# Model Theory

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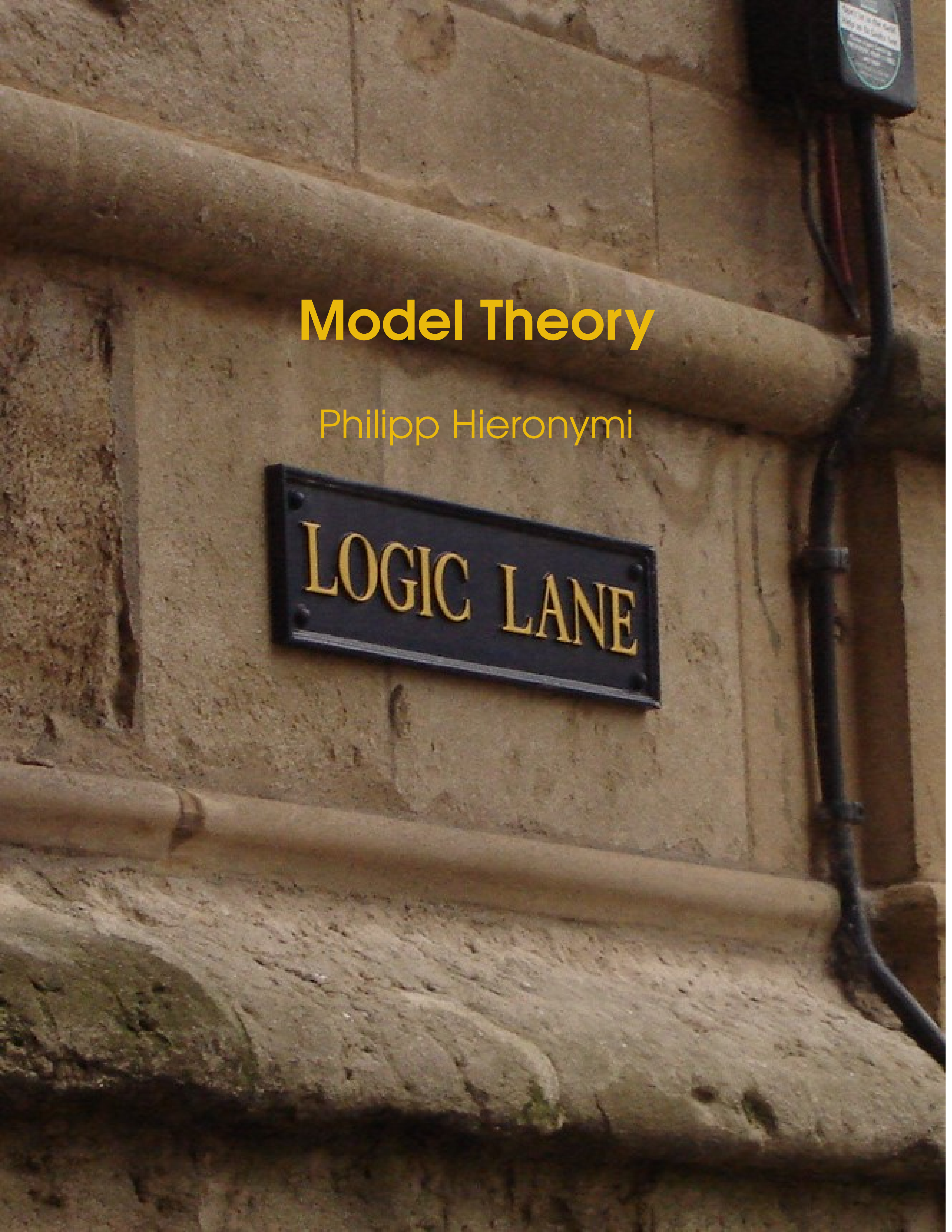

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# Contents

# 1. Introduction

These are lecture notes for a first-course in model theory. While not following it too closely, the content of this lecture is chosen with Aschenbrenner et al. [ADH17, Appendix B] in mind, focusing on quantifier-elimination and model completeness techniques and applications to algebra and geometry. Other excellent (and highly recommended) treatments of modern model theory are Marker [Mar02] and Tent and Ziegler [TZ12]. In addition, when discussing o-minimal, we follow van den Dries [Dri98], and recommend this book to everyone interested in learning more about o-minimality.

Unsurprisingly, model theory studies models of theories. This is a common theme within mathematics and algebra in particular: we are used to studying algebraic objects (called structures in model theory) that satisfy certain axioms. Algebraic examples are groups, rings, fields, vector spaces and so on. In each of these examples, we specify some basic operations (in the case of groups: the binary group operation), then state axioms (the usual group axioms) that a set together with such operations, has to satisfy. Model theory has this in common with universal algebra, but model theory focuses more on the connection between logical formulas and these structures. As Chang and Keisler [CK73] wrote, we can summaries this observation as

**"universal algebra + logic = model theory".**

An important focus, accelerated by the emergence of o-minimality in the early 1980's, has been the study of definable subsets in a given structure. Basically, a set $X$ is definable if there is a logically formula $\varphi$ in the language of the given structure such that for a tuple of elements $a$ the formula $\varphi(a)$ holds in this structure if and only if $a \in X$. The overall theme is to use and analyse this syntactic description in order to prove semantic, often geometric, consequences for the given definable set. Again, this is a common enterprise in mathematics. For example, in algebraic geometry we study algebraic varieties, which are solution sets of polynomial equations like

$$\{a \in \mathbb{C}^n \ : \ p(a) = 0\},$$

where $p$ is polynomial with coefficients from $\mathbb{C}$. Also in this setting, the syntactical description of the sets is used to prove results about its geometry. Model theory generalizes this idea by replacing the polynomial equation by an arbitrary formula, and hence the variety by a definable set. So in Hodges' words [Hod97],

> **“model theory = algebraic geometry - fields".**

In the case of algebraic geometry we are succesful in using the syntactical description of varieties to prove strong geometric properties of varieties. In general, being definable in some structure has essentially no consequences on a set, as we can simply choose structures whose languages are rich enough to express arbitrarily complicated objects. For example, adding a predicate for the set of integers to the field of real numbers makes every Borel - and even every projective set - definable in that structure. So even the question whether or not all definable sets in this structure are Lebesgue measurable, is independent of ZFC. Thus an important part of model theory, it is to determine which (and what kind of) structures have well-behaved definable sets. Van den Dries [Dri99], quoting Hrushovski, sums this up as follows:

> **“A lot of model theory is concerned with discovering and charting the “tame” regions of mathematics, where wild phenomena like space filling curves and Gödel incompleteness are absent, or at least under control. As Hrushovski put it recently: Model Theory = Geography of Tame Mathematics.”**

Here model theorists attempt to find notions of tameness common among classes of well-behaved structures. In this course, we will learn about **strong minimality** as an attempt to capture the tameness of algebraic geometry and about **o-minimality** as an attempt to capture the tameness of semi-algebraic geometry. Along the way, we also mention **$\omega$-stability**.

