!\!\!\!\mid}^{a}$ satisfies the following:

- (full existence) for all $a, C \subseteq B$ where $|a| = 1$ there exists $a' \equiv_C a$ with $a' \mathop{\smile\!\!\!\!\!\mid}^{a}_{C} B$;
- (stationarity) for all $a, a', C \subseteq B$ with $C = \operatorname{acl}(C)$ and $|a| = |a'| = 1$, if $a \equiv_C a'$, $a \mathop{\smile\!\!\!\!\!\mid}^{a}_{C} B$ and $a' \mathop{\smile\!\!\!\!\!\mid}^{a}_{C} B$ then $a \equiv_B a'$.

Remark 2.2.12 (Zilber's trichotomy conjecture). We have seen essentially 3 types of pregeometries: trivial ones (where $\operatorname{cl}(X) = X$), modular ones (as in vector spaces or divisible groups) and the one in ACF which is not modular. Zilber's trichotomy, which dates back to the 70's, essentially says that those are the only ones in a strongly minimal theory:

If T is strongly minimal, then the geometry induced by the pregeometry acl is either the trivial geometry ($\operatorname{acl}(X) = X$ for all X), either an affine or projective geometry over a division ring, or T interprets an algebraically closed field.

The first two cases are often translated into a single notion which corresponds to the case where the pregeometry is called *locally modular*. So the conjecture is often restated as: *every non locally modular strongly minimal theory interprets an algebraically closed field.* It was shown later that the conjecture fails, in 1993 Hrushovski introduced special forms of Fraïssé limits –now called *Hrushovski constructions*– to construct a counterexample to Zilber's conjecture. Nonetheless, the idea of classifying what sorts of pregeometry appear in reducts or in interpretable structures persisted and have been the source of many applications of model theory to other branches of mathematics. Those are called *restricted trichotomy conjectures* and are shown to hold in several theories. For instance, a form of the trichotomy conjecture proved for generic difference fields was used by Hrushovski to prove the Manin-Mumford conjecture in algebraic geometry in 2001. This field of research is still active today, in 2022 Castle proved the following restricted trichotomy, which was long overdue: if M is a non locally modular strongly minimal structure interpretable in an algebraically closed field K of characteristic 0 then M interprets K.

Exercise 28. Assume that $a_1, \ldots, a_n$ are such that $a_i \notin \operatorname{acl}(Ca_{<i})$ for all $1 \leq i \leq n$, then $a_1, \ldots, a_n$ are independent over C.

Exercise 29 (Tarski's Test). Let $S \subseteq \mathbb{M}$. The following are equivalent:

- $\phi(\mathbb{M}) \cap S \neq \emptyset$, for all $\phi(x) \in \mathscr{L}_1(S)$ with $\phi(\mathbb{M}) \neq \emptyset$;
- S is the domain of an elementary substructure of $\mathbb{M}$.

In a strongly minimal theory, we denote by $\dim(S)$ the dimension of any subset of $\mathbb{M}$, relatively to the pregeometry acl.

Corollary 2.2.13. *Let T be strongly minimal.*

(1) Every infinite algebraically closed set $S = \operatorname{acl}(S)$ is the domain of an elementary substructure of $\mathbb{M}$.
(2) Two models M and N are isomorphic if and only if $\dim(M) = \dim(N)$.

(3) *For all uncountable cardinal κ, T is κ-categorical: if $M, N \models T$ and $|M| = |N| = \kappa$ then $M \cong N$.*

Proof. *(1)* Let $\phi(x) \in \mathscr{L}_1(S)$. If $\phi(\mathbb{M})$ is finite it consists of elements algebraic over S hence $\phi(\mathbb{M}) \subseteq \operatorname{acl}(S) = S$. If $\phi(\mathbb{M})$ is infinite, it is cofinite in $\mathbb{M}$ hence intersects any infinite set, in particular S. We conclude by Exercise 29.

(2) Let M and N be two small models of T of the same dimension κ. Let $a = (a_i)_{i<\kappa}$ be a basis of M and $b = (b_i)_{i<\kappa}$ a basis of N. Then $a_i \mathop{\smile\!\!\!\!\!\!|}^{a} a_{<i}$ and $b_i \mathop{\smile\!\!\!\!\!\!|}^{a} b_{<i}$ for all $i < \kappa$ hence $a \equiv b$ by Proposition 2.2.8. Let σ be an automorphism of $\mathbb{M}$ sending a to b, it restricts to an elementary bijection between $\operatorname{acl}(a)$ and $\operatorname{acl}(b)$ (Proposition 2.1.9). We have $\operatorname{acl}(a) = M$ and $\operatorname{acl}(b) = N$. Any elementary bijection between two structures is an isomorphism, hence we get $M \cong N$. The converse implication is trivial.

(3) Observe that $|\operatorname{acl}(S)| \leq \max\{\aleph_0, |S|\}$ for all S. In particular, if $|M| \geq \aleph_1$ then $\dim(M) = |M|$ so we conclude by *(2)*. $\square$

Definition 2.2.14. We say that T has *uniform finiteness* or *eliminates* $\exists^\infty$ if for all $\phi(x, y) \in \mathscr{L}_{1+k}(\emptyset)$ with $|x| = 1$ and $|y| = k$ there is $n \in \mathbb{N}$ such that for all $c \in \mathbb{M}^k$ if $\phi(\mathbb{M}, c)$ is finite, then $|\phi(\mathbb{M}, c)| \leq n$.

Exercise 30. Prove that every strongly minimal theory has uniform finiteness.

Exercise 31 (*). Let M be a structure such that every definable subsets of M is finite or cofinite (M is called *minimal*). Prove that $\operatorname{Th}(M)$ is strongly minimal if and only if $\operatorname{Th}(M)$ has uniform finiteness.

Exercise 32. Let M be an ω-saturated model. Prove that T has uniform finiteness if and only if for all $\phi(x) \in \mathscr{L}_1(M)$ there is $n \in \mathbb{N}$ such that $\phi(M)$ is either infinite or $|\phi(M)| \leq n$.

2.2.3. Back to algebraically closed fields. Let T be any completion of ACF and $\mathbb{M} = \mathbb{K}$. Let $\mathbb{F}$ be the prime field ($\mathbb{Q}$ or $\mathbb{F}_p$ for some prime p).

Definition 2.2.15 (Algebraic disjointness). A tuple $a = (a_1, \ldots, a_n)$ is *(algebraically) dependent* over some set C if there exists a nontrivial polynomial $P(X_1, \ldots, X_n)$ over $\mathbb{F}(C)$ (the field generated by C) such that $P(a_1, \ldots, a_n) = 0$. For small subsets A, B, C of $\mathbb{K}$ we define the following relation

$$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}_{C} B \iff \text{every finite tuple from } \mathbb{F}(AC) \text{ which is dependent over } BC \text{ is dependent over } C$$

Exercise 33. Check that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}} = \mathop{\smile\!\!\!\!\!\!|}^{a\,m}$ for $\mathrm{cl} = \operatorname{acl} = (\cdot)^{\mathrm{alg}}$.

The dimension in this context is usually called the *transcendence degree*. By Theorem 1.2.12, we get

Proposition 2.2.16. *The relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}$ satisfies symmetry, finite character, existence, normality, monotonicity, base monotonicity, transitivity, anti-reflexivity, closure, strong closure, local character.*

Definition 2.2.17 (Linear disjointness). A tuple $(a_1, \ldots, a_n)$ is *linearly dependent* over some set C if there exists a nontrivial tuple $(c_1, \ldots, c_n)$ from $\mathbb{F}(C)$ such that $\sum_i a_i c_i = 0$. For small subsets A, B, C we define

$$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}_{C} B \iff \text{every finite tuple from } \mathbb{F}(AC) \text{ which is linearly dependent over } BC \text{ is linearly dependent over } C.$$

Exercise 34. Check that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}} \to \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}$. Also $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}_C B$ implies $\mathbb{F}(AC) \cap \mathbb{F}(BC) = \mathbb{F}(C)$ werease $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}_C B$ implies $(AC)^{\mathrm{alg}} \cap (BC)^{\mathrm{alg}} = C^{\mathrm{alg}}$, where $\mathbb{F}(C)$ is the field generated by C, for $\mathbb{F}$ the prime field.

Exercise 35. The relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}$ satisfies monotonicity, base monotonicity, transitivity, symmetry, local character (*hard*).

Recall that for a field C and two (unital) commutative C-algebras A and B we may form the tensor product $A \otimes_C B$ which is again a C-algebra which satisfy the universal property:

$$\begin{tikzcd}
 & A \arrow[dr] \arrow[drr, bend left, "f"] & & \\
C \arrow[ur] \arrow[dr] & & A \otimes_C B \arrow[r, dashed, "h"] & D \\
 & B \arrow[ur] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

To read as: for all C-algebra homomorphisms f and g from A (resp. B) to another C-algebra D there is a C-algebra homomorphism h from $A \otimes_C B$ to D which commutes with the canonical embeddings $A \to A \otimes_C B$ (mapping a to $a \otimes 1$) and $B \to A \otimes_C B$ (mapping b to $1 \otimes b$).

The main facts we will use concerning linear disjointness are the following:

FACT 2.2.18. *Let A, B and C be fields with $C \subseteq A \cap B$. Let $A \otimes_C B \to A[B]$ be the canonical mapping $a \otimes b \mapsto ab$. This map is an isomorphism iff $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}_C B$.*

FACT 2.2.19. *If C is algebraically closed, then for all A, B we have $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}_C B$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}_C B$.*

The proof of Fact 2.2.18 is simply unravelling the definition, it is not hard and you can do it as an exercise. Fact 2.2.19 is a particular case of a more general phenomenon: for fields $E \subseteq F \cap K$, if $F \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}_E K$ and F is a *regular* extension of E (which means that $F \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}_E E^{\mathrm{alg}}$) then $F \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}_E K$. The proof of this is more involved.

REMARK 2.2.20. Note that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}$ is not stronger than $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}$: take any $\alpha \in \mathbb{Q}^{\mathrm{alg}} \setminus \mathbb{Q}$ then $\alpha \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}_{\mathbb{Q}} \alpha$ but $\alpha \mathop{\not\smile\!\!\!\!\!\!|}^{\mathrm{ld}}_{\mathbb{Q}} \alpha$. In particular, in ACF, $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}$ do not satisfy right closure for cl = acl. It does satisfy closure for the closure operator given by the field generated by.

PROPOSITION 2.2.21. *Let A, A', B, C be small fields with $C \subseteq A \cap A' \cap B$. If $A \equiv_C A'$, $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}_C B$ and $A' \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}_C B$, then $A \equiv_B A'$.*

PROOF. We use abundantly Exercise 27. Let $f : A \to A'$ be a field isomorphism over C. Using Fact 2.2.18, we have $A[B] \simeq A \otimes_C B$ and $A'[B] \simeq A' \otimes_C B$. We extend f to an isomorphism of rings $A \otimes_C B \to A' \otimes_C B$ by preserving B pointwise, i.e. mapping $a \otimes b \mapsto f(a) \otimes b$. This yields an isomorphism of integral domaines $A[B] \to A'[B]$ preserving B pointwise, which extends to the field generated by AB and $A'B'$. $\square$

2.2.4. A second example: the random graph. Let $\mathscr{L}$ be the language of graphs, that is a single binary relation symbole R. Let Graph be the theory of graphs, i.e. expressing that R is antireflexive ($\forall x \neg R(x,x)$) and symmetryc ($\forall xy R(x,y) \leftrightarrow R(y,x)$). The theory RG is defined by adding to Graph the following axiom-scheme:

$$\forall x_0 \dots x_{m-1} \forall y_1 \dots y_{n-1} \left(\bigwedge_{i \neq j} x_i \neq y_j \to \exists z \bigwedge_{i<m} R(z, x_i) \wedge \bigwedge_{j<n} \neg R(z, y_j) \wedge z \neq y_j \right)$$

for all $n, m < \omega$.

A *random graph* (sometimes *Rado graph*, sometimes *Erdős–Rényi*) is a countable model of RG. Note that RG has no finite model.

THEOREM 2.2.22. *The theory RG has quantifier elimination and is complete. Further, it is model-complete.*

PROOF. This is an easy exercise. $\square$

EXAMPLE 2.2.23. *Define the following graph with vertex $\mathbb{N}$ and put an edge Γ between n and $m > n$ if and only if 2^n appears non-trivially in the binary development of m. Then $(\mathbb{N}, \Gamma)$ is a random graph.*

Exercise 36. Prove that for all small A, B, any graph isomorphism $f : A \to B$ is an elementary map. It follows that $A \equiv B$ if and only if the (restricted) graphs (A, R) and (B, R) are isomorphic.

Exercise 37. Prove that any model of Graph admits and extension which is a model of RG. We say that RG is the *model-companion* of Graph.

Corollary 2.2.24. *In RG,* $\mathrm{acl}(A) = \mathrm{dcl}(A) = A$*, for all* A*.*

It follows that in a monster model $\mathbb{M}$ of RG, the independence relation $\mathop{\smile\!\!\!\!\!\!|}^{a}$ is given by $A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} B$ if and only if $AC \cap BC = C$, or equivalently $A \cap B \subseteq C$.

Proposition 2.2.25. *In RG, the relation* $\mathop{\smile\!\!\!\!\!\!|}^{a}$ *satisfies symmetry, finite character, existence, normality, monotonicity, base monotonicity, transitivity, anti-reflexivity, closure, strong closure, local character.*

Proof. In RG, the algebraic closure defines a trivial pregeometry hence in particular modular hence the relation $\mathop{\smile\!\!\!\!\!\!|}^{a} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ satisfies all those properties by Theorem 1.2.12. $\square$

We define a strengthening of the relation $\mathop{\smile\!\!\!\!\!\!|}^{a}$:

$$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{st}}_{C} B \text{ if } A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} B \text{ and for all } u, v \in ABC, \text{ if } R(u, v) \text{ then } u, v \in AC \text{ or } u, v \in BC.$$

We call the relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{st}}$ the *free amalgamation in RG.*

Proposition 2.2.26. *The relations* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{st}}$ *satisfies symmetry, finite character, existence, normality, monotonicity, base monotonicity, transitivity, anti-reflexivity, closure, strong closure.*

Proof. The properties symmetry, finite character, existence, normality, monotonicity, base monotonicity and transitivity are easy to check. The properties anti-reflexivity, closure and strong closure are trivial since $\mathrm{acl}(A) = A$. $\square$

Example 2.2.27. *The property local character is not satisfied by* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{st}}$*. For a counterexample, take a singleton* a *and an arbitrary big infinite* B *such that* $R(a, b)$ *for all* $b \in B$*.*

CHAPTER 3

Axiomatic calculus with independence relations

3.1. Axioms for independence relations in an ambient theory

We work in the same setting as above, $\mathbb{M}$ is a monster model for some $\mathscr{L}$-theory T.

Definition 3.1.1. A ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ on $\mathbb{M}$ is *invariant under automorphisms* (or simply *invariant*) if for all small A, B, C and $\sigma \in \mathrm{Aut}(\mathbb{M})$ we have:

$$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff \sigma(A) \mathop{\smile\!\!\!\!\!\!|}_{\sigma(C)} \sigma(B)$$

Definition 3.1.2 (Axioms of independence relations, part 3). Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant ternary relation on small subsets of $\mathbb{M}$. We define the following axioms.

(12) (extension) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ then for any $D \supseteq B$ there is $A' \equiv_{BC} A$ with $A' \mathop{\smile\!\!\!\!\!\!|}_C D$.
(13) (full existence) For all A, B, C there exists $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B$.
(14) (the independence theorem over models) Let M be a small model, and assume $A \mathop{\smile\!\!\!\!\!\!|}_M B$, $C_1 \mathop{\smile\!\!\!\!\!\!|}_M A$, $C_2 \mathop{\smile\!\!\!\!\!\!|}_M B$, and $C_1 \equiv_M C_2$. Then there is a set C such that $C \mathop{\smile\!\!\!\!\!\!|}_M AB$, $C \equiv_{MA} C_1$, and $C \equiv_{MB} C_2$.
(15) (stationarity over models) Let M be a small model, and assume $C_1 \mathop{\smile\!\!\!\!\!\!|}_M A$, $C_2 \mathop{\smile\!\!\!\!\!\!|}_M A$, and $C_1 \equiv_M C_2$. Then $C_1 \equiv_{MA} C_2$.

Proposition 3.1.3. *Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant relation.*

(a) If $\mathop{\smile\!\!\!\!\!\!|}$ satisfies extension then $\mathop{\smile\!\!\!\!\!\!|}$ satifies right normality and right closure.
(b) If $\mathop{\smile\!\!\!\!\!\!|}$ satisfies existence, symmetry, monotonicity, base monotonicity, transitivity and extension, then $\mathop{\smile\!\!\!\!\!\!|}$ satisfies strong closure.
(c) If $\mathop{\smile\!\!\!\!\!\!|}$ satisfies extension and right monotonicity then $\mathop{\smile\!\!\!\!\!\!|}$ satisfies existence if and only if it satisfies full existence.
(d) If $\mathop{\smile\!\!\!\!\!\!|}$ satisfies full existence, right normality, right monotonicity and right transitivity, then $\mathop{\smile\!\!\!\!\!\!|}$ satisfies extension.

Proof. *(a)* Assume that $A \mathop{\smile\!\!\!\!\!\!|}_C B$. By extension there exists $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C BC$. As $A' \equiv_{BC} A$ we get $A \mathop{\smile\!\!\!\!\!\!|}_C BC$ by invariance. Similarly there exists $A'' \equiv_{BC} A$ with $A'' \mathop{\smile\!\!\!\!\!\!|}_C \mathrm{acl}(B)$. If σ is an automorphism over BC which sends A'' to A, then by invariance $A \mathop{\smile\!\!\!\!\!\!|}_C \sigma(\mathrm{acl}(B))$. As $\sigma(\mathrm{acl}(B)) = \mathrm{acl}(B)$ (as set) we conclude $A \mathop{\smile\!\!\!\!\!\!|}_C \mathrm{acl}(BC)$.

(b) First we show that $A \mathop{\smile\!\!\!\!\!\!|}_C B$ implies $\mathrm{acl}(AC) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(C)} \mathrm{acl}(BC)$. Assume that $A \mathop{\smile\!\!\!\!\!\!|}_C B$ then using *(a)* we have $A \mathop{\smile\!\!\!\!\!\!|}_C BC$ and $A \mathop{\smile\!\!\!\!\!\!|}_C \mathrm{acl}(BC)$. By base monotonicity we have $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(C)} \mathrm{acl}(BC)$ and we conclude $\mathrm{acl}(AC) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(C)} \mathrm{acl}(BC)$ by symmetry. Conversely if $\mathrm{acl}(AC) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(C)} \mathrm{acl}(BC)$ then by monotonicity and symmetry we have $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(C)} B$. Using existence we have $A \mathop{\smile\!\!\!\!\!\!|}_C C$ and again by *(a)* we have $A \mathop{\smile\!\!\!\!\!\!|}_C \mathrm{acl}(C)$. Using transitivity and normality with $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(C)} B$ we get $A \mathop{\smile\!\!\!\!\!\!|}_C B\mathrm{acl}(C)$ hence $A \mathop{\smile\!\!\!\!\!\!|}_C B$ by monotonicity.