A prerequisite for this course is basic knowledge of the syntax and semantics of first-order logic, and some experience with expressing mathematical statements in first-order logic. Although strictly speaking, much of this is reviewed in Chapter 2. We assume some basic knowledge of set theory, in particular ordinals and cardinals. Throughout, we will look at examples arising from algebra such as algebraically closed fields. Thus background knowledge as is usually obtained in a first course in algebra, is - at the very least - helpful. A good reference for the necessary background in set theory and basic first-order logic is Professor Koepke's script available at

`www.math.uni-bonn.de/ag/logik/teaching/2019WS/logik/Current_Scriptum.pdf`

# 2. Models and theories

## 2.1 Syntax

In this section, we will recall the definition of the syntax of first-order logic. While this section is self-contained, it is assumed that students have seen first-order logic before.

**Definition 2.1.1** A **language** $\mathscr{L}$ is a triple $(S_{\text{func}}, S_{\text{rel}}, \text{ar})$ such that $S_{\text{func}}$ and $S_{\text{rel}}$ are disjoint set and $\text{ar} : S_{\text{func}} \cup S_{\text{rel}} \to \mathbb{N}$. We call elements of $S_{\text{func}}$ **function symbols in $\mathscr{L}$**, and elements of $S_{\text{rel}}$ **relation symbols in $\mathscr{L}$**. For every element $s \in S_{\text{func}} \cup S_{\text{rel}}$, we call $ar(s)$ the **arity** of $s$ and say that $s$ is $ar(s)$-ary. A 0-ary function symbol is called a **constant symbol**.

Outside of model theory, languages are referred to as **signatures** (or vocabularies) and the set of all $\mathscr{L}$-sentence (or $\mathscr{L}$-formulas) is called the language of the signature. We will not use this terminology here, even though this would be more precise.

**Notation 2.1.** *We often use abbreviations when defining languages. For example, if $\mathscr{L}$ is the triple $\big(\{f,c\},\{R\},(f \mapsto 2, c \mapsto 0, R \mapsto 5)\big)$, we often simply say: $\mathscr{L} = \{f,c,R\}$ where $f$ is a binary function symbol, $c$ is a constant symbol and $R$ is a 5-ary relation symbol.*

■ **Example 2.1** We collect a few example of languages:

1. the **empty language** $\mathscr{L}_\emptyset := \emptyset$ (also called the language of pure sets),
2. the **language of groups** $\mathscr{L}_{\text{g}} := \{\cdot, ()^{-1}, e\}$, where $\cdot$ is binary function, $()^{-1}$ is a unary function symbol, and $e$ is a constant symbol,
3. the **language of rings** $\mathscr{L}_{\text{r}} := \{+,-,\cdot,0,1\}$, where $+,-$ and $\cdot$ are binary function symbols and 0 and 1 are constant symbols,
4. the **language of ordered rings** $\mathscr{L}_{\text{or}} := \mathscr{L}_{\text{r}} \cup \{<\}$, where $<$ is a binary relation symbol,
5. the **language of graphs** $\mathscr{L}_\sim := \{\sim\}$, where $\sim$ is a binary relation.

■

**Definition 2.1.2** Let $\mathscr{L}$ be a language. The set $\mathscr{T}(\mathscr{L})$ is defined as the smallest set such that

1. $c \in \mathscr{T}(\mathscr{L})$ for every constant symbol $c$ in $\mathscr{L}$,

2. $x \in \mathscr{T}(\mathscr{L})$ for every variable symbol $x$,
3. if $t_1, \dots, t_n \in \mathscr{T}(\mathscr{L})$ and $f$ is an $n$-ary function symbol in $\mathscr{L}$, then $f(t_1, \dots, t_n) \in \mathscr{T}(\mathscr{L})$.

An element of $\mathscr{T}(\mathscr{L})$ is called an **$\mathscr{L}$-term**.

**Notation 2.2.** *We often say that $t(x_1, \dots, x_n)$ is an $\mathscr{L}$-term. This means that $t$ is an $\mathscr{L}$-term and $x_1, \dots, x_n$ are variable symbol, and all variable symbols that appears in $t$ are among the $x_1, \dots, x_n$.*

**Definition 2.1.3** Let $\mathscr{L}$ be a language. The set $\mathscr{F}(\mathscr{L})$ is defined as the smallest set such that
1. $s = t$ for every $s, t \in \mathscr{T}(\mathscr{L})$,
2. $R(t_1, \dots, t_n)$ for $t_1, \dots, t_n \in \mathscr{T}(\mathscr{L})$ and $n$-ary relation symbol $R$,
3. $\neg\varphi$ for every $\varphi \in \mathscr{F}(\mathscr{L})$,
4. $(\varphi \vee \psi)$ for every $\varphi, \psi \in \mathscr{F}(\mathscr{L})$,
5. $(\exists x \varphi)$ for every $\varphi \in \mathscr{F}(\mathscr{L})$ and variable symbol $x$.

An element of $\mathscr{F}(\mathscr{T})$ is called an **$\mathscr{L}$-formula**. We call an $\mathscr{L}$-formula $\varphi$ **quantifier-free** if the symbol $\exists$ does not appear in $\varphi$, **atomic** if $\varphi$ is quantifier-free and further the symbols $\neg$ and $\vee$ do not appear in $\varphi$. We say variable $x$ is **free** in an $\mathscr{L}$-formula $\varphi$ if it appears in $\varphi$ outside the scope of the quantifier $\exists x$. A $\mathscr{L}$-formula $\varphi$ is called a **$\mathscr{L}$-sentence** if no variable symbol is free in $\varphi$.

**Notation 2.3.** *Let $\varphi, \psi$ be $\mathscr{L}$-formulas and let $x$ be a variable symbol. We write $(\forall x \varphi)$ for $\neg\exists x \neg\varphi$, we write $(\varphi \wedge \psi)$ for $\neg(\neg\varphi \vee \neg\psi)$, and we write $(\varphi \to \psi)$ for $(\neg\varphi \vee \psi)$. We also write $(\varphi \leftrightarrow \psi)$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. We often say that $\varphi(x_1, \dots, x_n)$ is an $\mathscr{L}$-formula. This means that $\varphi$ is an $\mathscr{L}$-term and $x_1, \dots, x_n$ are variable symbols, and all variable symbols that are free in $\varphi$ are among the $x_1, \dots, x_n$.*

## 2.2 Semantics

**Definition 2.2.1** Let $\mathscr{L}$ be a language. An **$\mathscr{L}$-structure** is a pair $\mathscr{M} = (M, I)$ where $M$ is a set, and $I$ is a function with domain $S_{\text{func}} \cup S_{\text{rel}}$ such that
1. $I(f) : M^n \to M$ for all $f \in S_{\text{func}}$ with $\text{ar}(f) = n$,
2. $I(R) \subseteq M^n$ for all $R \in S_{\text{rel}}$ with $\text{ar}(R) = n$.

For $f \in S_{\text{func}}$ and $R \in S_{\text{rel}}$, we also write $f^{\mathscr{M}}$ for $I(f)$ and $R^{\mathscr{M}}$ for $I(R)$, and call these the **interpretations** of these symbol in $\mathscr{M}$. We refer to $M$ as the **universe** (or: domain, or: underlying set) of $\mathscr{M}$.

**Notation 2.4.** *Given a language $(S_{\text{func}} \cup S_{\text{rel}}, \text{ar})$, we often simply write*

$$\mathscr{M} := \Big( (M, (f^{\mathscr{M}})_{f \in S_{\text{func}}}, (R^{\mathscr{M}})_{R \in S_{\text{rel}}} \Big)$$

*to define a new structure. For example, if $\mathscr{L} = \{f, R\}$, where $f$ is an $m$-ary function symbol and $R$ is an $n$-ary relation symbol, we would define an $\mathscr{L}$-structure $\mathscr{M}$ by specifying a set $M$, an $m$-ary function $f^{\mathscr{M}} : M^m \to M$ and a subset $R \subseteq M^n$. We simply write $\mathscr{M} = (M, f^{\mathscr{M}}, R^{\mathscr{M}})$.*

■ **Example 2.2** 1. Let $\mathscr{L} = \{R, f\}$, where $R$ is 2-ary relation symbol and $f$ is binary function symbol. Then all of the following are $\mathscr{L}$-structures:

(a) $(\mathbb{R}, <, +)$,
(b) $(\mathbb{R}_{>0}, <, \cdot)$,
(c) $(\mathbb{Q}, <, \cdot)$,
(d) $(\mathbb{Z}, <, +)$,
(e) $(\mathbb{Z}, |, +)$.

2. $(\mathbb{R},<,+,-,0,1)$ is an $\mathscr{L}_{\mathrm{or}}$-structure.
3. A graph $(V,E)$, where $V$ is the set of vertices and $E$ is the edge relation, is an $\mathscr{L}_{\sim}$-structure. Note that an ordered set $(D,\prec)$ is also an $\mathscr{L}_{\sim}$-structure.
4. A set $X$ is an $\mathscr{L}_{\emptyset}$-structure.

■

Given the pair $\mathscr{M} := (M,I)$, we can easily recover the language $\mathscr{L}$. We call $\mathscr{L}$ the **language of $\mathscr{M}$**. Indeed, we often define structures with explicitly mentioning the language. We simply consider structures as a set $M$ together with functions $(f_j : M^{n_j} \to M)_{j\in J}$ and subsets $(R_k \subseteq M^{n_k})_{k\in K}$, and construct the appropriate language only when necessary. It is clear that up renaming of the function, relation and constant symbols this language is uniquely determined by the given structure.

**Definition 2.2.2** Let $\mathscr{L}$ be a language and let $\mathscr{M}$ be an $\mathscr{L}$-structure. For each $\mathscr{L}$-term $t(x_1,\ldots,x_m)$ we define a function $t^{\mathscr{M}} : M^m \to M$ recursively as follows:
1. $t^{\mathscr{M}}(a) = c^{\mathscr{M}}$ if $t$ is the constant symbol $c$ in $\mathscr{L}$,
2. $t^{\mathscr{M}}(a) = a_i$ if $t$ is just the variable symbol $x_i$,
3. $t^{\mathscr{M}}(a) = f^{\mathscr{M}}(t_1^{\mathscr{M}}(a),\ldots,t_n^{\mathscr{M}}(a))$ if $t$ is $f(t_1,\ldots,t_n)$ for some $n$-ary function symbol $f$ in $\mathscr{L}$ and $\mathscr{L}$-terms $t_1,\ldots t_n$.

We call $t^{\mathscr{M}}$ the **interpretation of $t$ in $\mathscr{M}$**.

■ **Example 2.3** The interpretation of a term depends very much an the choice of the structure. Let $t$ be the $\mathscr{L}_{\mathrm{g}}$-term $e\cdot y\cdot(((x\cdot y)\cdot x))^{-1}$, and let $\mathscr{M} := (\mathbb{R},+,x\mapsto -x,0)$. Then $t^{\mathscr{M}} : \mathbb{R}^2 \to \mathbb{R}$ maps $(a,b)\in\mathbb{R}^2$ to $-2a$. If $\mathscr{N} := (\mathbb{N},\cdot,x^2,8)$, then $t^{\mathscr{N}} : \mathbb{N}^2 \to \mathbb{N}$ maps $(a,b)$ to $8a^4b^3$. ■

■ **Example 2.4** Let $\mathscr{R} := (R,+,-,\cdot,0,1)$ be a ring considered as an $\mathscr{L}_{\mathrm{r}}$-structure. Let $t(x_1,\ldots,x_n)$ be an $\mathscr{L}_{\mathrm{r}}$-term. Then there is a polynomial $p(X_1,\ldots,X_n)\in\mathbb{Z}[X_1,\ldots,X_n]$ such that $p(a_1,\ldots,a_n) = t^{\mathscr{R}}(a_1,\ldots,a_n)$ for all $(a_1,\ldots,a_n)\in R$. ■

**Definition 2.2.3** Let $\mathscr{M}$ be an $\mathscr{L}$-structure. For every $\mathscr{L}$-formula $\varphi(x_1,\ldots,x_m)$ and every $a\in M^m$ we define $\mathscr{M}\models\varphi(a)$ recursively as follows: $\mathscr{M}\models\varphi(a)$ holds if and only if
1. $t_1^{\mathscr{M}}(a) = t_2^{\mathscr{M}}(a)$ and $\varphi$ is $t_1 = t_2$ for some $\mathscr{L}$-terms $t_1,t_2$,
2. $(t_1^{\mathscr{M}}(a),\ldots,t_n^{\mathscr{M}}(a))\in R^{\mathscr{M}}$ and $\varphi$ is $R(t_1,\ldots,t_n)$ for some $n$-ary relation symbol $R$ in $\mathscr{L}$ and some $\mathscr{L}$-terms $t_1,\ldots,t_n$,
3. $\mathscr{M}\not\models\psi(a)$ and $\varphi$ is $\neg\psi$ for some $\mathscr{L}$-formula $\psi$,
4. $\mathscr{M}\models\psi(a)$ or $\mathscr{M}\models\chi(a)$, and $\varphi$ is $(\psi\vee\chi)$ for some $\mathscr{L}$-formulas $\psi,\chi$,
5. there is $b\in M$ such that $\mathscr{M}\models\psi(a,b)$, and $\varphi$ is $\exists x\psi$ for some $\mathscr{L}$-formula $\varphi$.

We say $\varphi(a)$ **holds in $\mathscr{M}$** if $\mathscr{M}\models\varphi(a)$. If $\varphi$ is a $\mathscr{L}$-sentence, we say $\mathscr{M}$ **satisfies** (or: models) $\varphi$.