(c) Assume that $\mathop{\smile\!\!\!\!\!\!|}$ satisfies extension and right monotonicity. Let A, B, C be given then by existence $A \mathop{\smile\!\!\!\!\!\!|}_C C$. By extension with $B = C$ and $D = BC$, there exists $A' \equiv_C A$ with $A' \mathop{\smile\!\!\!\!\!\!|}_C BC$, so $A' \mathop{\smile\!\!\!\!\!\!|}_C B$ by monotonicity. Conversely if full existence holds then in particular by taking $B = C$ we get that there exists $A' \equiv_C A$ with $A' \mathop{\smile\!\!\!\!\!\!|}_C C$. By invariance $A \mathop{\smile\!\!\!\!\!\!|}_C C$.

(d) Assume that $A \mathop{\smile\!\!\!\!\!\mid}_C B$ and $D \supseteq B$ is given. Using full existence there exists $A' \equiv_{BC} A$ with $A' \mathop{\smile\!\!\!\!\!\mid}_{BC} D$. By right normality we have $A' \mathop{\smile\!\!\!\!\!\mid}_{BC} CD$. By invariance $A' \mathop{\smile\!\!\!\!\!\mid}_C B$ and by normality $A' \mathop{\smile\!\!\!\!\!\mid}_C BC$. Using right transitivity $A' \mathop{\smile\!\!\!\!\!\mid}_C CD$ and by right monotonicity we have $A' \mathop{\smile\!\!\!\!\!\mid}_C D$. $\square$

EXERCISE 38 (Adler). Can you prove that in (d) the property right monotonicity is necessary? To answer this, you could try to find an invariant relation on some theory satisfying full existence, right normality, right transitivity but neither right monotonicity nor extension. The existence of such a theory and relation is an open problem.

LEMMA 3.1.4 (Baudisch). *Let $\mathop{\smile\!\!\!\!\!\mid}$ be an invariant relation satisfying full existence, right transitivity and stationarity over some set C. Then $\mathop{\smile\!\!\!\!\!\mid}$ satisfies base monotonicity over C: $A \mathop{\smile\!\!\!\!\!\mid}_C D \implies A \mathop{\smile\!\!\!\!\!\mid}_B D$ for all A, B, D with $C \subseteq B \subseteq D$.*

PROOF. Assume that $A \mathop{\smile\!\!\!\!\!\mid}_C D$ and $C \subseteq B \subseteq D$. By full existence, there exists $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\mid}_C B$. Again by full existence there exists $A'' \equiv_B A'$ with $A'' \mathop{\smile\!\!\!\!\!\mid}_B D$. By invariance, we also have $A'' \mathop{\smile\!\!\!\!\!\mid}_C B$. By right transitivity, we have $A'' \mathop{\smile\!\!\!\!\!\mid}_C D$. As $A \mathop{\smile\!\!\!\!\!\mid}_C D$ and $A'' \equiv_C A' \equiv_C A$ we get $A'' \equiv_D A$ by stationarity over C. In particular, by invariance, we conclude that $A \mathop{\smile\!\!\!\!\!\mid}_B D$. $\square$

EXERCISE 39. Prove that if $\mathop{\smile\!\!\!\!\!\mid}$ is invariant, then extension is equivalent to: if $A \mathop{\smile\!\!\!\!\!\mid}_C B$ then for all $D \supseteq B$ there exists $D' \equiv_{BC} D$ with $A \mathop{\smile\!\!\!\!\!\mid}_C D'$.

EXERCISE 40 (*). Assume that $\mathop{\smile\!\!\!\!\!\mid}$ satisfies right monotonicity and full existence. Prove that for all set C, stationarity over C implies the independence theorem over C.

EXERCISE 41. Prove that the following sets of axioms are equivalent for an invariant relation $\mathop{\smile\!\!\!\!\!\mid}$:

- symmetry, existence, normality, monotonicity, base monotonicity, transitivity, closure, strong closure, extension.
- symmetry, existence, monotonicity, base monotonicity, transitivity, extension.
- symmetry, monotonicity, base monotonicity, transitivity, normality, full existence.

Convention. For now on, the notation $\mathop{\smile\!\!\!\!\!\mid}^a$ refers to the independence relation relatively to the closure operator acl, i.e.

$$A \mathop{\smile\!\!\!\!\!\mid}^a_C B \iff \operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl}(C)$$

EXERCISE 42. Using Proposition 2.1.9, check that $\mathop{\smile\!\!\!\!\!\mid}^a$ is invariant.

We recall a classical Lemma from P.M. Neumann.

FACT 3.1.5 (P.M. Neuman). *Let X be a set and suppose G is a group of permutations of X. Let $P, Q \subseteq X$ be finite subsets such that no point in P has a finite orbit. Then there is some $g \in G$ such that $gP \cap Q = \emptyset$.*

PROPOSITION 3.1.6. *The relation $\mathop{\smile\!\!\!\!\!\mid}^a$ is invariant and satisfies symmetry, finite character, existence, normality, monotonicity, transitivity, anti-reflexivity and full existence.*

PROOF. All were checked before except full existence. We fix A, B, C. Let $B^* := \operatorname{acl}(BC) \setminus \operatorname{acl}(C)$. We want to find $A' \equiv_C A$ such that $\operatorname{acl}(A'C) \cap B^* = \emptyset$.

CLAIM 6. By compactness it is enough to show that for all finite $A_0 \subseteq \operatorname{acl}(AC)$ and $B_0 \subseteq B^*$, there is $A'_0 \equiv_C A_0$ such that $A'_0 \cap B_0 = \emptyset$.

PROOF OF THE CLAIM. I give this argument in details now. This sort of arguments are classical and generally just referred to as "by compactness". Let $(a_i)_{i<\alpha}$ be an enumeration of $\operatorname{acl}(AC)$ starting with A (i.e. there is $\beta \leq \alpha$ such that $\{a_i \mid i < \beta\} = A$). Let $\Sigma(x)$ be the type of $(a_i)_{i<\alpha}$ over C. Now observe that for any $a' = (a'_i)_{i<\alpha}$ realising Σ, the tuple a' is an enumeration of $\operatorname{acl}(A'C)$ for $A' = \{a'_i \mid i < \beta\}$, and $A' \equiv_C A$. To conclude the claim it suffices to prove that $\Delta(x) := \Sigma(x) \cup \{x_i \neq b \mid b \in B^*, i < \alpha\}$ is consistent. By compactness it suffices to prove that it is finitely consistent and each finite fragment of $\Delta(x)$ involves finitely many variables, hence is contained in $\mathrm{tp}(A_0/C)\bigcup\{x_i \neq b \mid b \in B_0\}$ for some finite $A_0 \subseteq \mathrm{acl}(AC)$ and $B_0 \subseteq B^*$. □

For such A_0 and B_0, we have $B_0 \cap \mathrm{acl}(C) = \emptyset$ hence the orbit of every element in B_0 over $\mathrm{Aut}(\mathbb{M}/C)$ is infinite. By Fact 3.1.5 there exists $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ such that $\sigma B_0 \cap A_0 = \emptyset$. Take $A_0' = \sigma^{-1}(A_0)$. □

PROPOSITION 3.1.7. *For $T = \mathrm{ACF}$, the relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{alg}}$ is invariant and satisfies extension and stationarity over algebraically closed sets.*

PROOF. We start with extension. Let a, B, C be given and we prove that there exists $a' \equiv_C a$ with $a' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{alg}}_C B$. We may assume that $C \subseteq B$ and that $a = (a_1, \ldots, a_n)$ is a finite tuple, B, C are fields. By Exercise 27, $\mathrm{tp}(a/C)$ is given by the field isomorphism type of $C(a)$ over C. Let $X = (X_1, \ldots, X_n)$ be a tuple of algebraically independent elements over B. Let $I = \{P(X) \in C[X] \mid P(a) = 0\}$. Observe that I is a prime ideal and let F be the fraction field of $C[X]/I$. For $a_i' := X_i + I$ we have that the map $f : C(a) \to F = C(a')$ which fixes C and $a_i \mapsto a_i'$ is a field isomorphism. To check that $a' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{alg}}_C B$, observe that if a' is algebraically dependent over B, there exists $Q(X) \in B[X]$ such that $Q(a') = 0$. Then $Q(X) + I = 0$ hence $Q(X) \in I \subseteq C[X]$ so a' is algebraically dependent over C. To conclude, consider $K = (FB)^{\mathrm{alg}}$, then K is an elementary extension of $(CB)^{\mathrm{alg}}$ (by model-completeness) hence the type given by the isomorphism type of K over CB is consistent so there exists such a' in $\mathbb{K}$ and $a' \equiv_C a$ (using f).

To conclude, stationarity over algebraically closed sets follows from Proposition 2.2.21 and Fact 2.2.19. □

REMARK 3.1.8. Note that in ACF, $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ld}}$ does not satisfy extension. To see this, use Proposition 3.1.3 (a) and Remark 2.2.20 or simply consider $a = \sqrt{2}, C = B = \mathbb{Q}$ and $D = \mathbb{Q}(\sqrt{2})$.

PROPOSITION 3.1.9. *If $T = \mathrm{RG}$ then $\mathop{\smile\!\!\!\!\!\!\mid}^{a}$ and $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{st}}$ are invariant relations. $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{st}}$ satisfies extension and stationarity over every set and $\mathop{\smile\!\!\!\!\!\!\mid}^{a}$ satisfies extension and the independence theorem over models.*

PROOF. Invariance is trivial. We prove extension for $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{st}}$ which implies it for $\mathop{\smile\!\!\!\!\!\!\mid}^{a}$ (although we already know that $\mathop{\smile\!\!\!\!\!\!\mid}^{a}$ satisfies it by Proposition 3.1.6). Let a, B, C be given and assume that B and C are contained live in a small model $M \prec \mathbb{M}$. We may assume that $a \cap C = \emptyset$. Let x be a tuple of new elements (i.e. outside of M) such that $|x| = |a|$. Let $N = Mx$. The relation $\Gamma = R \restriction M$ defines a graph on M. We define an edge relation on N which extends Γ. Let f be the bijection between Ca and Cx which fixes C pointwise and maps a_i to x_i. Define $\Gamma' = f(\Gamma)$ in the sense that $\Gamma(u, v)$ holds if and only if $\Gamma'(f(u), f(v))$ for all $u, v \in Ca$. Observe that $\Gamma' \cap \Gamma = \Gamma(C)$. Let $\Gamma_1 = \Gamma \cup \Gamma'$. Then (N, Γ_1) is an extension of (M, Γ) (i.e. $\Gamma_1 \cap M = \Gamma$) . By Exercise 37 there exists a model (M_1, Γ_2) of RG extending (N, Γ_1). By model-completeness, $(M, \Gamma) \prec (M_1, \Gamma_2)$ hence in particular the type $\mathrm{tp}^{M_1}(x/CB)$ (computed in M_1) is finitely satisfiable in $\mathbb{M}$. Let a' be a realisation of this type in $\mathbb{M}$. Clearly $a' \equiv_C a$. It is clear that $a' \cap B \subseteq C$. Further, by construction there is no new edge between a' and B hence $a' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{st}}_C B$. For stationarity, assume that a, a', C, B are given with $a \equiv_C a'$, $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{st}}_C B$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{st}}_C B$. Let $f : Ca \to Ca'$ be a graph isomorphism over C mapping a to a' and $g : B \to B$ be the identity. We claim that $h = f \cup g$ is elementary. First, as $a \cap B \subseteq C$, $a' \cap B \subseteq C$ and f and g agree on C, the map is well-defined. The second condition of $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{st}}$ implies that there is no edge between $Ca \setminus C$ (resp. $Ca' \setminus C$) and $B \setminus C$ hence h thus defined preserves the edge relation.

We turn to the independence theorem for $\mathop{\smile\!\!\!\!\!\!\mid}^{a}$. Let $C_1 \equiv_E C_2$, $C_1 \mathop{\smile\!\!\!\!\!\!\mid}^{a}_E A$ and $C_2 \mathop{\smile\!\!\!\!\!\!\mid}^{a}_E B$. By symmetry, monotonicity and normality we may assume that $E \subseteq C_1 \cap C_2 \cap A \cap B$. Let M be a small model containing A, B, C_1, C_2. Let X be a set of new points with $X \cap M = \emptyset$ and $|X| = |C_1 \setminus E|$. Let $C = X \cup E$. There is a graph isomorphism $h : C_1 \cong C_2$ fixing E pointwise. Let $f : C \to C_1$ be a bijection fixing E pointwise. Let $g : C \to C_2$ be $h \circ f$. As $C_1 \cap A = E$ the following extension

$f_A : CA \to C_1A$ of f is well-defined:

$$f_A(x) = \begin{cases} f(x) & \text{if } x \in C \\ x & \text{if } x \in A \end{cases}$$

Similarly we define the extension $g_B : CB \to C_2B$ of B by Id_B. Let $\Gamma_1 = f_A^{-1}(R)$ and $\Gamma_2 = g_B^{-1}(R)$, and $\Gamma = \Gamma_1 \cup \Gamma_2$. Similarly as above, let N be a model of RG extending $(MX, R(M) \cup \Gamma)$ (by Exercise 37). We have $M \prec N$ hence the type $\mathrm{tp}^N(C/AB)$ (computed in N) is finitely satisfiable in $\mathbb{M}$. Any realisation C of this type in $\mathbb{M}$ satisfies the conclusion of the independence theorem over E. $\square$

Example 3.1.10. *Note that in RG, $\mathop{\smile\!\!\!\!\!\!|}^{a}$ do not satisfy stationarity (over any set): let $a \equiv_C b$ and $B = C \cup \{d\}$ for $d \notin Cab$ with $R(a, d)$ and $\neg R(b, d)$.*

3.2. A Theorem of Adler

3.2.1. Indiscernible sequences.

Definition 3.2.1. Let $(I, <)$ be an infinite linear order and $a = (a_i)_{i \in I}$ a sequence of tuples of the same size α. Let $x = (x_i)_{i<\alpha}$ be a set of variables. The *Ehrenfeucht-Mostowski type of a over C*, denoted $\mathrm{EM}(a/C)$, is the set of $\mathscr{L}_\alpha(C)$-formulas $\phi(\vec{x}_1, \ldots, \vec{x}_n)$ with $\mathbb{M} \vDash \phi(a_{i_1}, \ldots, a_{i_n})$ for all $i_1 < \ldots < i_n \in I$, for all $n < \omega$ and $\vec{x}_i \subseteq x$. For each $\vec{x}_i$ there is a finite subset $\beta_i \subseteq \alpha$ such that $\vec{x}_i = (x_j)_{j \in \beta_i}$ and when we say $\mathbb{M} \vDash \phi(a_{i_1}, \ldots, a_{i_n})$ we really mean $\mathbb{M} \vDash \phi(a_{i_1} \restriction \beta_1, \ldots, a_{i_n} \restriction \beta_n)$.

Of course, by adding variables in the above we may assume that $\beta_i = \beta_j$ in the formula $\phi(\vec{x}_1, \ldots, \vec{x}_n)$.

Remark 3.2.2. We see $\mathrm{EM}(a/C)$ as a (partial) type Σ in variables

$$\underbrace{(x_i)_{i<\alpha}, (x_i)_{i<\alpha}, \ldots, (x_i)_{i<\alpha}, \ldots}_{\omega\text{-times}}$$

In particular one sees that the EM-type does not depend on the order type of $(I, <)$.

Definition 3.2.3. Let $(I, <)$ be an infinite linear order and $a = (a_i)_{i \in I}$ a sequence of tuples. We say that a is *indiscernible over C* if for all $i_1 < \ldots < i_n \in I$ and $j_1 < \ldots < j_n \in I$ we have $\mathbb{M} \vDash \phi(a_{i_1}, \ldots, a_{i_n})$ if and only if $\mathbb{M} \vDash \phi(a_{j_1}, \ldots, a_{j_n})$.

Exercise 43. Prove that $(a_i)_{i \in I}$ is indiscernible over C if and only if for all $n < \omega$, for all $i_1 < \ldots < i_n \in I$ and $j_1 < \ldots < j_n \in I$ we have

$$a_{i_1} \ldots a_{i_n} \equiv_C a_{j_1} \ldots a_{j_n}$$

Exercise 44. Assume that $a = (a_i)_{i<\omega}$ is indiscernible over C, then $\phi(x_1, \ldots, x_n) \in \mathrm{EM}(a/C)$ if and only if $\vDash \phi(a_0, \ldots, a_{n-1})$.

Exercise 45 (*). Prove that $\mathrm{EM}(a/C)$ is complete if and only if a is indiscernible over C.

Exercise 46. Let $(a_i)_{i<\alpha}$ be an indiscernible sequence over $\emptyset$. Then $|\{a_i \mid i < \alpha\}|$ is either $|\alpha|$ or 1.

We will need some basic fact from set theory, namely the Ramsey theorem and the Erdős-Rado theorem. In partition calculus, we denote by $[\kappa]^n$ the set of n-elements subsets of κ and the symbol

$$\kappa \to (\lambda)^n_\mu$$

stands for the statement: for any function $f : [\kappa]^n \to \mu$, there is $A \subseteq \kappa$ with $|A| = \lambda$ such that f is constant on $[A]^n$. Stated differently: every partition of $[\kappa]^n$ into μ pieces has a homogeneous set of size λ (meaning that there is an infinite subsets whose n-elements subsets are all in one component). Recall that the function $\beth$ is defined as follows: for α an ordinal and μ a cardinal: $\beth_0(\mu) = \mu$, $\beth_\alpha(\mu) = 2^{\beth_\beta(\mu)}$ if $\alpha = \beta + 1$ and $\beth_\alpha(\mu) = \sup_{\beta<\alpha} \beth_\beta(\mu)$ if α is limit. We define also $\beth_\alpha := \beth_\alpha(\aleph_0)$.

FACT 3.2.4. *Classical partition calculus facts.*

- *(Pigeonhole principle)* $\omega \to (\omega)^1_k$
- *(Ramsey Theorem)* $\omega \to (\omega)^n_k$. *In other words: if A is an infinite set and $C_1, \dots, C_k$ is a coloring of $[A]^n$, then there is an infinite subset of A whose n-elements subsets is of the same color C_i.*
- *(Erdős-Rado Theorem)* $\beth_n^+(\mu) \to (\mu^+)^{n+1}_\mu$.

Of course if $\kappa' \geq \kappa$ and $\lambda' \leq \lambda$ then

$$\kappa \to (\lambda)^n_\mu \implies \kappa' \to (\lambda')^n_\mu$$

so one looks for the smallest κ and the biggest λ.