■ **Example 2.5** Let $\mathscr{L}$ be $\{+,\cdot,0,1\}$, where $+,\cdot$ are binary function symbols and $0,1$ are constant symbols. Consider the following five $\mathscr{L}$-structures:

1. $\mathscr{A}_1 := (\mathbb{N},+,\cdot,0,1)$,
2. $\mathscr{A}_2 := (\mathbb{Z},+,\cdot,0,1)$,
3. $\mathscr{A}_3 := (\mathbb{Q},+,\cdot,0,1)$,
4. $\mathscr{A}_4 := (\mathbb{R},+,\cdot,0,1)$,
5. $\mathscr{A}_5 := (\mathbb{C},+,\cdot,0,1)$.

Consider the following $\mathscr{L}$-sentences $\varphi_1, \dots, \varphi_4$:

$$\begin{aligned}
\varphi_1 &: \quad \forall x \exists y\, x + y = 0 \\
\varphi_2 &: \quad \forall x\, (x \neq 0) \to (\exists y\, x \cdot y = 1) \\
\varphi_3 &: \quad \exists x\, x \cdot x = 1 + 1 \\
\varphi_4 &: \quad \forall x \exists y\, y \cdot y = x
\end{aligned}$$

It is easy to check that $\mathscr{A}_{i+1} \models \varphi_i$, but $\mathscr{A}_i \not\models \varphi_i$ for $i = 1, \dots, 4$. ■

**Exercise 2.1** Let $\mathscr{L}$ be the language that consists of a binary function symbol $\mathsf{f}$. Consider the $\mathscr{L}$-structures $\mathscr{M} = (\mathbb{Z}, +)$ and $\mathscr{N} = (\mathbb{Z} \times \mathbb{Z}, +)$, where addition in $\mathscr{N}$ is defined coordinate-wise. Find an $\mathscr{L}$-sentence that is true in $\mathscr{M}$, but not in $\mathscr{N}$.

**Exercise 2.2** Let $\mathscr{L}$ be the language consisting of a binary relation symbol $<$ and a unary relation symbol $R$. Then there is an $\mathscr{L}$-sentence $\sigma$ such that

$$(\mathbb{R}, <, X) \models \sigma \text{ if and only if } X \text{ is finite}$$

for all $X \subseteq \mathbb{R}$.

**Definition 2.2.4** Let $\mathscr{M}, \mathscr{N}$ be $\mathscr{L}$-structures and let $\mu : M \to N$ be injective. We say $\mu$ is **$\mathscr{L}$-embedding** if

1. $\mu(c^{\mathscr{M}}) = c^{\mathscr{N}}$ for each constant symbol $c$ in $\mathscr{L}$,
2. $\mu(f^{\mathscr{M}}(a_1, \dots, a_n)) = f^{\mathscr{N}}(\mu(a_1), \dots, \mu(a_n))$ for every function symbol $f$ in $\mathscr{L}$ and $a_1, \dots, a_n \in M$,
3. $(a_1, \dots, a_n) \in R^{\mathscr{M}}$ if and only if $(\mu(a_1), \dots, \mu(a_n)) \in R^{\mathscr{N}}$, for each relation symbol $R$ in $\mathscr{L}$.

If $\mu$ is also surjective, we say $\mu$ is an **$\mathscr{L}$-isomorphism**. In this situation, we say that $\mathscr{M}$ and $\mathscr{N}$ are **$\mathscr{L}$-isomorphic**. If $M \subseteq N$ and the inclusion map is an $\mathscr{L}$-embedding, then we say that $\mathscr{M}$ is a **substructures** of $\mathscr{N}$.

■ **Example 2.6** The structures $(\mathbb{R}, <, +, 0)$ and $(\mathbb{R}_{>0}, <, \cdot, 1)$ are isomorphic with $x \mapsto e^x$. ■

■ **Example 2.7** Let $\mathbb{Q}^{\mathrm{rc}}$ be the set of all real algebraic numbers[1]. Then $(\mathbb{Q}, <, +, \cdot)$ is a substructure of $(\mathbb{Q}^{\mathrm{rc}}, <, +, \cdot)$, and both these structures are substructures of $(\mathbb{R}, <, +, \cdot)$. ■

**Lemma 2.2.1** Let $\mathscr{M}, \mathscr{N}$ be $\mathscr{L}$-structures, and let $\mu : \mathscr{M} \to \mathscr{N}$ be an $\mathscr{L}$-embedding. Then for every $\mathscr{L}$-term $t(x_1, \dots, x_n)$ and $a \in M^n$

$$\mu(t^{\mathscr{M}}(a)) = t^{\mathscr{N}}(\mu(a)).$$

*Proof.* Using induction on terms, we show that $\mu(t^{\mathscr{M}}(a)) = t^{\mathscr{N}}(\mu(a))$ for all $\mathscr{L}$-terms $t(x_1, \dots, x_n)$ and all $a \in M^n$. The base cases when $t$ is a constant symbol or a variable symbol, follow immediately from the definition of $\mathscr{L}$-homomorphism. Now suppose that $t(x)$ is of the form $f(t_1, \dots, t_n)$ for some

[1] A real number is **algebraic** if it is the solution of a non-trivial polynomial equations with rational coefficients.

$n$-ary function symbol $f$ and $\mathscr{L}$-terms $t_1,\ldots,t_n$. By the induction hypothesis, we have

$$\begin{aligned}\mu(t^{\mathscr{M}}(a)) &= \mu(f^{\mathscr{M}}(t_1^{\mathscr{M}}(a),\ldots,t_n^{\mathscr{M}}(a)) \\ &= f^{\mathscr{N}}(\mu(t_1^{\mathscr{M}}(a)),\ldots,\mu(t_n^{\mathscr{M}}(a))) \\ &= f^{\mathscr{N}}(t_1^{\mathscr{N}}(\mu(a)),\ldots,t_n^{\mathscr{N}}(\mu(a))) = t^{\mathscr{N}}(\mu(a)).\end{aligned}$$

■

**Proposition 2.2.2** Let $\mathscr{M},\mathscr{N}$ be $\mathscr{L}$-structures, and let $\mu:\mathscr{M}\to\mathscr{N}$ be an $\mathscr{L}$-embedding. Then for every quantifier-free $\mathscr{L}$-formula $\varphi(x)$ and every $a\in M^{|x|}$

$$\mathscr{M}\models\varphi(a) \text{ if and only if } \mathscr{N}\models\varphi(\mu(a)).$$

*Proof.* We proceed by induction on quantifier-free $\mathscr{L}$-formulas. The cases that $\varphi$ is of the form $t_1=t_2$ for some $\mathscr{L}$-terms or of the form $R(t_1,\ldots,t_n)$ for some $n$-ary relation symbol $R$ and $\mathscr{L}$-terms $t_1,\ldots,t_n$, follow easily from Lemma 2.2.1.
Suppose that $\varphi$ is $\neg\psi$ for some quantifier-free $\mathscr{L}$-formula $\psi$. Then by the induction hypothesis and the definition of $\models$,

$$\begin{aligned}\mathscr{M}\models\neg\psi(a) &\text{ if and only if } \mathscr{M}\not\models\psi(a) \\ &\text{ if and only if } \mathscr{N}\not\models\psi(\mu(a)) \text{ if and only if } \mathscr{N}\models\neg\psi(\mu(a)).\end{aligned}$$