LEMMA 3.2.5 (Ramsey and Compactness). *Let C be a small set, $(I, <)$ and $(J, <)$ two ordered sets. For all sequence $a = (a_i)_{i\in I}$ there exists $b = (b_j)_{j\in J}$ such that b is indiscernible over C and b satisfies $\mathrm{EM}(a/C)$, by which we mean that $\phi(b_{j_1}, \dots, b_{j_n})$ for all $j_1 < \dots < j_n \in J$ and $\phi \in \mathrm{EM}(a/C)$.*

PROOF. Note that "satisfying $\mathrm{EM}(a/C)$" do not make sense in general since $\mathrm{EM}(a/C)$ is a type in $\alpha\times\omega$ variables and b is a tuple indexed by $\alpha\times J$. It does make sense to ask that $\vDash \phi(b_{j_1}, \dots, b_{j_n})$, for all $\phi(x_1, \dots, x_n) \in \mathrm{EM}(a/C)$ and $j_1 < \dots < j_n \in J$. As b will be indiscernible, we actually get that $\mathrm{EM}(a/C) \subseteq \mathrm{EM}(b/C)$. So $\mathrm{EM}(b/C)$ is a completion of $\mathrm{EM}(a/C)$. We need to show that the following partial type in the set of variables $(x_j)_{j\in J}$ with $|x_j| = |a_i|$ is consistent:

$$\{\phi(x_{j_1}, \dots, x_{j_n}) \mid \phi \in \mathrm{EM}(a/C), j_1 < \dots < j_n \in J\}$$
$$\cup \{\phi(x_{i_1}, \dots, x_{i_n}) \leftrightarrow \phi(x_{j_1}, \dots, x_{j_n}) \mid i_1 < \dots < i_n \in J, j_1 < \dots < j_n \in J, \phi \in \mathscr{L}_{\alpha\times n}(C), n < \omega\}$$

By compactness, it is enough to show consistency for all finite subsets of the above. As $\mathrm{EM}(a/C)$ is closed under conjunctions, it is enough to show that the following partial type is consistent:

$$\{\psi(x_1, \dots, x_n)\} \cup \{\phi(x_{i_1}, \dots, x_{i_n}) \leftrightarrow \phi(x_{j_1}, \dots, x_{j_n}) \mid i_1 < \dots < i_n \in J, j_1 < \dots < j_n \in J, \phi \in \Delta\}$$

for $\psi \in \mathrm{EM}(a/C)$ and a finite Δ over C. Let $A = \{a_i \mid i \in I\}$. Any element of $[A]^n$ is now considered as a tuple $\vec{a} = (a_{i_1}, \dots, a_{i_n})$ where $i_1 < \dots < i_n$. We define the following equivalence relation for $\vec{a}, \vec{a}' \in [A]^n$:

$$\vec{a} \sim \vec{a}' \iff \vDash \phi(\vec{a}) \iff \vDash \phi(\vec{a}'), \text{ for all } \phi \in \Delta.$$

Observe that there are at most $2^{|\Delta|}$ equivalent classes, hence by Ramsey's Theorem there exists an infinite subset A' included in one single class. By enumerating A' with the induced order, we get that the partial type above is finitely satisfiable. $\square$

EXERCISE 47 (*). Let $(a_i)_{i<\omega}$ be a C-indiscernible sequence and $\alpha > \omega$. Then there exists $(a_i)_{\omega\leq i<\alpha}$ such that $(a_i)_{i<\alpha}$ is C-indiscernible.

LEMMA 3.2.6 (Erdős-Rado and compactness). *Let $\kappa > |T| + |C|$ and $\lambda = \beth_{(2^\kappa)^+}$. If $(a_i)_{i<\lambda}$ is a sequence of tuples with $|a_i| \leq \kappa$. Then there is an C-indiscernible sequence $(b_i)_{i<\omega}$ such that for each n there are $i_0 < \dots < i_n < \lambda$ with*

$$b_0, \dots, b_n \equiv_C a_{i_0}, \dots, a_{i_n}.$$

In particular b satisfies the EM*-type of a over C.*

PROOF. See Exercise 48. $\square$

THEOREM 3.2.7. *Assume that $a = (a_i)_{i<\omega}$ is C-indiscernible. Then there exists a (small) model $M \supseteq C$ such that a is M-indiscernible. In particular a is $\mathrm{acl}(C)$-indiscernible.*

PROOF. Let N be any small model containing C. Let $\kappa = \max\{|a_i|, |N|, |T|\}$ and $\lambda = \beth_{(2^\kappa)^+}$. Let $(a_i)_{i<\lambda}$ be an extension of a of length λ. By Lemma 3.2.6 there exists $(b_i)_{i<\omega}$ indiscernible over N such that for all $n < \omega$ there exists $i_0 < \dots < i_n < \lambda$ with $b_0 \dots b_n \equiv_N a_{i_0} \dots a_{i_n}$. In particular, as $(a_i)_{i<\lambda}$ is C-indiscernible, $(b_i)_{i<\omega}$ satisfies the $\mathrm{EM}(a/C)$, which is complete as a is C-indiscernible. It follows that $b \equiv_C a_{<\omega}$. Let σ be an automorphism over C sending b to a, and let $M = \sigma(N)$. Observe that M is a small model containing C. Then $aM \equiv_C bN$ hence a is M-indiscernible. For the rest, observe that for any model $M \supseteq C$ we have $\mathrm{acl}(C) \subseteq M$. $\square$

Exercise 48 (Hard). We prove Lemma 3.2.6.

(1) We prove that there exists the following data by induction. For $n < \omega$, a type $p_n(x_1, \ldots, x_n) \in S_n(C)$, $I_n \subseteq (2^\kappa)^+$ with $|I_n| = (2^\kappa)^+$ and $(X_i^n)_{i \in I_n}$ with $X_i^n \subseteq \lambda$ such that
- $X_i^{n+1} \subseteq X_i^n$ for all $i \in I_{n+1}$
- $|X_i^n| > \beth_{2^\kappa+\alpha}$, if i is the α's element of I_n
- $(a_{j_1}, \ldots, a_{j_n}) \vDash p_n$ for all $j_1 < \ldots < j_n \in X_i^n$ for all $i \in I_n$.

(a) Start with $I_0 = (2^\kappa)^+$, $X_i^0 = \lambda$ and $p_0 = \emptyset$, check that the three conditions above are satisfied.

Assume that I_n, X_i^n ($i \in I_n$) and p_n have been constructed. Let $(\xi_\alpha)_{\alpha<(a2^\kappa)^+}$ be an increasing enumeration of I_n and set $I'_n = \{\xi_{\alpha+n} \mid \alpha < (2^\kappa)^+\}$. Fix $i = \xi_{\alpha+n} \in I'_n$.

(a) Prove that there are at most 2^κ types $\mathrm{tp}(a_{j_1}, \ldots, a_{j_n}/C)$, for $j_1 < \ldots < j_n \in X_i^n$.

(b) Deduce from Erdős-Rado that there is some $X_i^{n+1} \subseteq X_i^n$ and $p_{n+1}^i \in S_{n+1}(C)$ such that $|X_i^{n+1}| > \beth_{2^\kappa+\alpha}$ and $(a_{j_1}, \ldots, a_{j_n}) \vDash p_{n+1}^i$, for all $j_1 < \ldots < j_n \in X_i^{n+1}$. (*Hint.* Observe that $\beth_{2^\kappa+\alpha+n} = \beth_n(\beth_{2^\kappa+\alpha})$.)

(c) Deduce that there exists $I_{n+1} \subseteq I'_n$ such that $|I_{n+1}| = (2^\kappa)^+$ and $p_{n+1}^i = p_{n+1}^j$ for all $i, j \in I_{n+1}$. (*Hint.* Observe that $|I'_n| = (2^\kappa)^+$.)

(2) For $x = (x_i)_{i<\omega}$ and $p(x) = \bigcup_{n<\omega} p_n$. Prove that any $b \vDash p$ satisfies the conclusion of Lemma 3.2.6.

3.2.2. Morley sequences and Adler's theorem.

Definition 3.2.8. Let $\downarrow$ be in independence relation. We say that $(a_i)_{i<\alpha}$ is an $\downarrow$*-Morley sequence over* C if $a_i \downarrow_C a_{<i}$ for all $i < \alpha$. For $p \in S(B)$ we say that $(a_i)_{i<\alpha}$ is a $\downarrow$-Morley sequence over C *in* p if $a_i \vDash p$ for all $i < \alpha$. If $B = C$ we call $(a_i)_{i<\alpha}$ a *Morley sequence in* $\mathrm{tp}(a_0/C)$.

In general a Morley sequence is by default indexed by ω.

Exercise 49. Prove that if $\downarrow$ satisfies right monotonicity, extension and $a \downarrow_C B$ for $C \subseteq B$. Then there exists a Morley sequence over C in $\mathrm{tp}(a/B)$.

Exercise 50. Prove that if $\downarrow$ satisfies full existence then for all a, C there exists a Morley sequence in $\mathrm{tp}(a/C)$.

Exercise 51 (*). Prove that if $\downarrow$ is invariant and satisfies (right) monotonicity and stationarity, then any Morley sequence in $\mathrm{tp}(a/C)$ is indiscernible over C.

Definition 3.2.9. Let $\downarrow$ be in independence relation, we denote by $\downarrow^{\mathrm{opp}}$ the relation defined by

$$A \mathop{\downarrow}^{\mathrm{opp}}_{C} B \iff B \mathop{\downarrow}_{C} A.$$

Exercise 52. If $\downarrow$ satisfies left-sided monotonicity/normality/base monotonicity/transitivity then $\downarrow^{\mathrm{opp}}$satisfies the right-sided version of those axioms.

Lemma 3.2.10. *Assume that* $\downarrow$ *is invariant and satisfies (left and right) normality, right monotonicity, right base monotonicity, left transitivity. If there exists arbitrarily long* $\downarrow$*-Morley sequence in* $\mathrm{tp}(a/B)$ *over* C *then there is a* B*-indiscernible* $\downarrow^{\mathrm{opp}}$*-Morley sequence in* $\mathrm{tp}(a/B)$ *over* C.

Proof. Let $(a_i)_{i<\kappa}$ be a Morley sequence in $\mathrm{tp}(a/B)$ over C for κ big enough. Using Erdős-Rado, there exists a sequence $b = (b_i)_{i<\omega}$ which is indiscernible over B and such that for all $n < \omega$ there exists $i_1 < \ldots < i_n$ such that $b_0 \ldots b_{n-1} \equiv_B a_{i_1} \ldots a_{i_n}$. As $a_m \downarrow_C a_{<m}$ for all $m < \omega$, we have $b_n \downarrow_C b_{<n}$ by invariance and right monotonicity, hence $(b_i)_{i<\omega}$ is a $\downarrow$-Morley sequence over C in $\mathrm{tp}(a/B)$ which is B-indiscernible. We now "invert" the sequence.

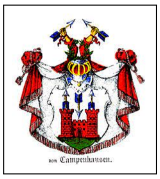
Claim 7. For any indiscernible sequence $(b_i)_{i<\omega}$ over B, there exists $(b'_i)_{i<\omega}$ such that $b'_0 \dots b'_n \equiv_B b_n \dots b_0$, for all $n < \omega$.

Proof. Let $x = (x_i)_{i<\omega}$ be variables with $|x_i| = |b_i|$. Let $p_n(x_0, \dots, x_n) = \mathrm{tp}(b_n, \dots, b_0)$. To prove the claim it is enough to find a realisation of $p(x) := \bigcup_n p(x_0, \dots, x_n)$. First observe that $p_n(x_0, \dots, x_n) \subseteq p_{n+1}(x_0, \dots, x_{n+1})$: as $b_n \dots b_0 \equiv_B b_{n+1} \dots b_1$, if $\phi(x_0, \dots, x_n) \in p_n$ then $\vDash \phi(b_{n+1}, \dots, b_1)$ hence $\phi(x_0, \dots, x_n) \in p_{n+1}$. It follows that p is finitely consistent since $b_n \dots b_0 \vDash p_n$, for any n, so we conclude by compactness. $\square$

Let $b' = (b'_i)_{i<\omega}$ be the inverse of b as in the claim. Then b' is also indiscernible over B. As $b'_0 \equiv_B b_0$, b' is a sequence of realisations of $\mathrm{tp}(a/B)$. It remains to check that b' is $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{opp}}$ Morley over C. By invariance, $b'_0 \mathop{\smile\hspace{-0.9em}|}_C b'_1 \dots b'_n$ and as $b'_0 \dots b'_n \equiv_C b'_1 \dots b'_{n+1}$ we have by invariance $b'_1 \mathop{\smile\hspace{-0.9em}|}_C b'_2 \dots b'_{n+1}$ and by right monotonicity we have $b'_1 \mathop{\smile\hspace{-0.9em}|}_C b'_2 \dots b'_n$. By right normality, right base monotonicity and right monotonicity we get $b'_0 \mathop{\smile\hspace{-0.9em}|}_{Cb'_1} b'_2 \dots b'_n$. Using left normality and left transitivity we have $b'_0 b'_1 \mathop{\smile\hspace{-0.9em}|}_C b'_2 \dots b'_n$. Iterating the last operation we get $b'_n \mathop{\smile\hspace{-0.9em}|}^{\mathrm{opp}}_C b'_{<n}$ so b' is an $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{opp}}$-Morley sequence. $\square$

Remark 3.2.11. One may ask the following question when confronted to the previous proof: do we really need Erdős-Rado or Ramsey is enough? If one get the sequence $(b_i)_{i<\omega}$ via Ramsey and compactness instead of Erdős-Rado, then $(b_i)_{i<\omega}$ satisfies the EM-type of $(a_i)_{i<\omega}$. The point would be to conclude that $(b_i)_i$ is also Morley over C. In other words the question is: if a is a $\mathop{\smile\hspace{-0.9em}|}$-Morley sequence over C and $b \vDash \mathrm{EM}(a/C)$, is b a Morley sequence over C? Of course here the EM-type is not necessarily complete (otherwise one conclude by invariance).

Remark 3.2.12.
Here is another proof of Claim 7 suggested by the Baron of Campenhausen. Start with a B-indiscernible sequence $b = (b_i)_{i<\omega}$. By Ramsey and compactness there exists a B-indiscernible sequence $b'' = (b''_i)_{i\in\mathbb{Z}}$ satisfying the EM-type of b over B. Then for each $n \in \mathbb{N}$ we have



$$b''_{-n} \dots b''_0 \equiv_B b''_0 \dots b''_n \equiv_B b_0 \dots b_n$$

hence by setting $b'_n := b''_{-n}$ for each $n \in \mathbb{N}$ we get $b'_0 \dots b'_n \equiv_B b_n \dots b_0$.

Fact 3.2.13. *Some set-theoretic reminders about regular cardinals.*

1. *If κ, λ are cardinals, we say that κ is* cofinal in λ *if there exists a subset $A \subseteq \lambda$ of cardinality κ such that for all $i < \lambda$ there is $j \in A$ such that $i \leq j$. The smallest κ which is cofinal in λ is called the cofinality of λ, denoted $\mathrm{cf}(\lambda)$. For example $\mathrm{cf}(n) = 1$ and $\mathrm{cf}(\aleph_0) = \aleph_0$. Of course, $\mathrm{cf}(\lambda) \leq \lambda$.*
2. *A cardinal κ is called* regular *if $\mathrm{cf}(\kappa) = \kappa$.*
3. *Every successor cardinal is regular. In particular, for any cardinal κ there exists $\lambda \geq \kappa$ which is regular.*

Lemma 3.2.14. *Assume that $\mathop{\smile\hspace{-0.9em}|}$ is invariant and satisfies left and right monotonicity, left normality, right base monotonicity, left transitivity, finite character and local character. If there is a B-indiscernible $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{opp}}$-Morley sequence over $C \subseteq B$ in $\mathrm{tp}(a/B)$, then $a \mathop{\smile\hspace{-0.9em}|}^{\mathrm{opp}}_C B$.*

Proof. Let $(a_i)_{i<\omega}$ is an $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{opp}}$-Morley sequence over C in $\mathrm{tp}(a/B)$ which is indiscernible over B. By Ramsey and compactness, we may take an extension $(a_i)_{i<\kappa}$ for some regular cardinal κ greater than or equal to $\kappa(B)$ as in local character. We know that $a_n \mathop{\smile\hspace{-0.9em}|}^{\mathrm{opp}}_C a_{<n}$ for all $n < \omega$. If $\alpha < \kappa$, then using indiscernibility over B, invariance and left monotonicity, we have $a_\alpha \mathop{\smile\hspace{-0.9em}|}^{\mathrm{opp}}_C a_{i_1} \dots a_{i_n}$, for any $i_1, \dots, i_n < \alpha$. By finite character of $\mathop{\smile\hspace{-0.9em}|}$, we conclude that $a_\alpha \mathop{\smile\hspace{-0.9em}|}^{\mathrm{opp}}_C a_{<\alpha}$ $(*)$.

Using local character, there exists $D \subseteq Ca_{<\kappa}$ with $|D| < \kappa$ with $B \mathop{\smile\hspace{-0.9em}|}_D C(a_i)_{i<\kappa}$. As $|D| < \kappa$ and κ is regular, D is not cofinal in κ hence there is some $\alpha < \kappa$ such that $D \subseteq Ca_{<\alpha}$. By right base monotonicity and right monotonicity we conclude $B \mathop{\smile\hspace{-0.9em}|}_{Ca_{<\alpha}} a_\alpha$ $(**)$.

By combining $(*)$ and $(**)$ we use left normality and left transitivity of $\mathop{\smile\hspace{-0.9em}|}$ to conclude $Ba_{<\alpha} \mathop{\smile\hspace{-0.9em}|}_C a_\alpha$ so $B \mathop{\smile\hspace{-0.9em}|}_C a_\alpha$ by left monotonicity and $B \mathop{\smile\hspace{-0.9em}|}_C a$ by invariance. $\square$

Theorem 3.2.15. *Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant relation satisfying left and right monotonicity, right base monotonicity, left normality, left transitivity, finite character, local character and extension. Then $\mathop{\smile\!\!\!\!\!\!|}$ satisfies symmetry.*

Proof. By Proposition 3.1.3, $\mathop{\smile\!\!\!\!\!\!|}$ also satisfies right normality. Assume that $a \mathop{\smile\!\!\!\!\!\!|}_C B$. By left and right normality we may assume that $C \subseteq B$. Iterating the use of extension, there exists arbitrarily long $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence in $\mathrm{tp}(a/B)$ over C hence by Lemma 3.2.10 there exists an $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{opp}}$-Morley sequence in $\mathrm{tp}(a/B)$ over C which is B-indiscernible. By Lemma 3.2.14 we have $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{opp}}_C B$. □

Definition 3.2.16. An *Adler independence relation* (in short *AIR*) is an invariant relation satisfying left and right monotonicity, right base monotonicity, left normality, left transitivity, finite character, local character and extension. It is called a *strict* AIR if it further satisfies anti-reflexivity.

Remark 3.2.17. By Adler's theorem of symmetry (Theorem 3.2.15) any Adler independence relation is symmetric, hence there is no need in mentioning the left and right attributes in the previous definition. Using Proposition 3.1.3, an invariant relation $\mathop{\smile\!\!\!\!\!\!|}$ is an AIR if and only if it satisfies symmetry, finite character, local character, normality, monotonicity, base monotonicity, transitivity, extension, existence, full existence and strong closure.

Remark 3.2.18 (Local character depends on the type). Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant relation satisfying local character, i.e. for all A there is $\kappa = \kappa(A)$ such that for all B there is $C \subseteq B$ with $|C| \leq \kappa$ and $A \mathop{\smile\!\!\!\!\!\!|}_C B$. Then κ actually does not depend on A but on $\mathrm{tp}(A)$. If $A' \equiv A$ witnessed by $\sigma \in \mathrm{Aut}(\mathbb{M})$ with $\sigma(A) = A'$, then for any B, apply local character for A with $\sigma^{-1}(B)$ and push back with σ to get a subset $C \subseteq B$ of size $\kappa(A)$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B$. Exercise 53 actually shows that under finite character, the cardinal $\kappa(A)$ in local character depends only on $|A|$.

In particular, using local character as in the proof of Lemma 3.2.14 yields the following convenient lemma.

Lemma 3.2.19. *Let $(b_i)_{i<\omega}$ be a C-indiscernible sequence and $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant relation satisfying right monotonicity, right base monotonicity and local character. Then there exists a model M containing C such that $(b_i)_{i<\omega}$ is an M-indiscernible $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence over M.*

Proof. Let κ be as in local character for b_0 and we may assume that κ is regular (if κ is not regular, consider κ^+). Note that κ only depends on $\mathrm{tp}(b_0)$. Let $(b_i)_{i\leq\kappa}$ be an extension of $(b_i)_{i<\omega}$.

We construct a chain of models $(M_i)_{i<\kappa}$ such that M_i contains $Cb_{<i}$ and $(b_n)_{i\leq n\leq\kappa}$ is M_i-indiscernible, for all $i < \kappa$. We use Theorem 3.2.7: for M_0 take any model containing C such that $(b_i)_{i<\kappa}$ is M_0-indiscernible. If M_i has been constructed, hence M_i contains $Cb_{<i}$ and $(b_n)_{i\leq n<\kappa}$ is M_i-indiscernible. Then $(b_n)_{i+1\leq n<\kappa}$ is M_ib_{i+1}-indiscernible hence by Theorem 3.2.7 there exists M_{i+1} containing M_ib_{i+1} such that $(b_n)_{i+1\leq n<\kappa}$ is M_{i+1}-indiscernible. If i is a limit ordinal, and $(M_j)_{j<i}$ has been constructed, define $M_i = \bigcup_{j<i} M_j$.

By local character, as $b_0 \equiv b_\kappa$, there exists a subset D of $\bigcup_{i<\kappa} M_i$ of size less than κ such that $b_\kappa \mathop{\smile\!\!\!\!\!\!|}_D \bigcup_{i<\kappa} M_i$. As κ is regular, we have that $D \subseteq M_{i_0}$ for some $i_0 < \kappa$ hence by base monotonicity we have $b_\kappa \mathop{\smile\!\!\!\!\!\!|}_{M_{i_0}} \bigcup_{i<\kappa} M_i$ and by monotonicity, $b_\kappa \mathop{\smile\!\!\!\!\!\!|}_{M_{i_0}} b_{<\kappa}$. By indiscernibility and invariance, we also have $b_i \mathop{\smile\!\!\!\!\!\!|}_{M_{i_0}} (b_j)_{i_0<j<i}$. Using an automorphism over C sending $(b_{i_0+j})_{j<\omega}$ to $(b_j)_{j<\omega}$, we get a model M as needed. Note that $i_0 + j < \kappa$ for all $j < \omega$ since otherwise κ would not be regular. □

Remark 3.2.20. Consider the following version of local character, that we call *chain local character*: for all A there exists κ such that for all chain of models $(M_i)_{i<\kappa}$ there exists $i_0 < \kappa$ such that $A \mathop{\smile\!\!\!\!\!\!|}_{M_{i_0}} \bigcup_{i<\kappa} M_i$. Then, in the statement of the previous Lemma, one might replace base monotonicity and local character by chain local character and have the same conclusion.

Exercise 53. We prove that if $\mathop{\smile\!\!\!\!\!\!|}\,$ is invariant and satisfies, finite character, right base monotonicity and local character, then for all A there exists κ depending only on $|A|$ such that for all B there exists $B_0 \subseteq B$ of size $< \kappa$ such that $A \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$. Fix A and B.

(a) Let S be a set of representative of $\bigcup_{n<\omega} \mathbb{M}^n$ quotiented by the equivalence relation $\equiv$. Prove that $|S| = |\bigcup_{n<\omega} S_n(\emptyset)|$.

(b) Let $\lambda = \sup\{\kappa(s) \mid s \in S\}$. Prove that for all finite tuple a from A, there exists $B(a) \subseteq B$ of size $< \lambda$ such that $a \mathop{\smile\!\!\!\!\!\!|}_{B(a)} B$.

(c) Take $B_0 = \bigcup_{a \subseteq_{\text{finite}} A} B(a)$. Check that $|B_0| \leq \lambda + A$ and that $A \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$.

(d) Conclude.

3.3. An easy criterion for NSOP_4 theories

3.3.1. Heirs and coheirs.

Definition 3.3.1 (Heirs and coheirs). Let $C \subseteq B$, $p(x)$ a type over C and q a type over B with $p \subseteq q$.

- q is an *heir* of p if for all $\phi(x, b) \in q$ there exists c from C such that $\phi(x, c) \in p$.
- q is a *coheir* of p if q is finitely satisfiable in C: for all $\phi(x, b) \in q$ there exists c from C such that $\models \phi(c, b)$.

We write $a \mathop{\smile\!\!\!\!\!\!|}^h_C b$ if $\mathrm{tp}(a/Cb)$ is an heir of $\mathrm{tp}(a/C)$ and $a \mathop{\smile\!\!\!\!\!\!|}^u_C b$ if $\mathrm{tp}(a/Cb)$ is a coheir of $\mathrm{tp}(a/C)$.

Exercise 54 (*). Let $\alpha > \omega$ and $(a_i)_{i<\alpha}$ be a C-indiscernible sequence. Prove that $(a_i)_{\omega \leq i < \alpha}$ is a $Ca_{<\omega}$-indiscernible $\mathop{\smile\!\!\!\!\!\!|}^h$-Morley sequence over $Ca_{<\omega}$, i.e. $(a_i)_{\omega \leq i<\alpha}$ is $Ca_{<\omega}$-indiscernible and $a_i \mathop{\smile\!\!\!\!\!\!|}^h_{Ca<\omega} a_{<i}$ for all $\omega \leq i < \alpha$.

Exercise 55. Prove that if $a \mathop{\smile\!\!\!\!\!\!|}^h_C b$ then $a \mathop{\smile\!\!\!\!\!\!|}^h_{\mathrm{dcl}(C)} b$.

Exercise 56. Prove that $a \mathop{\smile\!\!\!\!\!\!|}^h_C b$ if and only if $b \mathop{\smile\!\!\!\!\!\!|}^u_C a$, i.e. $(\mathop{\smile\!\!\!\!\!\!|}^h)^{\mathrm{opp}} = \mathop{\smile\!\!\!\!\!\!|}^u$.

Exercise 57. If $\mathop{\smile\!\!\!\!\!\!|}\,$ satisfying symmetry and $\mathop{\smile\!\!\!\!\!\!|}^0$ is a relation with $\mathop{\smile\!\!\!\!\!\!|}^0 \to \mathop{\smile\!\!\!\!\!\!|}$, then $(\mathop{\smile\!\!\!\!\!\!|}^0)^{\mathrm{opp}} \to \mathop{\smile\!\!\!\!\!\!|}$. In particular $\mathop{\smile\!\!\!\!\!\!|}^h \to \mathop{\smile\!\!\!\!\!\!|}$ if and only if $\mathop{\smile\!\!\!\!\!\!|}^u \to \mathop{\smile\!\!\!\!\!\!|}$.

Exercise 58. Prove that $\mathop{\smile\!\!\!\!\!\!|}^h$ (and $\mathop{\smile\!\!\!\!\!\!|}^u$) are invariant under automorphisms.

Proposition 3.3.2. *The relation $\mathop{\smile\!\!\!\!\!\!|}^u$ is invariant and satisfies finite character, left and right monotonicity, left and right base monotonicity, left transitivity and left normality.*

Proof. We check left transitivity and left normality, the other properties are easy and left as an exercise. Let $C \subseteq B \subseteq D$ and A such that $D \mathop{\smile\!\!\!\!\!\!|}^u_B A$ and $B \mathop{\smile\!\!\!\!\!\!|}^u_C A$. Let $\phi(x, a) \in \mathrm{tp}(D/AC)$. As $D \mathop{\smile\!\!\!\!\!\!|}^u_B A$, there exists b from B such that $\models \phi(b, a)$, hence $\phi(x, a) \in \mathrm{tp}(B/AC)$. As $B \mathop{\smile\!\!\!\!\!\!|}^u_C A$ there exists c from C such that $\models \phi(c, a)$. It follows that $D \mathop{\smile\!\!\!\!\!\!|}^u_C A$. For normality, assume that $A \mathop{\smile\!\!\!\!\!\!|}^u_C B$ and assume that $\phi(ac, b)$ holds, for some $\mathscr{L}(C)$-formula $\phi(x, y, z)$, i.e. $\phi(x, y, b) \in \mathrm{tp}(AC/BC)$. Then $\phi(xc, b) \in \mathrm{tp}(A/BC)$ hence as $A \mathop{\smile\!\!\!\!\!\!|}^u_C B$, there exists c' from C such that $\models \phi(c'c, b)$, hence $AC \mathop{\smile\!\!\!\!\!\!|}^u_C B$. $\square$

Recall that a filter $\mathcal{F}$ on a set S is a subset of the powerset of S which does not contain $\emptyset$, is closed under finite intersections and under supersets (if $a \in \mathcal{F}$ and $a \subseteq b$ then $b \in \mathcal{F}$). An ultrafilter $\mathcal{U}$ on S is a filter on S such that for all subset a of S we have either a or its complement is in $\mathcal{U}$. A principal ultrafilter is given by all subsets containing a given element. Using Zorn's Lemma, every filter is contained in an ultrafilter.

Definition 3.3.3. Let A, B be small sets and let $\mathcal{U}$ be an ultrafilter on A^n. The *average type of $\mathcal{U}$ over B* is the type defined by:

$$\mathrm{Av}(\mathcal{U}/B) := \{\phi(x, b) \in \mathscr{L}(B) \mid \{a \in A^n \mid \models \phi(a, b)\} \in \mathcal{U}\}$$

for $x = (x_1, \dots, x_n)$.

EXERCISE 59. Check that the definition above defines a (consistent/complete) type over B.

THEOREM 3.3.4. *$a \mathop{\smile\!\!\!\!\!\!\;\mid}^u_C B$ if and only if* $\mathrm{tp}(a/BC) = \mathrm{Av}(\mathcal{U}/BC)$ *for some ultrafilter* $\mathcal{U}$ *on* $C^{|a|}$. *In particular,* $\mathop{\smile\!\!\!\!\!\!\;\mid}^u$ *satisfies extension.*

PROOF. Assume first that $a \mathop{\smile\!\!\!\!\!\!\;\mid}^u_C B$. Let $n = |a|$. Define $\mathcal{F}$ to be the set of $C' \subseteq C^n$ such that $\phi(C, b) \subseteq C'$, for some $\phi(x, b) \in \mathrm{tp}(a/BC)$. As $a \mathop{\smile\!\!\!\!\!\!\;\mid}^u_C B$, $\phi(C, b)$ is non-empty for any $\phi(x, b) \in \mathrm{tp}(a/BC)$. As $\mathcal{F}$ is clearly closed under finite intersections and closed under supersets, $\mathcal{F}$ defines a filter on C^n. Let $\mathcal{U}$ be an ultrafilter on C^n extending $\mathcal{F}$. Clearly, $\phi(x, b) \in \mathrm{Av}(\mathcal{U}/BC)$ for any $\phi(x, b) \in \mathrm{tp}(a/BC)$ hence $\mathrm{tp}(a/BC) \subseteq \mathrm{Av}(\mathcal{U}/BC)$. As $\mathrm{tp}(a/BC)$ is a complete type, we have $\mathrm{tp}(a/BC) = \mathrm{Av}(\mathcal{U}/BC)$. Conversely, if $\mathcal{U}$ is an ultrafilter on C^n and $\mathrm{tp}(a/BC) = \mathrm{Av}(\mathcal{U}/BC)$, then for all $\phi(x, b) \in \mathrm{tp}(a/BC)$, the set $\phi(C, b) = \{c \in C^n \models \phi(c, b)\} \in \mathcal{U}$ hence is not empty, so $a \mathop{\smile\!\!\!\!\!\!\;\mid}^u_C B$.

To conclude, we first prove extension for finite tuples. Assume that $a \mathop{\smile\!\!\!\!\!\!\;\mid}^u_C B$ and let $\mathcal{U}$ be an ultrafilter on C^n such that $\mathrm{tp}(a/BC) = \mathrm{Av}(\mathcal{U}/BC)$. Assume further that $BC \subseteq D$. Let $q(x, D)$ be the type $\mathrm{Av}(\mathcal{U}/D)$. Clearly, for any $\phi(x, b) \in \mathrm{tp}(a/BC)$ we have $\phi(x, b) \in \mathrm{Av}(\mathcal{U}/BC)$ if and only if $\phi(x, b) \in \mathrm{Av}(\mathcal{U}/D)$ hence $\mathrm{tp}(a/BC) \subseteq q(x, D)$. For any $a' \models q(x, D)$, we have $a' \equiv_{BC} a$ and $a' \mathop{\smile\!\!\!\!\!\!\;\mid}^u_C D$. Using finite character, we have that $\mathop{\smile\!\!\!\!\!\!\;\mid}^u$ satisfies extension. $\square$

REMARK 3.3.5. The characterisation of finitely satisfiable types via average types explains the index "u" in $\mathop{\smile\!\!\!\!\!\!\;\mid}^u$: it stands for *ultrafilter*.

COROLLARY 3.3.6. *$\mathop{\smile\!\!\!\!\!\!\;\mid}^u$ satisfies existence and full existence* over models*: for any A, B, M there exists $A' \equiv_M A$ with $A' \mathop{\smile\!\!\!\!\!\!\;\mid}^u_M B$.*

PROOF. To get existence, observe that if $\phi(x, m) \in \mathrm{tp}(A/M)$ then $\mathbb{M} \models \exists x \phi(x, m)$ hence $M \models \exists x \phi(x, m)$. There is m' from M such that $\models \phi(m', m)$, hence $A \mathop{\smile\!\!\!\!\!\!\;\mid}^u_M M$. The rest is from Theorem 3.3.4 and Proposition 3.3.2. $\square$

THEOREM 3.3.7. *$\mathop{\smile\!\!\!\!\!\!\;\mid}^h$ satisfies local character. In particular, if $a = (a_i)_{i<\omega}$ is C-indiscernible, then there exists a model M containing C such that $a = (a_i)_{i<\omega}$ is an M-indiscernible $\mathop{\smile\!\!\!\!\!\!\;\mid}^h$-Morley sequence over M, i.e. $a_n \mathop{\smile\!\!\!\!\!\!\;\mid}^h_M a_{<n}$ for all $n < \omega$.*

PROOF. By finite character it is enough to prove that for all finite tuple a and for all B there exists $C \subseteq B$ with $|C| \leq |T|$ such that $a \mathop{\smile\!\!\!\!\!\!\;\mid}^h_C B$. Assume that a and B are given. We construct a sequence $(C_n)_{n<\omega}$ with $C_n \subseteq B$ such that $|C_{n+1}| \leq |T| + |C_n|$ and for all $\phi(x, b) \in \mathrm{tp}(aC_n/B)$ there exists $c \in C_{n+1}$ such that $\phi(x, c) \in \mathrm{tp}(aC_n/C_{n+1})$. We proceed by induction. Let $C_0 = \emptyset$ and if C_n has been constructed, then enumerates all formulas $\phi(x, b) \in \mathrm{tp}(aC_n/B)$ (i.e. such that there exists $d \in aC_n$ such that $\models \phi(d, b)$) and extend C_n by adding one tuple b for each such formula $\phi(x, b)$. Then $|C_{n+1}| \leq |T| + |C_n|$. By construction we have $a \mathop{\smile\!\!\!\!\!\!\;\mid}^h_C B$ for $C = \bigcup_{i<\omega} C_i$ hence as $|C| \leq |T|$ we conclude local character for $\mathop{\smile\!\!\!\!\!\!\;\mid}^h$. As $\mathop{\smile\!\!\!\!\!\!\;\mid}^h$ satisfies right monotonicity and right base monotonicity, we conclude by Lemma 3.2.19. $\square$

EXERCISE 60 (Ramsey). We give an alternate proof of the fact that if $a = (a_i)_{i<\omega}$ is C-indiscernible, then there exists a model M containing C such that $a = (a_i)_{i<\omega}$ is an M-indiscernible $\mathop{\smile\!\!\!\!\!\!\;\mid}^h$-Morley sequence over M. The advantage of this proof due to N. Ramsey (not the old Ramsey of "Ramsey and compactness", a recent Ramsey) is that it does not use the Erdös-Rado theorem.

(1) Consider the expansion $\mathbb{M}^{\mathrm{Sk}}$ of $\mathbb{M}$ by Skolem functions in the expansion $\mathscr{L}^{\mathrm{Sk}}$ of the language $\mathscr{L}$: for each $\mathscr{L}$-formula $\phi(x, y)$ there is a $|y|$-ary function f_ϕ such that $\mathbb{M}^{\mathrm{Sk}} \models \phi(f_\phi(y), y)$. Prove that $\mathbb{M}^{\mathrm{Sk}}$ is a monster model of its theory.
(2) Prove that every dcl-closed set in $\mathbb{M}^{\mathrm{Sk}}$ is an elementary substructure of $\mathbb{M}^{\mathrm{Sk}}$ and the reduct to $\mathscr{L}$ is an elementary substructure of $\mathbb{M}$ (*Hint.* Use Exercise 29.)
(3) By Ramsey (the old one) and compactness, there exists an $\mathscr{L}^{\mathrm{Sk}}$-indiscernible sequence $(a'_i)_{i<\omega+\omega}$ over C satisfying the $\mathscr{L}^{\mathrm{Sk}}$-EM type of $(a_i)_{i<\omega}$ over C.
   (a) Prove that there exists a model N such that $(a'_i)_{\omega<i<\omega+\omega}$ is N-indiscernible and $a'_i \mathop{\smile\!\!\!\!\!\!\;\mid}^h_N a'_{<i}$ in the sense of $\mathbb{M}^{\mathrm{Sk}}$ (*Hint.* Use Exercises 54 and 55.)