The case that $\varphi$ is of the from $\psi\vee\chi$ for some quantifier-free $\mathscr{L}$-formulas $\psi,\chi$ can be handled similarly. ■

**Proposition 2.2.3** Let $\mathscr{M},\mathscr{N}$ be $\mathscr{L}$-structures, and let $\mu:\mathscr{M}\to\mathscr{N}$ be an $\mathscr{L}$-isomorphism. Then for every $\mathscr{L}$-formula $\varphi(x)$ and every $a\in M^{|x|}$

$$\mathscr{M}\models\varphi(a) \text{ if and only if } \mathscr{N}\models\varphi(\mu(a)).$$

*Proof.* We proceed by induction on $\mathscr{L}$-formulas. The base cases follow from Proposition 2.2.2, and the induction step for $\vee$ and $\neg$ follows by the same argument as in the proof of Proposition 2.2.2.
Suppose that $\varphi$ is $\exists x\ \psi(x,x_1,\ldots,x_n)$ for some $\mathscr{L}$-formula $\psi$. By induction, we have that for all $b\in M$ and all $a\in M^n$

$$\mathscr{M}\models\psi(b,a) \text{ if and only if } \mathscr{N}\models\psi(\mu(b),\mu(a)).$$

Since $\mu$ is bijective, we have that for all $a\in M^n$

$$\text{there is } b\in M \text{ such that } \mathscr{M}\models\psi(b,a) \text{ if and only if } \text{ there is } c\in N \text{ such that } \mathscr{N}\models\psi(c,\mu(a))$$

Thus $\mathscr{M}\models\exists x\psi(a)$ if and only if $\mathscr{N}\models\exists x\psi(\mu(a))$. ■

## 2.3 Definable sets

Let $\mathscr{L}$ be a language.

**Definition 2.3.1** Let $\mathscr{M}$ be an $\mathscr{L}$-structure with universe $M$. A set $X \subseteq M^m$ is **definable in $\mathscr{M}$** if there is an $\mathscr{L}$-formula $\varphi(x_1,\dots,x_m,y_1,\dots,y_n)$ and $b \in M^n$ such that

$$X = \{a \in M^m \ : \ \mathscr{M} \models \varphi(a,b)\}.$$

For $A \subseteq M$, we say $X$ is **$A$-definable in $\mathscr{M}$** if in the above definition the tuple $b$ can be chosen to be in $A^n$. A function $f : M^m \to M^n$ is said to be **$A$-definable in $\mathscr{M}$** if its graph is $A$-definable in $\mathscr{M}$.

We often drop the reference to $\mathscr{M}$ if it can be deduced from the context; that is often write definable instead of definable in $\mathscr{M}$. When we say **definable without parameters**, we mean $\emptyset$-definable. If $a \in M^m$ and $A \subseteq M$, we say $a$ is $A$-definable in $\mathscr{M}$ if $\{a\}$ is $A$-definable in $\mathscr{M}$.

■ **Example 2.8** Consider the structure $(\mathbb{R},+,\cdot)$. Then 0 is defined by the formula $x+x=x$. The order relation $<$, that is the set $\{(a,b) \in \mathbb{R}^2 \ : \ a<b\}$, is defined by

$$\exists z (z \neq 0 \wedge y = x + z^2).$$

Since 0 is $\emptyset$-definable, so is $<$. From this, we can see that all sets of the form

$$\{a \in \mathbb{R}^n \ : \ p(a)=0, q_1(a)>0, \dots q_k(a)>0\},$$

where $p,q_1,\dots,q_k \in \mathbb{R}[X_1,\dots,X_n]$, are definable, and so are finite unions of such sets. The later sets are called **semialgebraic**. ■

■ **Example 2.9** Consider a graph $(V,E)$ as an $\mathscr{L}_\sim$-structure. Then the set of isolated vertices is defined by the $\mathscr{L}_\sim$

$$\forall y \, \neg x \sim y.$$

A **$k$-dominant set** $S$ in a graph is a set of $k$ vertices such that all other vertices have a neighbor in $S$. The set of all $k$-tuples that form a $k$-dominant set is defined by the $\mathscr{L}_\sim$-formula

$$\forall y \bigvee_{i=1}^{k} (y = x_i \vee x \sim y_i))$$

■

■ **Example 2.10** Let $\mathscr{R}$ be the structure with universe $\mathbb{R}$ and language $\mathscr{L}$ that contains a binary relation symbol $<$ whose interpretation in $\mathscr{R}$ is usual order on $\mathbb{R}$. Let $\varphi(x,y)$ be an $\mathscr{L}$-formula. Then the set

$$\{a \in \mathbb{R} \ : \ \{b \in \mathbb{R} \ : \ \mathscr{R} \models \varphi(a,b)\} \text{ is open}\}$$

is definable by the $\mathscr{L}$-formula

$$\forall y \varphi(x,y) \to \Big(\exists z_1 \exists z_2 \; z_1 < y < z_2 \wedge \forall z_3 (z_1 < z_3 < z_2) \to \varphi(x,z_3)\Big)$$

■

**Exercise 2.3** Consider the structure $(\mathbb{R},<,+,\cdot,H)$, where $H:\mathbb{R}\to\mathbb{R}$ is arbitrary. Show that the set

$$\{a\in\mathbb{R} \ : \ H \text{ is differentiable at } a\}$$

is definable in this structure.

**Exercise 2.4** Let $c\in\mathbb{R}\setminus\{0\}$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_{n+1}-a_n=c$. Let $A=\{a_n \ : \ n\in\mathbb{N}\}$. View $A$ as a unary relation $A(-)$ over $\mathbb{R}$ and show that $\mathbb{Z}$ is definable in $(\mathbb{R},<,+,\cdot,A)$.

**Proposition 2.3.1** Let $\mathscr{M}$ be an $\mathscr{L}$-structure, let $X\subseteq M^m$ be $B$-definable, and let $\mu$ be an $\mathscr{L}$-automorphism of $\mathscr{M}$ such that $\mu(b)=b$ for all $b\in B$. Then $\mu(X)=X$.