(*b*) Prove that $(a'_i)_{\omega<i<\omega+\omega} \equiv_C (a_i)_{i<\omega}$ in $\mathbb{M}$.
(*c*) Let N_0 be the reduct of N to $\mathscr{L}$. Prove that there exists $M \succ \mathbb{M}$ such that $(a'_i)_{\omega<i<\omega+\omega}N_0 \equiv_C (a_i)_{i<\omega}M$.
(*d*) Conclude.

Exercise 61 (*). We define the following property

- (strong finite character). If $a \not\mathop{\smile\hskip -0.9em ^| \ }_C b$, then there is a formula $\phi(x,b) \in \operatorname{tp}(a/Cb)$ such that for all a', if $a' \vDash \phi(x,b)$ then $a' \not\mathop{\smile\hskip -0.9em ^| \ }_C b$.

Assume that $\mathop{\smile\hskip -0.9em ^| \ }$ satisfies left monotonicity, strong finite character and that $\mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{opp}}$ satisfies existence. Prove that $\mathop{\smile\hskip -0.9em ^| \ }^u \to \mathop{\smile\hskip -0.9em ^| \ }$.

Exercise 62 (*). Prove that any relation satisfying monotonicity, symmetry, existence and strong finite character also satisfies local character. Deduce from Example 2.2.27 that in RG, $\mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{st}}$ do not satisfy strong finite character.

Exercise 63. In ACF prove that $\mathop{\smile\hskip -0.9em ^| \ }^h \to \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{alg}}$ and $\mathop{\smile\hskip -0.9em ^| \ }^u \to \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{alg}}$. Actually, one can prove that in ACF, $\mathop{\smile\hskip -0.9em ^| \ }^h = \mathop{\smile\hskip -0.9em ^| \ }^u = \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{alg}}$.

Exercise 64. Prove that if $\mathop{\smile\hskip -0.9em ^| \ }$ is invariant and satisfies existence, symmetry, monotonicity, base monotonicity, transitivity, extension and strong finite character, then $\mathop{\smile\hskip -0.9em ^| \ }$ is an AIR.

Exercise 65 (Open). Is there an AIR which does not satisfy strong finite character?

3.3.2. NSOP$_4$ theories.

Definition 3.3.8. Given $n \geq 3$, we say that T has the *n-strong order property* (n-SOP) if there is an indiscernible sequence $(a_i)_{i<\omega}$ such that if $p(x,y) = \operatorname{tp}(a_0,a_1)$ then the type $q(x_1,\ldots,x_n)$ defined by

$$p(x_1,x_2) \cup p(x_2,x_3) \cup \ldots \cup p(x_{n-1},x_n) \cup p(x_n,x_1)$$

is inconsistent. We say that T is NSOP$_n$ if it does not have the n-SOP.

Remark 3.3.9. If T is NSOP$_n$ then T is NSOP$_{n+1}$. Indeed if T is NSOP$_n$, then for any indiscernible sequence $(a_i)_{i<\omega}$ there is $c_1,\ldots,c_n$ such that $c_ic_{i+1} \equiv a_0a_1$ for $i<n$ and $c_nc_1 \equiv a_0a_1$. Then $c_1c_2 \equiv a_0a_1 \equiv a_0a_2$ hence there exists $c_{\frac{3}{2}}$ such that $c_1c_{\frac{3}{2}}c_2 \equiv a_0a_1a_2$ and $c_1, c_{\frac{3}{2}}, c_2, \ldots, c_n$ witness NSOP$_{n+1}$.

Example 3.3.10. *DLO has the SOP$_n$ for all $n \geq 3$. Indeed, for an increasing sequence $(a_i)_{i<\omega}$ were $a_i < a_j$ iff $i<j$, then $p(x,y) = \operatorname{tp}(a_0,a_1)$ is isolated by the formula $x<y$, and clearly if the type q above was consistent, it would contradicts transitivity of $<$.*

Lemma 3.3.11. *Assume that $\mathop{\smile\hskip -0.9em ^| \ }$ is invariant and satisfies symmetry and stationarity over models. If $a \mathop{\smile\hskip -0.9em ^| \ }_M b$ and $a \equiv_M b$ then $ab \equiv_M ba$.*

Proof. By an automorphism, let a' be such that $ab \equiv_M ba'$. By invariance, $b \mathop{\smile\hskip -0.9em ^| \ }_M a'$ and by symmetry $a' \mathop{\smile\hskip -0.9em ^| \ }_M b$. Using stationarity we conclude $a' \equiv_{Mb} a$ hence $a'b \equiv_M ab$. As $ab \equiv_M ba'$, we have $ba \equiv_M ab$. ☐

Theorem 3.3.12. *Let $\mathop{\smile\hskip -0.9em ^| \ }$ be an invariant relation satisfying: symmetry, full existence, stationarity over models, and the following weak transitivity over models:*

$$a \mathop{\smile\hskip -0.9em ^| \ }_{Md} b \text{ and } a \mathop{\smile\hskip -0.9em ^| \ }^h_M d \text{ and } b \mathop{\smile\hskip -0.9em ^| \ }^u_M d \implies a \mathop{\smile\hskip -0.9em ^| \ }_M b$$

Then T is NSOP$_4$.

Proof. Let $(a_i)_{i<\omega}$ be an indiscernible sequence and $p(x,y) = \operatorname{tp}(a_0,a_1)$. We show that

$$p(x_0,x_1) \cup p(x_1,x_2) \cup p(x_2,x_3) \cup p(x_3,x_0)$$

is a consistent partial type. By Theorem 3.3.7, $a_i \mathop{\smile\hskip -0.9em ^| \ }^h_M a_{<i}$ for all $i<\omega$ and some small model M.

By full existence, there exists $a_0^* \equiv_{Ma_1} a_0$ such that $a_0^* \mathop{\smile\!\!\!\!\!\!\mid}_{Ma_1} a_2$. By symmetry, we have $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_{Ma_1} a_0^*$. As $a_2 \mathop{\smile\!\!\!\!\!\!\mid}^h_M a_1$ and $a_0^* \mathop{\smile\!\!\!\!\!\!\mid}^u_M a_1$, we conclude $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_M a_0^*$ using the weak transitivity assumption.

We have $a_0^* \equiv_M a_2$ and $a_0^* \mathop{\smile\!\!\!\!\!\!\mid}_M a_2$ hence by stationarity and symmetry (Lemma 3.3.11), we have $a_0^* a_2 \equiv_M a_2 a_0^*$. Then, there exists a_3^* such that $a_0^* a_2 a_1 \equiv_M a_2 a_0^* a_3^*$. We claim that (a_0^*, a_1, a_2, a_3^*) satisfies the type above. First, $a_0^* a_1 \equiv a_0 a_1$ hence $p(a_0^*, a_1)$. By indiscernability, $a_0 a_1 \equiv_M a_1 a_2$ hence $p(a_1, a_2)$. By choice, $a_2 a_3^* \equiv_M a_0^* a_1$ hence $p(a_2, a_3^*)$. Finally $a_3^* a_0 \equiv_M a_1 a_2$ hence $p(a_3^*, a_0)$. ☐

Corollary 3.3.13. *ACF is NSOP$_4$.*

Proof. In ACF, consider the relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{alg}}$. By Proposition 2.2.16 and 3.1.7, the invariant relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{alg}}$ satisfies symmetry, transitivity, full existence and stationarity over models. Further it is easy to check that $\mathop{\smile\!\!\!\!\!\!\mid}^h \to \mathop{\smile\!\!\!\!\!\!\mid}^{\text{alg}}$ (see Exercise 63) hence the conditions of Theorem 3.3.12 are satisfied by transitivity and monotonicity. ☐

Corollary 3.3.14. *RG is NSOP$_4$.*

Proof. We check that $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{st}}$ satisfies the hypotheses of Theorem 3.3.12. By Propositions 2.2.26 and 3.1.9 it remains to prove the weak transitivity property. Assume that $A \mathop{\smile\!\!\!\!\!\!\mid}^h_M B$, $B \mathop{\smile\!\!\!\!\!\!\mid}^h_M D$ and $A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{st}}_{MB} D$. From $A \mathop{\smile\!\!\!\!\!\!\mid}^h_M B$, $B \mathop{\smile\!\!\!\!\!\!\mid}^h_M D$ we deduce that $A \cap B \subseteq M$ and $B \cap D \subseteq M$. Let $uv \in AMD$ be such that $\vDash R(u, v)$. As $A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{st}}_{MB} D$, in particular $uv \subseteq AMB$ or $uv \subseteq MBD$. However $AMB \cap AMD \subseteq AM \cup (B \cap D) \subseteq AM$ and $MBD \cap AMD \subseteq MD$ hence hence $uv \subseteq AM$ or $uv \subseteq MD$. We conclude that $A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{st}}_M B$. ☐

Exercise 66. Using Lemma 3.2.19 and the proof of Theorem 3.3.12, prove the following: if $\mathop{\smile\!\!\!\!\!\!\mid}$ is an invariant relation satisfying symmetry, full existence, stationarity over models and $\mathop{\smile\!\!\!\!\!\!\mid}^0$ is invariant and satisfies right monotonicity, right base monotonicity and local character, if further:

$$a \mathop{\smile\!\!\!\!\!\!\mid}_{Md} b \text{ and } a \mathop{\smile\!\!\!\!\!\!\mid}^0_M d \text{ and } d \mathop{\smile\!\!\!\!\!\!\mid}^0_M b \implies a \mathop{\smile\!\!\!\!\!\!\mid}_M b$$

then T is NSOP$_4$.

Exercise 67. A formula $\phi(x, y)$ with $|x| = |y|$ has the *n-strong order property* if there exists $(a_i)_{i<\omega}$ such that $\phi(a_i, a_j)$ for all $i < j$ and

$$\phi(x_1, x_2) \wedge \phi(x_2, x_3) \wedge \dots \phi(x_n, x_1)$$

is inconsistent in T. Prove that T has SOP$_n$ if and only if there is a formula $\phi(x, y)$ which has SOP$_n$ modulo T.

Exercise 68 (*). Let $\mathop{\smile\!\!\!\!\!\!\mid}$ be an invariant relation satisfying: symmetry, monotonicity, transitivity, full existence, strong finite character and stationarity over models then T is NSOP$_4$. Deduce that ACF is NSOP$_4$.

Exercise 69 (*). We define the following property

- (freedom). If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and $C \cap AB \subseteq D \subseteq C$ then $A \mathop{\smile\!\!\!\!\!\!\mid}_D B$.

(1) In any set S the relation $A \cap B \subseteq C$ satisfies freedom.
(2) Prove that in RG, the relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{st}}$ satisfies freedom.
(3) We propose to prove the following criterion, due to Conant:

if T is a theory where there is an invariant relation $\mathop{\smile\!\!\!\!\!\!\mid}$ on small sets satisfying symmetry, full existence, stationarity over every set and freedom, then T is NSOP$_4$.

The proof is a variant of the one of Theorem 3.3.12.

(a) Let $(a_i)_{i<\omega}$ be an indiscernible sequence and $p(x, y) = \text{tp}(a_0, a_1)$. Let $C = a_0 \cap a_1$. Prove that $a_i \cap a_j = C$ for all i, j and that the sequence $(b_i)_{i<\omega}$ defined by $b_i = a_i \setminus C$ is indiscernible over C.

(b) Prove that there exists b_0^* with $b_0^* \equiv_{Cb_1} b_0$ and $b_0^* \mathop{\smile\!\!\!\!\!\!\mid}_C b_2$ (*Hint.* Use full existence and freedom.)
(c) Prove that $b_0^* b_2 \equiv_C b_2 b_0^*$.
(d) Conclude as in Theorem 3.3.12 that $p(x_0, x_1) \cup p(x_1, x_2) \cup p(x_2, x_3) \cup p(x_3, x_0)$ is consistent.

(4) Deduce that RG is NSOP_4.
(5) (*Bonus.* The property stationarity over every set is a strong property, but using Theorem 3.2.7 one might redo the whole proof above assuming only stationarity over models.)

CHAPTER 4

Forking and dividing

4.1. Generalities on dividing and forking

4.1.1. Dividing.

DEFINITION 4.1.1 (Dividing). Let $k \in \mathbb{N}$, b a tuple and C a set. We say that a formula $\phi(x, b)$ *k-divides over C* if there exists a sequence $(b_i)_{i<\omega}$ in $\text{tp}(b/C)$ such that the set $\{\phi(x, b_i) \mid i < \omega\}$ is k-inconsistent, i.e. the conjunction $\bigwedge_{j=1}^{k} \phi(x, b_{i_j})$ is inconsistent for all $i_1 < \ldots < i_k < \omega$. A formula divides if there is some k such that it k-divides.

A (partial) type $\Sigma(x)$ *divides over C* if there exists $\phi(x, b)$ which divides over C and $\Sigma(x) \vDash \phi(x, b)$. In particular if Σ is closed under conjunction, Σ divides over C if it contains a formula that does.

EXAMPLE 4.1.2. *In DLO, the formula $\phi(x, b_1, b_2)$ defined by $b_1 < x < b_2$ 2-divides over $\emptyset$. In ACF, let $\phi(x, b)$ be any nontrivial polynomial equation $P(X, b) = 0$ which solutions are not in $\mathbb{Q}^{\text{alg}}$. Then $\phi(x, b)$ 2-divides over $\emptyset$.*

EXERCISE 70. If $\phi(\mathbb{M}, b) \subseteq \psi(\mathbb{M}, d)$ and $\psi(x, d)$ divides over C then $\phi(x, b)$ divides over C.

EXERCISE 71. If T is strongly minimal and $\phi(x, b) \subseteq \mathbb{M}$ is infinite, then $\phi(x, b)$ does not divide over $\emptyset$.

LEMMA 4.1.3. *$\pi(x, b)$ divides over C if and only if there exists a C-indiscernible sequence $(b_i)_{i<\omega}$ such that $\text{tp}(b_0/C) = \text{tp}(b/C)$ and $\bigcup_{i<\omega} \pi(x, b_i)$ is inconsistent.*

PROOF. Assume that $\pi(x, b)$ divides over C, so there is $\phi(x, b)$ with $\pi(x, b) \vDash \phi(x, b)$ ($\phi(x, b)$ is a conjunction of formula from π, by compactness) and such that $\phi(x, b)$ divides over C. Let $(b_i)_{i<\omega}$ be a sequence with $\text{tp}(b_0/C) = \text{tp}(b/C)$ and such that $\{\phi(x, b_i) \mid i < \omega\}$ is k-inconsistent, for some k. Then the formula $\theta(y_1, \ldots, y_k)$ defined by $\neg(\exists x \bigwedge_i \phi(x, y_i)$ is in the EM-type of $(b_i)_{i<\omega}$ over C (even over $\emptyset$). By Ramsey and compactness, let $(b'_i)_{i<\omega}$ be an indiscernible sequence satisfying $\text{EM}((b_i)_{i<\omega}/C)$, then $\{\phi(x, b_i) \mid i < \omega\}$ is also k-inconsistent. In particular it is inconsistent and so is $\bigcup_{i<\omega} \pi(x, b'_i)$.

Conversely, assume that $(b_i)_{i<\omega}$ is an indiscernible sequence such that $\text{tp}(b_0/C) = \text{tp}(b/C)$ and that $\bigcup_{i<\omega} \pi(x, b_i)$ is inconsistent. By compactness, there is a finite conjunction $\phi(x, b)$ of formulas from $\pi(x, b)$ such that $\{\phi(x, b_i) \mid i < \omega\}$ is inconsistent. By compactness, there exists $k < \omega$ such that $\phi(x, b_{i_1}) \wedge \ldots \wedge \phi(x, b_{i_k})$ is inconsistent, so $\vDash \theta(b_{i_1}, \ldots, b_{i_k})$. As $(b_i)_{i<\omega}$ is C-indiscernible, if $\theta(b_{i_1}, \ldots, b_{i_k})$ holds for some ordered tuple $i_1 < \ldots < i_k$, then it holds for all such tuples, hence $\{\phi(x, b_i) \mid i < \omega\}$ is k-inconsistent. As $\text{tp}(b_0/C) = \text{tp}(b/C)$, we conclude that $\pi(x, b)$ divides over C. □

PROPOSITION 4.1.4. *The following are equivalent:*

1. $\text{tp}(a/Cb)$ *does not divide over C*
2. *for any C-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 = b$ there is $a' \equiv_{Cb} a$ such that $(b_i)_{i<\omega}$ is Ca'-indiscernible.*
3. *for any C indiscernible sequence $(b_i)_{i<\omega}$ there exists $(b'_i)_{i<\omega}$ with $(b'_i)_{i<\omega} \equiv_{Cb} (b_i)_{i<\omega}$ such that $(b'_i)_{i<\omega}$ is Ca-indiscernible.*

PROOF. (1) $\implies$ (2). Let $(b_i)_{i<\omega}$ be a C-indiscernible sequence with $b_0 = b$. By Lemma 4.1.3, for $p(x, y) = \text{tp}(ab/C)$, the partial type $\bigcup_{i<\omega} p(x, b_i)$ is consistent, let a'' be a realisation. By Ramsey and compactness, there exists a Ca''-indiscernible sequence $(b'_i)_{i<\omega}$ realising

$\mathrm{EM}((b_i)_{i<\omega}/Ca'')$. As $\vDash p(a'', b_i)$ for all $i < \omega$, $p(a'', x) \subseteq \mathrm{EM}((b_i)_{i<\omega}/Ca'')$, hence $b'_i \vDash p(a'', y)$ for all $j < \omega$. In particular, $\mathrm{tp}(a''b'_0/C) = p(x, y)$ i.e. $a''b'_0 \equiv_C ab$. As $(b_i)_{i<\omega}$ is C-indiscernible, we have $(b'_i)_{i<\omega} \equiv_C (b_i)_{i<\omega}$. Using an automorphism, there exists a' such that $(b'_i)_{i<\omega}a'' \equiv_C (b_i)_{i<\omega}a'$, so $(b_i)_{i<\omega}$ is Ca'-indiscernible. As $a'b_0 \equiv_C a''b'_0$ and $a''b'_0 \equiv_C ab_0$ we have $a' \equiv_{Cb} a$.

(2) $\implies$ (1). Let $p(x, b) = \mathrm{tp}(a/Cb)$ where $p(x, y)$ is a type over C. Let $(b_i)_{i<\omega}$ be any C-indiscernible sequence and a' as in (2). As $a' \equiv_{Cb} a$ we have $p(a', b)$. As $(b_i)_{i<\omega}$ is Ca'-indiscernible, we also have $p(a', b_i)$, for all $i < \omega$, so $\bigcup_{i<\omega} p(x, b_i)$ is consistent. By Lemma 4.1.3, $\mathrm{tp}(a/Cb)$ does not divide over C.

(2) $\iff$ (3). Clear via an appropriate automorphism. $\square$

Definition 4.1.5. We define the *non-dividing independence relation* $\mathop{\perp\!\!\!\!\perp}^d$ by

$$A \mathop{\perp\!\!\!\!\perp}^d_C B \iff \mathrm{tp}(A/BC) \text{ does not divide over } C.$$

Equivalently, $A \mathop{\perp\!\!\!\!\perp}^d_C B$ if for any enumeration a of A and b of B we have: for any indiscernible sequence $(b_i)_{i<\alpha}$ with $b_0 = b$ there is a' such that $a'b_i \equiv_C ab$ for all $i < \omega$.

Oddly enough, the non-dividing independence relation is also sometimes orally called the dividing independence relation.

Theorem 4.1.6. *The relation $\mathop{\perp\!\!\!\!\perp}^d$ is invariant and satisfies existence, finite character, left and right monotonicity, left and right normality, right base monotonicity, left transitivity and anti-reflexivity.*

Proof. The relation $\mathop{\perp\!\!\!\!\perp}^d$ is invariant because for any $\sigma \in \mathrm{Aut}(\mathbb{M})$, $\phi(x, b)$ divides over C if and only if $\phi(x, \sigma(b))$ divides over $\sigma(C)$. The property existence is trivial since $\mathrm{tp}(a/C)$ does not divides over C. The property finite character holds for $\mathop{\perp\!\!\!\!\perp}^d$ since dividing is witnessed at the level of formulas, which only mention a finite number of elements from A. Assume that $A \mathop{\perp\!\!\!\!\perp}^d_C B$. If $A' \subseteq A$ and $B' \subseteq B$ then $\mathrm{tp}(A'/B'C)$ consist of formulas in $\mathrm{tp}(A/BC)$ hence no formula in $\mathrm{tp}(A'/B'C)$ divides over C hence $A' \mathop{\perp\!\!\!\!\perp}^d_C B'$, which yields left and right monotonicity. Right normality holds by definition. For left normality, if $a \mathop{\perp\!\!\!\!\perp}^d_C b$ and $(b_i)_{i<\omega}$ is C-indiscernible with $b_0 = b$, then there is a' such that $a'b_i \equiv_C ab$. Then $a'Cb_i \equiv_C aCb$ hence $aC \mathop{\perp\!\!\!\!\perp}^d_C b$. We prove base monotonicity: assume $a \mathop{\perp\!\!\!\!\perp}^d_C b$ and $C \subseteq D \subseteq b$. If $(b_i)_{i<\omega}$ is a D-indiscernible sequence with $b_0 = b$, then $(b_iD)_{i<\omega}$ is C-indiscernible hence there exists a' with $a'b_iD \equiv_C abD$, so $a'b_i \equiv_D ab$ hence $a \mathop{\perp\!\!\!\!\perp}^d_D b$. We prove left transitivity: assume that $C \subseteq B \subseteq D$ and $D \mathop{\perp\!\!\!\!\perp}^d_B a$ and $B \mathop{\perp\!\!\!\!\perp}^d_C a$. Let $(a_i)_{i<\omega}$ be a C-indiscernible sequence with $a_0 = a$. As $B \mathop{\perp\!\!\!\!\perp}^d_C a$ there exists $B' \equiv_{aC} B$ such that $(a_i)_{i<\omega}$ is B'-indiscernible. Using an automorphism, there is D' such that $D'B' \equiv_{aC} DB$. By invariance, $D' \mathop{\perp\!\!\!\!\perp}^d_{B'} a$ hence as $(a_i)_{i<\omega}$ is B'-indiscernible, there is $D'' \equiv_{B'} D'$ such that $(a_i)_{i<\omega}$ is D''-indiscernible. Then $D''B' \equiv_{aC} D'B' \equiv_{aC} DB$ hence $D'' \equiv_{aC} D$ so we conclude $D \mathop{\perp\!\!\!\!\perp}^d_C a$. For anti-reflexivity, if $a \notin \mathrm{acl}(C)$ then there exists a C-indiscernible sequence $(a_i)_{i<\omega}$ of distinct elements with $a_0 = a$. This sequence witnesses that the formula $x = a \in \mathrm{tp}(a/Ca)$ 2-divides over C, hence $a \not\mathop{\perp\!\!\!\!\perp}^d_B a$. $\square$

In general, $\mathop{\perp\!\!\!\!\perp}^d$ is not an Adler independence relation, for instance, it need not be symmetric. It will be in the context of simple theories. However, we have the following connection between $\mathop{\perp\!\!\!\!\perp}^d$ and any Adler independence relation.

Proposition 4.1.7. $\mathop{\perp\!\!\!\!\perp}^d \to \mathop{\perp\!\!\!\!\perp}$ *for any Adler independence relation* $\mathop{\perp\!\!\!\!\perp}$.

Proof. Assume that $\mathop{\perp\!\!\!\!\perp}$ is an Adler independence relation and $a \mathop{\perp\!\!\!\!\perp}^d_C b$. Let $(b_i)_{i<\omega}$ be an $\mathop{\perp\!\!\!\!\perp}$-Morley sequence over C with $b_0 = b$. We may assume that $(b_i)_{i<\omega}$ is C-indiscernible using Lemma 3.2.10 and $\mathop{\perp\!\!\!\!\perp} = \mathop{\perp\!\!\!\!\perp}^{opp}$. As $a \mathop{\perp\!\!\!\!\perp}^d_C b$ there exists $a' \equiv_{Cb} a$ such that $(b_i)_{i<\omega}$ is Ca'-indiscernible, by Proposition 4.1.4. The sequence $(b_i)_{i<\omega}$ is a Ca'-indiscernible $\mathop{\perp\!\!\!\!\perp}$-Morley sequence over C in $\mathrm{tp}(b/Ca')$ so by Lemma 3.2.14 we have $a' \mathop{\perp\!\!\!\!\perp}_C b$. By invariance, we have $a \mathop{\perp\!\!\!\!\perp}_C b$. $\square$

Another use of Neuman's Lemma is the following:

Proposition 4.1.8. $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d \to \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^a$

Proof. Suppose $A \not\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^a_C B$ and let $a_0 \in (\mathrm{acl}(AC) \cap \mathrm{acl}(BC)) \setminus \mathrm{acl}(C)$. Let $a = (a_0, \ldots, a_n)$ be a tuple enumerating the orbit of a over $\mathrm{Aut}(\mathbb{M}/BC)$. Let $\phi(x, b)$ be a formula with $\phi(x, y) \in \mathscr{L}(C)$ which isolates $\mathrm{tp}(a/BC)$, so that a enumerates $\phi(\mathbb{M})$. As $a_0 \notin \mathrm{acl}(C)$ the orbit of every coordinate of a under $\mathrm{Aut}(\mathbb{M}/C)$ is infinite. By Neuman's Lemma (Fact 3.1.5) there exists $a^1 \equiv_C a$ such that $a^1 \cap a = \emptyset$ (as set of coordinates). By iterating applications of Neuman's Lemma, there is a sequence $(a^i)_{i<\omega}$ of pairwise disjoint tuples such that $a^i \equiv_C a$. By shifting by an automorphism, let b_i be such that $a^i b_i \equiv_C ab$. Then a^i enumerates the solutions of $\phi(x, b_i)$, so $\{\phi(x, b_i) \mid i < \omega\}$ is 2-inconsistent, so $a \not\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d_C B$. □

Remark 4.1.9. It was believed for a long time (at least since 2008!) that the implication $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d \to \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^{a\,m}$ were true in general. According to Proposition 1.2.14, it is enough that $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d$ satisfies the right version of normality, monotonicity, base monotonicity and closure. By Theorem 4.1.6, one only needs to check right closure. It was recently observed that $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d$ does not satisfy right closure, even worst, the implication $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d \to \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^{a\,m}$ does not hold in general.

Exercise 72. Prove that $a \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d_C b$ if and only if for any C-indiscernible sequence $(b_i)_{i<\alpha}$ with $b_0 = b$ there is a' such that $a'b_i \equiv_C ab$ for all $i < \omega$.

Exercise 73. In DLO, assume that $b_1 < c < a < b_2$. Check that $a \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d_c b_1 b_2$ and $a \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d_\emptyset c$, deduce that $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d$ does not satisfy right transitivity.

Exercise 74. In DLO, prove that $A \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d_C B$ if and only if $A \cap [b_1, b_2] \neq \emptyset$ implies $C \cap [b_1, b_2] \neq \emptyset$, for all $b_1 < b_2 \in B$.

Exercise 75. Prove that $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d$ satisfies strong finite character. Deduce that if $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d$ satisfies symmetry then $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d$ satisfies local character.

Exercise 76. Is there an independence relation $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}} \neq \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d$ such that $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d = \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^m$?

4.1.2. Digression: dividing in fields. In this subsection, T is a theory of fields in a language extending $\mathscr{L}_{\mathrm{ring}}$. Let $\mathbb{K}$ be a monster model of T.

Lemma 4.1.10. *Let F, A, B be a small subfields of T with $F \subseteq A \cap B$. If $A \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^h_F B$ or $A \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^u_F B$ then $A \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^{\mathrm{ld}}_F B$.*

Proof. As $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^h = (\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^u)^{\mathrm{opp}}$ and $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^{\mathrm{ld}}$ is symmetric, it is enough to prove that $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^h \to \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^{\mathrm{ld}}$. Let $a_1, \ldots, a_n \in A$ be linearly dependent over B. Then for some $b_1, \ldots, b_n$ we have $\sum_i a_i b_i = 0$. As $\mathrm{tp}(A/B)$ is an heir, there exists $c_1, \ldots, c_n \in F$ such that $\sum_i a_i c_i = 0$, hence $a_1, \ldots, a_n$ are linearly independent over F. □

Lemma 4.1.11. *Let $b = (b_1, \ldots, b_n)$ be a tuple linearly independent over a model F of T. Then the formula $\phi(x, y)$ defined by*

$$(x_1, \ldots, x_n) \neq (0, \ldots, 0) \wedge x_1 b_1 + \ldots + x_n b_n = 0$$

n-divides over F.

Proof. Using Corollary 3.3.6 there exist a $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^u$-Morley sequence $(b^i)_{i<\omega}$ in $\mathrm{tp}(b/F)$ with $b^0 = b$. Let $\Sigma(x)$ be the partial type consisting of all equations $\phi(x, b^j) = (x_1, \ldots, x_n) \neq (0, \ldots, 0) \wedge (\sum_i x_i b_i^j = 0)$, for all $j < \omega$. Using Lemma 4.1.10 we have that $b^j \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^{\mathrm{ld}}_F (b^i)_{i<j}$ hence the determinant of each n of those equations $(\sum_i x_i b_i^j = 0)$ is nonzero. It follows that $(0, \ldots, 0)$ is the only realisation of n of those equations. As the tuple $(0, \ldots, 0)$ does not satisfy $\phi(x, b_i)$ by definition, the partial type $\Sigma(x)$ is n-inconsistent. □

Proposition 4.1.12. *Let F be a small model of T and A, B be* acl*-closed small subsets of $\mathbb{K}$ containing F. If $A \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^d_F B$ then $A \mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}^{\mathrm{ld}}_C B$.*

PROOF. Assume that $A \mathop{\perp\!\!\!\perp}^{d}_{F} B$. If $A \mathop{\not\perp\!\!\!\perp}^{\mathrm{ld}}_{F} B$, then there exists a tuple $b = (b_1, \ldots, b_n)$ from B such that $\sum_i a_i b_i$ for some $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$ from A and $b_1, \ldots, b_n$ linearly independent over F. Hence the formula $(x_1, \ldots, x_n) \neq (0, \ldots, 0) \wedge (\sum_i x_i b_i = 0)$ is in $\mathrm{tp}(A/B)$, which contradicts Lemma 4.1.11. $\square$

REMARK 4.1.13. Chatzidakis also proved that in any theory of fields if $A \mathop{\perp\!\!\!\perp}^{d}_{F} B$ (or $A \mathop{\perp\!\!\!\perp}^{u}_{F} B$) then $\mathbb{K}$ is a separable extension of the field compositum $A(B)$ and $\mathrm{acl}(AB) \cap A^{\mathrm{alg}}(B^{\mathrm{alg}}) = A(B)$.

4.1.3. Forking.

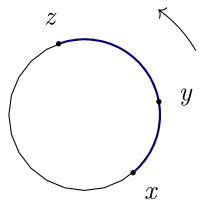
DEFINITION 4.1.14. A formula $\psi(x, b)$ *forks over* C if $\psi(x, b) \vDash \bigvee_{i=1}^{n} \phi(x, b_i)$ and each $\phi(x, b_i)$ divides over C. A (partial) type $\pi(x)$ forks over C if it implies a formula that forks over C.

We define the *non-forking independence relation* $\mathop{\perp\!\!\!\perp}^{f}$ by

$$A \mathop{\perp\!\!\!\perp}^{f}_{C} B \iff \mathrm{tp}(A/BC) \text{ does not fork over } C$$

We clearly have $\mathop{\perp\!\!\!\perp}^{f} \rightarrow \mathop{\perp\!\!\!\perp}^{d}$.

EXAMPLE 4.1.15 (A formula which forks and does not divides). *We describe the theory of an oriented circle, which we will denote* Circle.



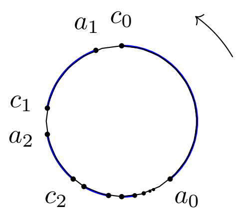
The domain is an infinite set and the language consists of a single ternary relation $C(x, y, z)$ such that

- $C(x, y, z) \iff C(y, z, x) \iff C(z, x, y)$
- *for all x the relation on (y, z) defined by $C(x, y, z)$ is a strict linear order which is dense without endpoints on the domain*

One easily shows that the theory Circle *has quantifier elimination in the language $\{C\}$. An easy and concrete model of this theory is given by defining the relation*

$$C(x, y, z) := (x < y < z) \vee (y < z < x) \vee (z < x < y)$$

in a dense linear order without endpoint.

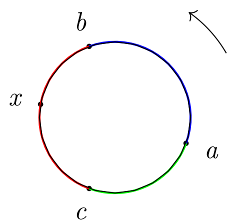


As in DLO, for any a, c the formula $\phi(y; a, c)$ given by $C(a, y, c)$ divides over $\emptyset$. To see this easily, consider pairs $(a_i, c_i)_{i<\omega}$ such that $C(a_0, a_i, c_i)$ and each $a_i c_i$ defines disjoint arcs via ϕ. It is easy to see that the type of a pair of distinct elements over $\emptyset$ is unique hence $a_i c_i \equiv ac$ for all $i < \omega$. The family obtained $(\phi(x, a_i, b_i))_{i<\omega}$ is 2*-inconsistent.*

The formula $x = x$ does not divide over $\emptyset$. Now observe that for any a, b, c such that $C(a, b, c)$ we have

$$(x = x) \vDash C(a, x, b) \vee C(b, x, c) \vee C(c, x, a) \vee x = a \vee x = b \vee x = c$$

hence as the formulas in the disjonct divides over $\emptyset$ the formula $x = x$ forks over $\emptyset$.



DEFINITION 4.1.16. Given any relation $\mathop{\perp\!\!\!\perp}$, we define the relation $\mathop{\perp\!\!\!\perp}^{*}$:

$$A \mathop{\perp\!\!\!\perp}^{*}_{C} B \iff \text{for all } D \supseteq B, \text{ there exists } A' \equiv_{BC} A \text{ with } A' \mathop{\perp\!\!\!\perp}_{C} D$$

We always have $\mathop{\perp\!\!\!\perp}^{*} \rightarrow \mathop{\perp\!\!\!\perp}$. By definition, if $\mathop{\perp\!\!\!\perp}^{0} \rightarrow \mathop{\perp\!\!\!\perp}$ and $\mathop{\perp\!\!\!\perp}^{0}$ satisfies extension, then $\mathop{\perp\!\!\!\perp}^{0} \rightarrow \mathop{\perp\!\!\!\perp}^{*}$.

PROPOSITION 4.1.17. *If $\mathop{\perp\!\!\!\perp}$ is invariant and satisfies left and right monotonicity then $\mathop{\perp\!\!\!\perp}^{*}$ is invariant and satisfies left and right monotonicity, right normality and extension. If $\mathop{\perp\!\!\!\perp}$ satisfies one of the following property: right base monotonicity, left transitivity, left normality, anti-reflexivity then so does $\mathop{\perp\!\!\!\perp}^{*}$.*

Proof. Assume that $\mathop{\downarrow}^*$ is invariant and satisfies left and right monotonicity. Assume that $ABC \equiv A'B'C'$ and let D' be such that $B' \subseteq D'$. There exists D such that $B \subseteq D$ and an automorphism σ such that $\sigma(ABCD) = A'B'C'D'$. If $A \mathop{\downarrow}^*_C B$ there exists $A'' \equiv_{BC} A$ such that $A'' \mathop{\downarrow}^*_C D$. By invariance $\sigma(A'') \mathop{\downarrow}^*_{C'} D'$ and as $A'' \equiv_{BC} A$ we have $\sigma(A'') \equiv_{B'C'} A'$, hence $A' \mathop{\downarrow}^*_{C'} B'$.

If $A \mathop{\downarrow}^*_C B$, $A_0 \subseteq A$ and $B_0 \subseteq B$. Let D be such that $B_0 \subseteq D$, then $B \subseteq DB_0$ so there exists $A' \equiv_{BC} A$ witnessed by σ such that $A' \mathop{\downarrow}_C DB_0$. By right monotonicity we have $A' \mathop{\downarrow}_C D$. Also, let $A_0' = \sigma(A)$, then $A_0' \subseteq A'$ is such that $A_0' \equiv_{BC} A_0$. In particular $A_0' \equiv_{B_0C} A$ and by left monotonicity we have $A_0' \mathop{\downarrow}_C D$. We conclude that for all $B_0 \subseteq D$ there exists $A_0' \equiv_{B_0C} A_0$ such that $A_0 \mathop{\downarrow}_C D$ so $A_0 \mathop{\downarrow}^*_C B_0$.

We prove extension. Assume that $A \mathop{\downarrow}^*_C B$ and $B \subseteq D$. Let $a = (a_i)_{i<\alpha}$ be an enumeration of A.

Claim 8. There exists $p \in S_\alpha(CD)$ extending $\mathrm{tp}(a/BC)$ such that for all κ and for all κ-saturated model M containing CD there exists $a'' \vDash p$ with $a'' \mathop{\downarrow}_C M$.