*Proof.* Pick $b\in B^n$ and an $\mathscr{L}$-formula $\varphi(x,y)$ such that $X=\{c\in M^m \ : \ \mathscr{M}\models\varphi(c,b)\}$. Let $a\in M^m$. By Proposition 2.2.3 and since $\mu(b)=b$, we have

$$\mathscr{M}\models\varphi(\mu(a),b) \text{ if and only if } \mathscr{M}\models\varphi(\mu(a),\mu(b)) \text{ if and only if } \mathscr{M}\models\varphi(a,b).$$

Thus $\mu(a)\in X$ if and only if $a\in X$. It follows that $\mu(X)=X$. ■

■ **Example 2.11** Consider $(\mathbb{Z},+)$ and the automorphism $-:\mathbb{Z}\to\mathbb{Z}$ mapping $a$ to $-a$. It follows immediately that $\mathbb{N}$ is not $\emptyset$-definable. ■

■ **Example 2.12** Consider the complex field $(\mathbb{C},+,\cdot)$. We are going to show that $\mathbb{R}$ is not definable. Suppose $\mathbb{R}$ is using parameters from $B\subseteq\mathbb{C}$. We can assume $B$ is finite. Take $a\in\mathbb{R}, b\in\mathbb{C}\setminus\mathbb{R}$ that are algebraically independent over $B$. Then there is an automorphism $\mu:\mathbb{C}\to\mathbb{C}$ that fixes $B$ and maps $a$ to $b$ and $b$ to $a$. ■

**Exercise 2.5** Let $\mathscr{M}\models T_\infty$, and let $A\subseteq M$ be finite. Which subsets of $M$ are $A$-definable in $\mathscr{M}$?

**Exercise 2.6** Let $\mathscr{L}=\{P\}$ be a language where $P$ is a unary relation symbol. Consider the $\mathscr{L}$-structure $\mathscr{A}=(\mathbb{N},E)$, where $E$ is the set of even natural numbers. Let $D$ be the set of natural numbers divisible by 3. Prove that $D$ is not definable in $\mathscr{A}$, even when parameters are allowed.

**Definition 2.3.2** Let $\mathscr{L}'$ be a language such that $\mathscr{L}'\supseteq\mathscr{L}$. Let $\mathscr{M}'$ be an $\mathscr{L}'$-structure and $\mathscr{M}$ be an $\mathscr{L}$-structure on the same universe. We say that $\mathscr{M}'$ is an **expansion** of $\mathscr{M}$ (or: $\mathscr{M}$ is a **reduct** of $\mathscr{M}'$) if

1. $c^{\mathscr{M}'}=c^{\mathscr{M}}$ for each constant symbol $c$ in $\mathscr{L}$,
2. $f^{\mathscr{M}'}=f^{\mathscr{M}}$ for each function symbol $f$ in $\mathscr{L}$,
3. $R^{\mathscr{M}'}=R^{\mathscr{M}}$ for each relation symbol $R$ in $\mathscr{L}$.

If $f^{\mathscr{M}'}$ and $R^{\mathscr{M}'}$ are definable in $\mathscr{M}$ for every function symbol $f$ and relation symbol $R$ in $\mathscr{L}'\setminus\mathscr{L}$, we say that $\mathscr{M}'$ is an **expansion by definitions** of $\mathscr{M}$.

**Notation 2.5.** *We use … in the definition of a structure to indicate that we mean an expansion. For example, when we write $\mathscr{R}=(R,<,+,\ldots)$, we mean that $\mathscr{R}$ is an expansion of $(R,<,+)$*

■ **Example 2.13** 1. By Example 2.8 $(\mathbb{R},<,+,\cdot)$ is an expansion by definitions of $(\mathbb{R},+,\cdot)$.

2. The real exponential field $(\mathbb{R}, <, +, \cdot, \exp)$ is an expansion of $(\mathbb{R}, <, +)$. We will later see that it is not an expansion by definitions.

■

**Exercise 2.7** Let $\mathcal{M}'$ be an $\mathcal{L}'$-structure and $\mathcal{M}$ be an $\mathcal{L}$-structure such that $\mathcal{L}' \supseteq \mathcal{L}$ and $\mathcal{M}'$ is an expansions of $\mathcal{M}$ by definitions. Show that every subset of $M^n$ is definable in $\mathcal{M}'$ if and only it is definable in $\mathcal{M}$.

## 2.4 Theories

Let $\mathcal{L}$ be a language. We call a set of $\mathcal{L}$-sentences an **$\mathcal{L}$-theory**.

**Definition 2.4.1** Let $T$ be an $\mathcal{L}$-theory and let $\mathcal{M}$ be an $\mathcal{L}$-structure. We say $\mathcal{M}$ is a **model of $T$** (or: models $T$) if $\mathcal{M} \models \sigma$ for each $\sigma \in T$.
Let $\sigma$ be an $\mathcal{L}$-sentence. We say $\sigma$ is a **logical consequence** of $T$ (written: $T \models \sigma$) if every model $\mathcal{M}$ of $T$ satisfies $\mathcal{M} \models \sigma$.

**Definition 2.4.2** Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The **theory of $\mathcal{M}$** (written: $\mathrm{Th}(\mathcal{M})$) is the set of all $\mathcal{L}$-sentences $\sigma$ such that $\mathcal{M} \models \sigma$. Similarly, for a class $\mathcal{K}$ of $\mathcal{L}$-structures, the **theory of $\mathcal{K}$** (written: $\mathrm{Th}(\mathcal{K})$) is the set of all $\mathcal{L}$-sentences $\sigma$ such that $\mathcal{M} \models \sigma$ for all $\mathcal{M} \in \mathcal{K}$.
Let $\mathcal{N}$ be an $\mathcal{L}$-structure. Then we say $\mathcal{M}$ and $\mathcal{N}$ are **elementary equivalent** (written: $\mathcal{M} \equiv \mathcal{N}$) if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

We collect the following consequence of Proposition 2.2.3.

**Proposition 2.4.1** Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures. If $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, then they are elementary equivalent.

■ **Example 2.14** Every bijection between two $\mathcal{L}_\emptyset$-structures is an $\mathcal{L}_\emptyset$-isomorphism. Thus two $\mathcal{L}_\emptyset$-structures of the same cardinality are elementary equivalent. ■

■ **Example 2.15** We consider the empty language $\mathcal{L}_\emptyset$, and define for each $n \in \mathbb{N}$ the $\mathcal{L}_\emptyset$-sentence $\varphi_n$ given by

$$\exists x_1 \exists x_2 \dots \exists x_n \bigwedge_{i<j\leq n} x_i \neq x_j.$$

The **theory of infinite sets** $T_\infty$ is the $\mathcal{L}_\emptyset$-theory $\{\varphi_i \ : \ i \in \mathbb{N}\}$. Note that an $\mathcal{L}_\emptyset$-structures $\mathcal{M}$ is a model of this theory if and only if its universe is infinite. ■

■ **Example 2.16** Let $\mathcal{L}_<$ be the language of consisting of single binary relation symbol $<$. Let $T_{\mathrm{lo}}$ be the $\mathcal{L}_<$-theory consisting of the following $\mathcal{L}_<$-sentences:

$$\forall x \, \neg x < x$$
$$\forall x \forall y \forall z (x < y \wedge y < z) \to (x < z)$$
$$\forall x \forall y (x \neq y) \to (x < y \vee y < x)$$

An $\mathcal{L}_<$-structure $\mathcal{M} = (M, <^{\mathcal{M}})$ is a model of $T_{\mathrm{lo}}$ if and only if $<^{\mathcal{M}}$ is a (strict) linear order on $M$.