Proof of the claim. Assume not. Then for all $p \in S_\alpha(CD)$ extending $\mathrm{tp}(a/BC)$ there exists $\kappa(p)$ and a $\kappa(p)$-saturated model $M(p)$ containing CD such that for all $a'' \vDash p$ we have $a'' \not\mathop{\downarrow}_C M(p)$. As $S_\alpha(CD)$ is small, let $\kappa = \max\{\kappa(p) \mid p \in S_\alpha(CD)\}$ and let M be a κ-saturated model. By the assumption $a \mathop{\downarrow}^*_C B$, there exists $a' \equiv_{BC} a$ such that $a' \mathop{\downarrow}_C M$. Let $q = \mathrm{tp}(a'/CD)$. Clearly q extends $\mathrm{tp}(a/BC)$. As M is κ-saturated, it is κ-universal, hence as $\kappa \geq \kappa(q)$ there is an automorphism σ over CD such that $\sigma(M(q)) \subseteq M$. As $a' \mathop{\downarrow}_C M$ we have $a' \mathop{\downarrow}_C \sigma(M(q))$ by monotonicity hence by applying σ^{-1} and by invariance, we have $\sigma^{-1}(a') \mathop{\downarrow}_C M(q)$. As $\sigma^{-1}(a') \vDash q$ we get a contradiction. $\square$

Let p be as in the claim and let a' be a realisation of p. Let E be a superset of D. Let M be a $|E|$-saturated model containing CD. Then by the claim there exists $a'' \equiv_{CD} a'$ such that $a'' \mathop{\downarrow}_C M$. Using $|E|$-saturation of M and invariance we may assume that $E \subseteq M$ (as in the proof of the claim) hence by monotonicity we have $a'' \mathop{\downarrow}_C E$. We conclude that $a' \mathop{\downarrow}^*_C D$ hence as $a' \equiv_{BC} a$, $\mathop{\downarrow}^*$ satisfies extension.

The property right normality follows from extension. We now assume that $\mathop{\downarrow}$ is invariant and satisfies left and right monotonicity.

Assume that $\mathop{\downarrow}$ satisfies right base monotonicity. Assume that $A \mathop{\downarrow}^*_C B$ with $C \subseteq B_0 \subseteq B$. Then for any D with $B \subseteq D$ there exists $A' \equiv_{BC} A$ such that $A' \mathop{\downarrow}_C D$, hence $A' \mathop{\downarrow}_{B_0} D$ so $A \mathop{\downarrow}^*_{B_0} B$.

Assume that $\mathop{\downarrow}$ satisfies left transitivity. We use here the clearly equivalent alternative definition of $\mathop{\downarrow}^*$: $A \mathop{\downarrow}^*_C B$ if for all $B \subseteq D$ there exists $D' \equiv_{AC} D$ such that $A \mathop{\downarrow}_C D'$. Assume that $D \mathop{\downarrow}^*_B A$ and $B \mathop{\downarrow}^*_C A$ for $C \subseteq B \subseteq D$. Let $\hat{A}$ be a superset of A. As $B \mathop{\downarrow}^*_C A$ and $D \mathop{\downarrow}^*_B A$, there exists $\hat{A}' \equiv_A \hat{A}$ with $B \mathop{\downarrow}_C \hat{A}'$ and there exists $\hat{A}'' \equiv_D \hat{A}$ such that $D \mathop{\downarrow}_B \hat{A}''$. Then $\hat{A}'' \equiv_B \hat{A}'$ hence $B \mathop{\downarrow}_C \hat{A}''$ by invariance. By left transitivity we have $D \mathop{\downarrow}_C \hat{A}''$ hence we conclude $D \mathop{\downarrow}^*_C A$.

Assume that $\mathop{\downarrow}$ satisfies left normality. Assume that $A \mathop{\downarrow}^*_C B$. Then for all $B \subseteq D$ there is $A' \equiv_{BC} A$ such that $A' \mathop{\downarrow}_C D$. As $A'C \equiv_B CAC$ and $AC \mathop{\downarrow}_C D$ we conclude $AC \mathop{\downarrow}^*_C B$

If $\mathop{\downarrow}$ satisfies anti-reflexivity then so does $\mathop{\downarrow}^*$ since $\mathop{\downarrow}^* \to \mathop{\downarrow}$. $\square$

Lemma 4.1.18. *Suppose that $\mathop{\downarrow}$ is an invariant relation. The following are equivalent:*

(1) $\mathop{\downarrow}^*$ satisfies existence;
(2) $\mathop{\downarrow}$ satisfies full existence;
(3) there exists an invariant relation $\mathop{\downarrow}^0$ satisfying full existence with $\mathop{\downarrow}^0 \to \mathop{\downarrow}$.

Proof. (1) $\implies$ (2). Let A, B, C are given. By (1), we have $A \mathop{\downarrow}^*_C C$. In particular for $D = B$, there exists $A' \equiv_C A$ such that $A' \mathop{\downarrow}_C B$ so $\mathop{\downarrow}$ satisfies full existence.

(2) $\implies$ (3) is clear, take $\mathop{\downarrow}^0 = \mathop{\downarrow}$.

(3) $\implies$ (1). Let A, C be given, we want to show that $A \mathop{\smile\hspace{-0.9em}|}^{*}_{C} C$. Let $D \supseteq C$, and by full existence for $\mathop{\smile\hspace{-0.9em}|}^{0}$ there exists $A' \equiv_C A$ with $A' \mathop{\smile\hspace{-0.9em}|}^{0}_{C} D$. As $\mathop{\smile\hspace{-0.9em}|}^{0} \to \mathop{\smile\hspace{-0.9em}|}$, $A' \mathop{\smile\hspace{-0.9em}|}_{C} D$ hence $A \mathop{\smile\hspace{-0.9em}|}^{*}_{C} C$. $\square$

Exercise 77. Prove that if $\mathop{\smile\hspace{-0.9em}|}$ is invariant and satisfies monotonicity and strong finite character then $\mathop{\smile\hspace{-0.9em}|}^{*}$ satisfies strong finite character.

Exercise 78 (*). Prove that if $\mathop{\smile\hspace{-0.9em}|}$ is invariant and satisfies monotonicity and finite character then $\mathop{\smile\hspace{-0.9em}|}^{*}$ satisfies finite character.

Theorem 4.1.19. $\mathop{\smile\hspace{-0.9em}|}^{f} = \mathop{\smile\hspace{-0.9em}|}^{d\,*}$.

Proof. Observe that $a \mathop{\smile\hspace{-0.9em}|}^{f}_{C} b$ if and only if for all $B \subseteq D$ the (partial) type $\pi_D(x)$ defined by

$$\mathrm{tp}(a/Cb) \cup \{\neg\phi(x,d) \mid d \subseteq D, \phi(x,d) \text{ divides over } C\}$$

By compactness, π_D is inconsistent if and only if there exist a finite number of formula $\phi_i(x,d)$ which divides over C and such that $\mathrm{tp}(a/Cb) \vDash \bigvee_i \phi_i(x,d)$, the latter is equivalent to $\mathrm{tp}(a/Cb)$ forking over C. The theorem follows. $\square$

Corollary 4.1.20. *$\mathop{\smile\hspace{-0.9em}|}^{f}$ always satisfies finite character, left and right monotonicity, left normality, right base monotonicity, left transitivity, finite character and extension. In particular, $\mathop{\smile\hspace{-0.9em}|}^{f}$ is an AIR if and only if $\mathop{\smile\hspace{-0.9em}|}^{f}$ satisfies local character.*

Proof. By putting together Proposition 4.1.17, Theorem 4.1.6 and Theorem 4.1.19, it remains to check that $\mathop{\smile\hspace{-0.9em}|}$ satisfies finite character, which is trivial by the definition of $\mathop{\smile\hspace{-0.9em}|}^{f}$. The property local character is the only one missing from the definition of an AIR. $\square$

Remark 4.1.21. If $\mathop{\smile\hspace{-0.9em}|}^{f}$ is an AIR, then $\mathop{\smile\hspace{-0.9em}|}^{d} = \mathop{\smile\hspace{-0.9em}|}^{f}$ by Proposition 4.1.7 and Theorem 4.1.19.

Example 4.1.22. *We come back to Example 4.1.15, which cumulates several pathological behaviours around forking and dividing and $\mathop{\smile\hspace{-0.9em}|}^{*}$.*

(1) $\mathop{\smile\hspace{-0.9em}|}^{f} \neq \mathop{\smile\hspace{-0.9em}|}^{d}$ in general: $a \mathop{\smile\hspace{-0.9em}|}^{d}_{\emptyset} \emptyset$ whereas $a \not\mathop{\smile\hspace{-0.9em}|}^{f}_{\emptyset} \emptyset$ (even at the level of formula, there is a forking formula that does not divides).

(2) $\mathop{\smile\hspace{-0.9em}|}^{f}$ does not satisfy existence: $a \not\mathop{\smile\hspace{-0.9em}|}^{f}_{\emptyset} \emptyset$, equivalently $\mathop{\smile\hspace{-0.9em}|}^{d}$ does not satisfy full existence (by Lemma 4.1.18)

(3) $\mathop{\smile\hspace{-0.9em}|}^{d}$ does not satisfy extension[1]: for any distinct a, b, c, we have $a \mathop{\smile\hspace{-0.9em}|}^{d}_{\emptyset} \emptyset$ but there is no $a' \equiv_{\emptyset} a$ such that $a' \mathop{\smile\hspace{-0.9em}|}^{d}_{\emptyset} bc$, because $\mathrm{tp}(a'/bc)$ contains either the formula $C(b,x,c)$ or $C(c,x,b)$ or $x = b$ or $x = c$ all of which divides over $\emptyset$.

(4) Forcing the extension axiom $\mathop{\smile\hspace{-0.9em}|}^{*}$ does not preserve existence in general: $\mathop{\smile\hspace{-0.9em}|}^{d}$ always satisfy existence and $\mathop{\smile\hspace{-0.9em}|}^{f}$ does not.

Corollary 4.1.23. *If $\mathop{\smile\hspace{-0.9em}|}$ is invariant and satisfies left and right monotonicity then $\mathop{\smile\hspace{-0.9em}|}^{m*}$ is invariant and satisfies left and right monotonicity, right normality, right closure, right base monotonicity and extension. Further $\mathop{\smile\hspace{-0.9em}|}^{m*} \to \mathop{\smile\hspace{-0.9em}|}^{m} \to \mathop{\smile\hspace{-0.9em}|}$.*

Proof. By Proposition 1.2.14, $\mathop{\smile\hspace{-0.9em}|}^{m}$ satisfies right monotonicity and right base monotonicity. By Proposition 4.1.17, $\mathop{\smile\hspace{-0.9em}|}^{m*}$ satisfies right monotonicity, right base monotonicity and extension. By Proposition 3.1.3, $\mathop{\smile\hspace{-0.9em}|}^{m*}$ further satisfies right normality and right closure. In particular Proposition 1.2.14 applies and $\mathop{\smile\hspace{-0.9em}|}^{m*} \to \mathop{\smile\hspace{-0.9em}|}^{m}$. $\square$

Exercise 79. Prove that $\mathop{\smile\hspace{-0.9em}|}^{f}$ satisfies strong finite character. Deduce that $\mathop{\smile\hspace{-0.9em}|}^{f}$ satisfies local character if and only if $\mathop{\smile\hspace{-0.9em}|}^{f}$ satisfies symmetry and existence.

Exercise 80. Is there an independence relation $\mathop{\smile\hspace{-0.9em}|}$ different from $\mathop{\smile\hspace{-0.9em}|}^{u}$ such that $\mathop{\smile\hspace{-0.9em}|}^{*} = \mathop{\smile\hspace{-0.9em}|}^{u}$?

[1] Note that full existence and extension would be equivalent if $\mathop{\smile\hspace{-0.9em}|}^{d}$ satisfies right transitivity by Proposition 3.1.3 (c,d), but by Example 73, $\mathop{\smile\hspace{-0.9em}|}^{d}$ does not satisfy right transitivity in general.

4.1.4. Forking and the independence theorem.

THEOREM 4.1.24. *Let $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0$ be an invariant relation satisfying right monotonicity, right base monotonicity and local character.*

Let $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}$ be an invariant relation satisfying left and right monotonicity and the following property ($\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0$-amalgamation over models):

if $c_1 \equiv_M c_2$ and $c_1 \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}_M a$, $c_2 \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}_M b$ and $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0_M b$ then there exists c with $c \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}_E ab$ and $c \equiv_{Ma} c_1$, $c \equiv_{Mb} c_2$.

Then $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{m*} \to \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^f$.

PROOF. By Corollary 4.1.23, the relation $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{m*}$ satisfies satisfies left and right monotonicity, right normality, right closure, right base monotonicity and extension.

We show that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{m*} \to \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^d$, the result follows from $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^f = \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{d\,*}$ (Theorem 4.1.19).

Assume that $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{m*}_C b$, for some a, b, C. Let $(b_i)_{i<\omega}$ be a C-indiscernible sequence with $b = b_0$. By Lemma 3.2.19, there exists a model $M \supseteq C$ such that $(b_i)_{i<\omega}$ is an M-indiscernible $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0$-Morley sequence over M, i.e. $b_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0_M b_{<i}$ for all $i < \omega$.

By extension there exists a' such that $a' \equiv_{Cb} a$ and $a' \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{m*}_C bM$. It follows from base monotonicity and right monotonicity that

$$a' \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}_M b.$$

For each $i \geq 0$ there exists an automorphism σ_i over M sending $b = b_0$ to b_i, so setting $a'_i = \sigma_i(a')$ we have: $a'_i b_i \equiv_M a'b$ hence by invariance $a'_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}_M b_i$. Note that $a'b \equiv_C ab$.

CLAIM 9. There exists a'' such that $a''b_i \equiv_M a'b$ for all $i < \omega$.

PROOF OF THE CLAIM. By induction and compactness, it is sufficient to show that for all $i < \omega$, there exists a''_i such that for all $k \leq i$ we have $a''_i b_k \equiv_M a'b$ and $a''_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}_M b_{\leq i}$. For the case $i = 0$ take $a''_0 = a'$. Assume that a''_i has been constructed, we have

$$a'_{i+1} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}_M b_{i+1} \text{ and } b_{i+1} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0_M b_{\leq i} \text{ and } a''_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}_M b_{\leq i}.$$

As $a'_{i+1} \equiv_M a''_i$, by $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0$-amalgamation over models, there exists a''_{i+1} such that

(1) $a''_{i+1} b_{i+1} \equiv_M a'_{i+1} b_{i+1}$
(2) $a''_{i+1} b_{\leq i} \equiv_M a''_i b_{\leq i}$
(3) $a''_{i+1} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}_M b_{\leq i+1}$.

By induction and compactness, there exists a'' such that $a''b_i \equiv_M ab$ for all $i < \omega$, which proves the claim. ☐

Let a'' be as in the claim, then as $a'b \equiv_C ab$ we have $a''b_i \equiv_C ab$ for all $i < \omega$, hence $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^d_C b$. ☐

REMARK 4.1.25. Observe that the relations $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}$ and $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0$ in the previous result may not be symmetric. For instance, it could be that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0 = \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^h$ (see below). Then, the parameters a and b in the statement of $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0$-amalgamation do not play a symmetrical role a priori. However the role of (c_1, a) and (c_2, b) are symmetric hence if a relation satisfies $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^h$-amalgamation, it means that $tp(c_1/Ma)$ and $tp(c_2/Mb)$ can be amalgamated *whenever* $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0_M b$ *or* $b \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0_M a$ (i.e. $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{0\,\mathrm{opp}}_M b$). It follows that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0$-amalgamation and $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{0\,\mathrm{opp}}$-amalgamation are the same. Observe that the property $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}$-amalgamation has a contravariant behaviour: if $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^1 \to \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^2$, then if a relation satisfies $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^2$-amalgamation, it also satisfies $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^1$-amalgamation. In particular, if $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0 \to \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}$ then if $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}$ satisfies the independence theorem, it satisfies $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^0$-amalgamation.

COROLLARY 4.1.26. *Some consequences of Theorem 4.1.24.*

(1) *Let $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}$ be an invariant relation which satisfies left and right monotonicity and $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^h$-amalgamation over models. Then $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{m*} \to \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^f$.*

(2) Let $\mathop{\smile\!\!\!\!\!|}\,$ be an invariant relation, which satisfies left and right monotonicity, right base monotonicity, local character and the independence theorem over models. Then $\mathop{\smile\!\!\!\!\!|}^{*} \to \mathop{\smile\!\!\!\!\!|}^{f}$.

(3) Let $\mathop{\smile\!\!\!\!\!|}\,$ be an invariant relation satisfying left and right monotonicity, extension and stationarity over models, then $\mathop{\smile\!\!\!\!\!|}^{m*} \to \mathop{\smile\!\!\!\!\!|}^{f}$.

PROOF. (1) From Proposition 3.3.2 and Theorem 3.3.7, the relation $\mathop{\smile\!\!\!\!\!|}^{h}$ satisfies monotonicity, right base monotonicity and local character, hence we apply Theorem 4.1.24.

(2) We apply Theorem 4.1.24. As $\mathop{\smile\!\!\!\!\!|}\,$ satisfies right monotonicity, base monotonicity and local character we consider $\mathop{\smile\!\!\!\!\!|}^{0} = \mathop{\smile\!\!\!\!\!|}\,$. Observe that $\mathop{\smile\!\!\!\!\!|}\,$-amalgamation is the same as the independence theorem hence by Theorem 4.1.24 we get $\mathop{\smile\!\!\!\!\!|}^{m*} \to \mathop{\smile\!\!\!\!\!|}^{f}$. As $\mathop{\smile\!\!\!\!\!|}\,$ satisfies base monotonicity we have $\mathop{\smile\!\!\!\!\!|}^{m} = \mathop{\smile\!\!\!\!\!|}\,$, so we conclude.

(3) As $\mathop{\smile\!\!\!\!\!|}\,$ satisfies right monotonicity, extension and stationarity over models, it satisfies $\mathop{\smile\!\!\!\!\!|}^{0}$-amalgamation for *any* independence relation $\mathop{\smile\!\!\!\!\!|}^{0}$. Pick any $\mathop{\smile\!\!\!\!\!|}^{0}$ satisfying right monotonicity, right base monotonicity and local character (for instance $A \mathop{\smile\!\!\!\!\!|}^{0}_{C} B$ iff $A \cap B \subseteq C$) and apply Theorem 4.1.24. □

EXERCISE 81. If $\mathop{\smile\!\!\!\!\!|}\,$ satisfies symmetry, existence, monotonicity, base monotonicity, extension, the independence theorem and strong finite character then $\mathop{\smile\!\!\!\!\!|}\, \to \mathop{\smile\!\!\!\!\!|}^{f}$.

EXERCISE 82. In DLO, define $A \mathop{\smile\!\!\!\!\!|}^{>}_{C} B$ if and only if $a > b$ for all $a \in A$, $b \in BC$. Check that $\mathop{\smile\!\!\!\!\!|}^{>}$ satisfies left and right monotonicity, right base monotonicity, extension and stationarity (over every set). Deduce that $\mathop{\smile\!\!\!\!\!|}^{>} \to \mathop{\smile\!\!\!\!\!|}^{f}$. Is it faster than proving that the formula $x > b$ does not fork over C for any C?

4.2. Simple theories

4.2.1. Simple theories and dividing. We denote by $\omega^{<\omega}$ the set of all finite sequences of finite ordinals, considered as a tree with infinite branching at each node. Given $s, t \in \omega^{<\omega}$, we write $s \leq t$ if s is a prefix of t. We denote by ω^{ω} the set of all sequences of finite ordinals of length ω.

DEFINITION 4.2.1 (Shelah).) Let $k \geq 2$. A formula $\phi(x, y)$ has the *k-tree property* if there exists $(b_\nu)_{s \in \omega^{<\omega}}$ such that

(a) $\{\phi(x, b_s) \mid s \subseteq \nu\}$ is consistent for each $\nu \in \omega^{\omega}$;
(b) $\{\phi(x, b_{s^\frown i}) \mid i < \omega\}$ is k-inconsistent, for all $s \in \omega^{<\omega}$.

A formula has the *tree property* (TP) if it has the k-tree property for some $k < \omega$.

A theory T is *simple* (or NTP[2]) if no formula has the tree property.

EXAMPLE 4.2.2. *In DLO, the formula $\phi(x, yz)$ given by $y < x < z$ has the 2-tree property: start with $(b_i c_i)_{i<\omega}$ with $c_0 < b_0 < c_1 < b_1 < \ldots$. Inductively if $b_s c_s$ has been constructed, we find $\{b_{s^\frown j} c_{s^\frown j} \mid j < \omega\}$ such that*

$$b_s < b_{s^\frown 0} < c_{i^\frown 0} < b_{i^\frown 1} < c_{i^\frown 1} < \ldots < c_s.$$

Then $(b_s c_s)_{s \in \omega^{<\omega}}$ is a witness of 2-TP for $\phi(x, yz)$.

REMARK 4.2.3.
- If a formula $\phi(x, y, c)$ has the tree property, witnessed by a tree of parameters $(b_s)_{s \in \omega^{<\omega}}$, then the formula $\phi(x; yz)$ has the tree property with the tree $(b_s c)_{s \in \omega^{<\omega}}$. Hence for simplicity, it suffices to check that no formula without parameters has the tree property.
- Any theory definable (even interpretable) in a simple theory is again simple. This is because simplicity is defined at the level of formulas. In particular, any reduct of a simple theory is again simple.

[2]Nobody uses the determination NTP. The only interest in this notion is the following. There exists two notions: the *tree property of the first kind* (TP_1) and the *tree property of the second kind* (TP_2) such that a formula has the TP if and only if this formula has the TP_1 or the TP_2 (we also say *is* TP_1 or *is* TP_2). Then a theory is NTP_i if no formula in T is TP_i. In this sense, a theory is NTP (simple) if an only if it is NTP_1 and NTP_2.

REMARK 4.2.4. Assume that $\phi(x,y)$ has the k-tree property, witnessed by $b = (b_s)_{s\in\omega^{<\omega}}$. Then the tuple b satisfies the (partial) type $\Sigma^{\omega,\omega}((y_s)_{s\in\omega^{<\omega}})$ given by the union of (every branch is consistent)

$$\bigcup_{\nu\in\omega^\omega} \{\exists x\phi(x,y_{s_1}) \wedge \ldots \wedge \phi(x,y_{s_n}) \mid s_1,\ldots,s_n \subseteq \nu, n<\omega\}$$

and (every level is k-inconsistent)

$$\bigcup_{s\in\omega^{<\omega}} \left\{\neg\exists x \bigwedge_{j=1}^{k} \phi(x,y_{s^\frown i_j}) \mid i_1 < \ldots < i_k < \omega\right\}$$

Conversely, if this type is consistent then the formula $\phi(x,y)$ has the k-tree property. Note the importance of having k-inconsistency instead of inconsistency on each level otherwise the tree property would not be expressable as a type. From $\Sigma^{\omega,\omega}$, one derives easily a type $\Sigma^{\kappa,\mu}$ which witnesses the tree property for a tree shaped like $\kappa^{<\mu}$, for some limit ordinal μ. To do so, simply by add more variables and change $\omega^{<\omega},\omega^\omega$ by $\kappa^{<\mu},\kappa^\mu$. Then by compactness, the formula $\phi(x,y)$ has the tree property if an only if $\Sigma^{\kappa,\mu}$ is consistent, for any infinite κ and limit ordinal μ.

DEFINITION 4.2.5. A *ϕ-k dividing sequence over C, of length μ* is a sequence $(\phi(x,b_i))_{i<\mu}$ such that $\phi(x,b_i)$ divides over $Cb_{<i}$ and $\bigwedge_{i<\mu}\phi(x,b_i)$ is consistent.

LEMMA 4.2.6. *$\phi(x,y)$ has the k-tree property if and only if there exists arbitrary long ϕ-k dividing sequence over $\emptyset$.*

PROOF. Assume that $\phi(x,y)$ has the k-tree property and μ is given. We may assume that μ is a limit ordinal. By Remark 4.2.4 there exists an infinite tree of parameters $(b_s)_{s\in\kappa^{<\mu}}$ for some $\kappa > 2^{\max\{|T|,\mu\}}$. We choose recursively a branch $\nu\in\kappa^\mu$ such that the sequence of $\phi(x,b_s)$ for initial $s\subseteq\nu$, forms a ϕ-k dividing sequence. Let $n=|y|$. As $\kappa > 2^{\max\{|T|\}}$ and $|S_n(\emptyset)| \leq 2^{\max\{|T|\}}$ there are infinitely many b_i's (for $i<\kappa$, in the lowest level of the tree) which have the same type over $\emptyset$, so choose one of the indexes of those i_0. Clearly $\phi(x,b_{i_0})$ is a k-dividing formula over $\emptyset$. By induction, assume that $s = i_0^\frown \ldots ^\frown i_j$ is such that $\phi(x,b_s)$ divides over $b_{i_0},\ldots,b_{i_0^\frown\ldots^\frown i_{j-1}}$ and consider elements $(b_{s^\frown i})_{i<\kappa}$. As $\kappa > 2^{\max\{|T|,\mu\}} > |S_n(b_{i_0},\ldots,b_s|)$, infinitely many of $(b_{s^\frown i})_{i<\kappa}$ have the same type over $b_{i_0},\ldots,b_s$ hence choose the index i_{j+1} of one of those an set $s' = s^\frown i_{j+1}$. Then $\phi(x,b_{s'})$ divides over $b_{i_0},\ldots,b_s$. The limit case is done similarly. By induction, we find a branch $\nu\in\kappa^\mu$ such that the sequence $(\phi(x,b_{\nu\restriction\{0,\ldots,i\}})_{i<\mu}$ is a ϕ-k dividing sequence.

Conversely, assume that there is an infinite a $\phi(x,y)$-k dividing sequence $(\phi(x,b_i))_{i<\omega}$. For each $i<\omega$, let $(b_i^n)_{n<\omega}$ be a sequence in $\mathrm{tp}(b_i/b_{<i})$ with $b_i^0 = b_i$ witnessing that $\phi(x,b_i)$ divides over $b_{<i}$. We define a tree of parameters $(c_s)_{s\in\omega^{<\omega}}$ as follows. Start by defining $c_0 = b_0^0, c_1 = b_0^1, c_n = b_0^n$ for each $n<\omega$, which will be the first level of the tree. We describe level 2: define $c_{0n} := b_1^n$ and for each $n<\omega$. There is an automorphism σ_n sending b_0^0 to $c_n = b_0^n$ and define $c_{ni} := \sigma_n(b_1^i)$, so that the level above c_n is $\sigma_n((b_1^i)_{i<\omega})$. Once a tree $(c_s)_{s\subseteq\omega^{<m}}$ of height m has been constructed, such that $c_{0\ldots0} = b_m^0$ and for any s, $c = (c_{s(0)},\ldots,c_{s(m)}) \equiv (b_0,\ldots,b_n)$, witnessed by an automorphism $\sigma_s(b_0,\ldots,b_m) = c$. Then define $c_{s^\frown n} := \sigma_s(b_{m+1}^n)$, so that the level above c_s is the image of the level above b_m. This ensures that set of formula along the branches is consistent, and that the levels are k-inconsistent. $\square$

THEOREM 4.2.7. *T is simple if and only if $\mathop{\smile\!\!\!\!\!\!|}^d$ satisfies local character.*

PROOF. By Lemma 4.2.6, it is enough to prove that $\mathop{\smile\!\!\!\!\!\!|}^d$ satisfies local character if and only if there exists no infinite ϕ-k dividing sequence.

Assume that $(\phi(x,b_i))_{i<\kappa}$ is a ϕ-k dividing sequence, for some regular cardinal κ and $a \models \bigwedge_i \phi_i(x,b_i)$. Let $C\subseteq B := \{b_i \mid i<\kappa\}$ with $|C|<\kappa$. Then there exists $i<\kappa$ such that $C\subseteq b_{<i}$ so that $\phi(x,b_i)$ divides over $b_{<i}$ and hence $\phi(x,b_i)$ divides over C. It follows that for all $C\subseteq B$ such that $|C|<\kappa$ we have $a \not\mathop{\smile\!\!\!\!\!\!|}^d_C B$ hence local character fails.

Conversely, assume that $\mathop{\smile\!\!\!\!\!\!|}^d$ fails local character. Hence there exists a finite tuple a, a cardinal κ and a set B such that $a \not\mathop{\smile\!\!\!\!\!\!|}^d_C B$ for all $C \subseteq B$ with $|C| \leq \kappa$ [3]. As $a \mathop{\smile\!\!\!\!\!\!|}^d_B B$ we have $|B| > \kappa$. We construct a sequence of formula $(\phi_i(x, b_i))_{i<\kappa^+}$ as follows. Start with any $b_0 \subseteq B$ such that $a \not\mathop{\smile\!\!\!\!\!\!|}^d_\emptyset b_0$ witnessed by a k_0-dividing formula $\phi_0(x, b_0)$, for some $k_0 < \omega$. For any $i < \kappa^+$, if $b_{<i}$ has been constructed, then $|b_{<i}| \leq \kappa$ hence there exists b_i such that $a \not\mathop{\smile\!\!\!\!\!\!|}^d_{b_{<i}} b_i$ and a formula $\phi_i(x, b_i)$ which k_i-divides over $b_{<i}$. As $\kappa^+ > |T|$, there is an infinite (of size κ^+) subsequence with the same ϕ, and again an infinite subsequence with the same k_i, which is an infinite ϕ-k dividing sequence. $\square$

4.2.2. The Kim-Pillay theorem. Let T be a theory and $\mathbb{M}$ a monster model of T.

THEOREM 4.2.8. *Assume that there exists an invariant relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfying left and right monotonicity, right base monotonicity, local character, extension and the independence theorem over models, then:*

- $\mathop{\smile\!\!\!\!\!\!|} \to \mathop{\smile\!\!\!\!\!\!|}^f$
- *T is simple;*
- $\mathop{\smile\!\!\!\!\!\!|}^f = \mathop{\smile\!\!\!\!\!\!|}^d$*;*
- $\mathop{\smile\!\!\!\!\!\!|}^f$ *is symmetric.*

Assume further that $\mathop{\smile\!\!\!\!\!\!|}$ satisfies finite character, left normality and left transitivity, then $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^f = \mathop{\smile\!\!\!\!\!\!|}^d$ and $\mathop{\smile\!\!\!\!\!\!|}$ is symmetric.

PROOF. Under the first assumptions, we have by Corollary 4.1.26 (2) that $\mathop{\smile\!\!\!\!\!\!|}^* \to \mathop{\smile\!\!\!\!\!\!|}^f$. As $\mathop{\smile\!\!\!\!\!\!|}$ also satisfies extension, we have $\mathop{\smile\!\!\!\!\!\!|}^* = \mathop{\smile\!\!\!\!\!\!|}$ hence $\mathop{\smile\!\!\!\!\!\!|} \to \mathop{\smile\!\!\!\!\!\!|}^f$. As $\mathop{\smile\!\!\!\!\!\!|}$ satisfies local character, so does $\mathop{\smile\!\!\!\!\!\!|}^f$ hence by Corollary 4.1.20, $\mathop{\smile\!\!\!\!\!\!|}^f$ is an AIR and $\mathop{\smile\!\!\!\!\!\!|}^f = \mathop{\smile\!\!\!\!\!\!|}^d$ and $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies symmetry. Theorem 4.2.7 T is simple since $\mathop{\smile\!\!\!\!\!\!|}^d$ satisfies local character. If $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the other properties, then $\mathop{\smile\!\!\!\!\!\!|}$ is also an AIR hence satisfies symmetry and $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^d$ follows from Proposition 4.1.7. $\square$

COROLLARY 4.2.9 (The Kim-Pillay theorem). *Assume that there exists an invariant relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfying symmetry, normality, monotonicity, base monotonicity, transitivity, finite character, local character, full existence and the independence theorem over models. Then T is simple and $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^f$.*

REMARK 4.2.10. The statement of Corollary 4.2.9 is not the full statement of what people generally call the Kim-Pillay theorem. The full version is a characterisation of simplicity: T is simple *if and only if* there is an independence relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfying all those axioms. To recover the full statement, it remains to prove that if T is simple, then $\mathop{\smile\!\!\!\!\!\!|}^f = \mathop{\smile\!\!\!\!\!\!|}^d$ and that $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies the independence theorem. The Kim-Pillay characterisation of simple theories is a deep result as it yields an equivalence between two *a priory* very different definitions in nature. Let us rephrase the Kim-Pillay result: a theory is simple if and only if there exists an AIR which satisfies the independence theorem, and if so, it is unique (and equals $\mathop{\smile\!\!\!\!\!\!|}^f$). The local, or syntactic Definition 4.2.1 and the existence of an AIR satisfying the independence theorem are properties of very different nature. One nontrivial corollary of this equivalence is that the existence of an AIR satisfying the independence theorem is closed under reduct, which is, at first sight, not clear at all.

REMARK 4.2.11. The Kim-Pillay theorem is a crucial result in model theory and is very useful from a practical point of view. Assume that you stumble upon some theory in the bend of a road and that you would like to know whether this theory is simple. You then have a choice between checking that no formula $\phi(x, y)$ has the tree property, for this you would need a very strong understanding of definable sets, probably a strong quantifier elimination result. The other solution is to find a ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ and check that it satisfies the axioms of the Kim-Pillay theorem. The main obstacle in general is the independence theorem but it is still much easier to prove in general than checking TP. If you succeed in using the Kim-Pillay theorem, you would know much more about this independence relation, it coincides with $\mathop{\smile\!\!\!\!\!\!|}^f$ then you would have a new definition of $\mathop{\smile\!\!\!\!\!\!|}$, in terms of formulas.

[3] In all generality, local character is equivalent to the same statement assuming $\leq$ instead of $<$.

Exercise 83. Assume that there exists an invariant relation $\mathop{\smile\!\!\!\!\!\!\!\;|}$ which satisfies left and right monotonicity, right base monotonicity, local character, extension and the independence theorem over models, then T is simple.

Exercise 84. Assume that there exists an invariant relation $\mathop{\smile\!\!\!\!\!\!\!\;|}$ satisfying symmetry, existence, monotonicity, base monotonicity, strong finite character and $\mathop{\smile\!\!\!\!\!\!\!\;|}^u$-amalgamation over models. If $\mathop{\smile\!\!\!\!\!\!\!\;|}^*$ is symmetric , then:

- $\mathop{\smile\!\!\!\!\!\!\!\;|}^* \to \mathop{\smile\!\!\!\!\!\!\!\;|}^f$
- T is simple;
- $\mathop{\smile\!\!\!\!\!\!\!\;|}^f = \mathop{\smile\!\!\!\!\!\!\!\;|}^d$;
- $\mathop{\smile\!\!\!\!\!\!\!\;|}^f$ is symmetric.

Assume further that $\mathop{\smile\!\!\!\!\!\!\!\;|}$ satisfies normality and transitivity, then $\mathop{\smile\!\!\!\!\!\!\!\;|}^* = \mathop{\smile\!\!\!\!\!\!\!\;|}^f = \mathop{\smile\!\!\!\!\!\!\!\;|}^d$.

4.2.3. Back to the examples.

Corollary 4.2.12. *ACF and RG are simple theories and the forking independence relations is given respectively by* $\mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{alg}}$ *and* $\mathop{\smile\!\!\!\!\!\!\!\;|}^a$.

Proof. Using the Kim-Pillay theorem (Corollary 4.2.9) with $\mathop{\smile\!\!\!\!\!\!\!\;|} = \mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{alg}}$ for ACF (Propositions 2.2.16 and 3.1.7) and $\mathop{\smile\!\!\!\!\!\!\!\;|} = \mathop{\smile\!\!\!\!\!\!\!\;|}^a$ for RG (Propositions 2.2.25 and 3.1.9). □

Remark 4.2.13. Observe that concerning RG, we used two different independence relations to prove that it is NSOP_4 ($\mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{st}}$, see Corollary 3.3.14) or that it is simple.

4.3. A few words on Kim-Pillay style results

4.3.1. Stable theories. We did not have time to go through stable theories, but let me mention a few words on this central notion in model theory. Essentially we saw that simple theories are characterised by those theories where some Adler independence relation satisfies the independence theorem over models, and if so such independence relation is unique and given by $\mathop{\smile\!\!\!\!\!\!\!\;|}^f$. The property stationarity over models implies the independence theorem over models hence any theory where some AIR satisfies stationarity over models is simple. Define a theory to be *stable* if there is some AIR which satisfies stationarity over models. As in simple theories, this independence relation is unique and is $\mathop{\smile\!\!\!\!\!\!\!\;|}^f$. As for simple theories, stable theories have a *local* definition (i.e. formula-by-formula, as in Definition 4.2.1) which is that not formula $\phi(x,y)$ has the *order property*: there are no pairs of tuples $(a_i,b_i)_{i<\omega}$ such that $\phi(a_i,b_j)$ if and only if $i<j$. The result stating that this local definition is equivalent to the existence of an AIR which satisfied stationarity over models is known as the Harnik-Harrington principle and dates back to 1984! Note that our criterion Theorem 3.3.12 implies that every stable theory is NSOP_4. It is true that any simple theory is also NSOP_4 although it does not follow from Theorem 3.3.12.

4.3.2. NSOP_1 theories. A larger class than simple theories –NSOP_1 theories– have been more recently (2016/2017) characterised by the existence of a certain independence relation. The result of Chernikov-Kaplan-Ramsey can *almost* be stated as follows: a theory is NSOP_1 if and only if there exists an AIR satisfying the independence theorem but may fail base monotonicity. With this criterion available, plenty of theories otherwise considered as wild were recently proved to be NSOP_1.

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