The theory obtained by adding

$$\forall x \forall y (x < y) \to (\exists z\, x < z \wedge z < y)$$
$$\forall x \exists y\, x < y$$
$$\forall x \exists y\, y < x$$

is called DLO. It is easy to check an $\mathscr{L}_<$-structure $\mathscr{M} = (M, <^{\mathscr{M}})$ is a model of DLO if and only if $<^{\mathscr{M}}$ is a dense linear order on $M$ without endpoints. ■

■ **Example 2.17** Let $\mathscr{L}_\mathrm{g} = \{\cdot, e\}$, where $\cdot$ is binary function symbol and $e$ is a constant symbol. The **theory of groups** $T_\mathrm{g}$ is the set of the following three $\mathscr{L}_\mathrm{g}$-sentences:

$$\forall x \forall y \forall z\, x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$\forall x\, (x \cdot e = x \wedge e \cdot x = x)$$
$$\forall x \exists y (x \cdot y = e \wedge y \cdot x = e).$$

Obviously an $\mathscr{L}_\mathrm{g}$-structure $\mathscr{M}$ is a group (in the traditional sense) if and only if $\mathscr{M}$ is a model of the theory of groups. We can also consider groups in the extended language $\mathscr{L}_\mathrm{gr} = \{\cdot, ()^{-1}, e\}$. In this language, we add the sentence

$$\forall x (x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e).$$

to get the theory of groups in this extended language $T_\mathrm{gr}$.
When considering abelian groups, it is convenient to use a different (although equivalent) language: let $\mathscr{L}_\mathrm{ab} = \{+, 0\}$, where $+$ is binary function symbol and 0 is a constant symbol. Let $T_\mathrm{ab}$ be the $\mathscr{L}_\mathrm{ab}$-theory consisting of $T_\mathrm{g}$ (translated into $\mathscr{L}_\mathrm{ab}$-sentences) and the axiom

$$\forall x \forall y (x + y = y + x).$$

It is clear that an $\mathscr{L}_\mathrm{ab}$-structure $\mathscr{M}$ is an abelian group (in the traditional sense) if and only if $\mathscr{M}$ is a model of $T_\mathrm{ab}$. ■

■ **Example 2.18** The **theory of fields** $T_\mathrm{fields}$ in the language $\mathscr{L}_\mathrm{r} = \{+, -, \cdot, 0, 1\}$ is the set consisting of the following $\mathscr{L}_\mathrm{r}$-sentences:

$$\forall x \forall y \forall z\, x + (y + z) = (x + y) + z$$
$$\forall x\, x + 0 = x$$
$$\forall x \forall y \forall z (x - y = z \leftrightarrow x = z + y)$$
$$\forall x \forall y\, x + y = y + x$$
$$\forall x\, x \cdot 0 = 0$$
$$\forall x \forall y \forall z\, x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$\forall x\, x \cdot 1 = x$$
$$\forall x\, (x \neq 0) \to \exists y\, x \cdot y = 1$$
$$\forall x \forall y\, x \cdot y = y \cdot x$$
$$\forall x \forall y \forall z\, x \cdot (y + z) = x \cdot y + x \cdot z$$

Clearly, an $\mathscr{L}_{\mathrm{r}}$-structure $\mathscr{M}$ is a field if and only if $\mathscr{M} \models T_{\text{fields}}$. For each $n \in \mathbb{N}$, consider the $\mathscr{L}_{\mathrm{r}}$-sentence $\varphi_n$ defined as

$$\forall y_0 \dots \forall y_{n-1} \exists x\, x^n + \sum_{i=0}^{n-1} y_i x^i = 0$$

Set ACF to be $T_{\text{fields}} \cup \{\varphi_n \ : \ n \in \mathbb{N}\}$, the **theory of algebraically closed fields**. We can also use this language to restrict the characteristic of the field. For $p \in \mathbb{N}$, let $\psi_p$ be the $\mathscr{L}_{\mathrm{r}}$-sentence

$$\forall x\ \underbrace{x + \cdots + x}_{p\text{-times}} = 0$$

For a prime number $p$, we define $\mathrm{ACF}_p$ to be the $\mathscr{L}_{\mathrm{r}}$-theory $\mathrm{ACF} \cup \{\psi_p\}$. We let $\mathrm{ACF}_0$ be the $\mathscr{L}_{\mathrm{r}}$-theory $\mathrm{ACF} \cup \{\neg \psi_p : \ p \in \mathbb{N}_{>0}\}$. ■

■ **Example 2.19** The **theory of ordered fields** $T_{\text{ofields}}$ is the set of $\mathscr{L}_{\mathrm{or}}$-sentences containing $T_{\mathrm{lo}} \cup T_{\text{fields}}$ and

$$\forall x \forall y \forall z (x < y \to x + z < y + z)$$
$$\forall x \forall y (0 < x \wedge 0 < y \to 0 < x \cdot y)$$

Again, it is clear that an $\mathscr{L}_{\mathrm{or}}$-structure $\mathscr{M} \models T_{\text{ofields}}$ if and only $\mathscr{M}$ is an ordered field. ■

■ **Example 2.20** Fix a field $K$, and $\mathscr{L}_K$ be the language $\mathscr{L}_{\mathrm{ab}}$ togehter with unary function symbols $\lambda_k$ for each $k \in K$. Let $T_{\mathrm{VS}}(K)$ be the union of $T_{\mathrm{ab}}$, $T_\infty$ and the set of the following $\mathscr{L}_K$-sentences: for all $k, \ell \in K$

$$\forall x\ \lambda_k(\lambda_\ell(x)) = \lambda_{k\ell}(x)$$
$$\forall x\ \lambda_k(x) + \lambda_\ell(x) = \lambda_{k+\ell}(x)$$
$$\forall x \forall y\ \lambda_k(x) + \lambda_k(y) = \lambda_k(x+y)$$
$$\forall x\ \lambda_1(x) = x.$$

We observe that $\mathscr{M} \models T_{\mathrm{VS}}(K)$ if and only if $\mathscr{M}$ is an infinite $K$-vector space Suppose $\mathscr{N} \models T_{\mathrm{VS}}(K)$ and let $\mathscr{M}$ be $\mathscr{L}_K$-structure such that $M \subseteq N$. It is easy to see that $\mathscr{M}$ is a substructure of $\mathscr{N}$ if and only if $\mathscr{M}$ is $K$-subspace of $\mathscr{N}$. Also note that $\mathscr{L}_K$-isomorphism between two model $T_{\mathrm{VS}}(K)$ is just a bijective $K$-linear map. ■

**Exercise 2.8** Let $\mathscr{L}$ be the language whose symbol is a unary function symbol $f$. For each $\mathscr{L}$-sentence $\sigma$, let $S(\sigma)$ be the set of all cardinalities of finite models of $\sigma$. (The cardinality of an $\mathscr{L}$-structure is the cardinality of its underlying set.) Give an $\mathscr{L}$-sentence $\sigma$ such that $S(\sigma)$ is the set of odd natural numbers.

**Definition 2.4.3** Let $T$ be an $\mathscr{L}$-theory. We say that $T$ is **satisfiable** (or: has a model) if there is an $\mathscr{L}$-structure $\mathscr{M}$ such that $\mathscr{M} \models T$.

**Definition 2.4.4** Let $T$ be an $\mathscr{L}$-theory. We say $T$ is **complete** if $T \models \sigma$ or $T \models \neg\sigma$ for every $\mathscr{L}$-sentence $\sigma$.

**Notation 2.6.** *It will be convenient to sometimes use the following abbreviation: if $\varphi(x_1,\dots,x_n)$ is a $\mathscr{L}$-formula, $\mathscr{M}$ is an $\mathscr{L}$-structure, and $T$ is an $\mathscr{L}$-theory, then we write $\mathscr{M} \models \varphi$ if $\mathscr{M} \models \forall x_1 \dots \forall x_n \varphi$, and write $T \models \varphi$ if $T \models \forall x_1 \dots \forall x_n \varphi$.*

■ **Example 2.21** The theory of groups $T_g$ is not complete. Consider the following sentence $\sigma$ in the language $\mathscr{L}$ of the theory of groups:

$$\forall x \forall y\, x \cdot y = y \cdot x.$$

Clearly, a model $\mathscr{M}$ of $T_g$ satisfies $\sigma$ if and only if $\mathscr{M}$ is abelian. Since some groups are abelian, while other are not, we have that $T_g \not\models \sigma$ and $T_g \not\models \neg\sigma$. ■

■ **Example 2.22** Let $\mathscr{L}$ be a language and $\mathscr{M}$ an $\mathscr{L}$-structure. Then the theory of $\mathscr{M}$ is complete. Simply because for every $\mathscr{L}$-sentence $\sigma$, we have that $\mathscr{M} \models \sigma$ or $\mathscr{M} \models \neg\sigma$. Thus either $\sigma \in \mathrm{Th}(\mathscr{M})$ or $\neg\sigma \in \mathrm{Th}(\mathscr{M})$. ■

**Lemma 2.4.2** Let $T$ be an $\mathscr{L}$-theory such that every two models of $T$ are elementary equivalent. Then $T$ is complete.

*Proof.* Let $\mathscr{M} \models T$ and let $\sigma$ be an $\mathscr{L}$-sentence. We show that $\mathscr{M} \models \sigma$ if and only if $T \models \sigma$. Completeness follows, since $\mathscr{M} \models \sigma$ or $\mathscr{M} \models \neg\sigma$. From the definitions, we directly have that $\mathscr{M} \models \sigma$ whenever $T \models \sigma$. So now suppose that $\mathscr{M} \models \sigma$. Let $\mathscr{N}$ be another model of $T$. Since $\mathscr{M}$ and $\mathscr{N}$ are elementary equivalent, then $\mathscr{N} \models \sigma$. Since $\mathscr{N}$ was arbitrary, we have that $\sigma$ holds in every model of $T$. Thus $T \models \sigma$. ■

**Definition 2.4.5** Let $\mathscr{L}, \mathscr{L}'$ be languages such that $\mathscr{L} \subseteq \mathscr{L}'$, let $T$ be an $\mathscr{L}$-theory and $T'$ be an $\mathscr{L}'$-theory such that $T \subseteq T'$. We say $T'$ is an **extension of $T$ by definitions** if

- for every constant $c \in \mathscr{L}' \setminus \mathscr{L}$ there is $\mathscr{L}$-formula $\varphi(x)$ such that
$$T' \models (x = c) \leftrightarrow \varphi(x)$$
- for every $n$-ary function symbol $f \in \mathscr{L}' \setminus \mathscr{L}$ there is $\mathscr{L}$-formula $\varphi(x_1,\dots,x_{n+1})$ such that
$$T' \models (f(x_1,\dots,x_n) = x_{n+1}) \leftrightarrow \varphi(x_1,\dots,x_{n+1})$$
- for every $n$-ary relation symbol $R \in \mathscr{L}' \setminus \mathscr{L}$ there is $\mathscr{L}$-formula $\varphi(x_1,\dots,x_n)$ such that
$$T' \models R(x_1,\dots,x_n) \leftrightarrow \varphi(x_1,\dots,x_n)$$

■ **Example 2.23** Recall Example 2.17. We have that $T_{gr}$ is an extension of $T_g$ by definitions, because by the uniqueness of inverses

$$T_{gr} \models (x_1^{-1} = x_2) \leftrightarrow (x_1 \cdot x_2 = e \wedge x_2 \cdot x_1 = e).$$

■

## 2.5 Compactness via ultrafilters

In this section, we will give the prove the compactness theorem.

**Theorem 2.5.1 — Compactness theorem.** Let $T$ be an $\mathscr{L}$-theory such that every finite subset of $T$ is satisfiable. Then $T$ is satisfiable.

The compactness theorem is a foundational result in mathematical logic and of crucial importance in model theory. In a first course in logic the compactness theorem is obtained as a rather easy consequence of Gödel completeness theorem, which states that a theory is satisfiable if and only if no contradiction can be derived from it. To deduce the compactness theorem, just observe that every proof is finite and hence only uses finitely many assumptions. Thus if a contradiction can be deduced from a theory, then there is finite subtheory with the same property. This is rather a syntactic proof based on the formalization of what a proof precisely is. Here we give a semantic, very model-theoretic proof based on ultrafilter.

**Definition 2.5.1** Let $I$ be a nonempty set. A **proper filter on $I$** is a subset $U$ of $\mathscr{P}(I)$ such that

1. $\emptyset \notin U$ and $I \in U$,
2. $A \cap B \in U$ for all $A, B \in U$,
3. $B \in U$ for all $A, B \subseteq I$ with $A \in U$ and $A \subseteq B$.

A proper filter $U$ on $I$ is an **ultrafilter** if for all $A \subseteq I$, either $A \in U$ or $I \setminus A \in U$.

■ **Example 2.24**

1. Let $I$ be a nonempty set, and let $a \in I$. Then the set of all the subsets of $I$ that contain $a$, is an ultrafilter. Such an ultrafilter is called a **principal** ultrafilter.
2. The set of all cofinite subset $\mathbb{N}$ is a proper filter on $\mathbb{N}$. We will later see that this can be extended on ultrafilter.

■

**Exercise 2.9** Let $U$ be an ultrafilter on $I$. Show that the following are equivalent:

1. There exists $A \in U$ such that for all $B \in U$ we have $A \subseteq B$.
2. There is a finite $A \in U$.
3. $U$ is a principal ultrafilter.

For the rest of this section, fix a nonempty set $I$ and an ultrafilter $U$ on $I$.

Because $\emptyset \notin U$, the *or* in the definition of an ultrafilter is exclusive. So for every $A \subseteq I$, we have that $A \in U$ if and only if $I \setminus A \notin U$. If $A, B \subseteq I$, it is easy to see that $A \cup B \in U$ if and only if $A \in U$ or $B \in U$. Indeed, one direction follows immediately from 3. in Definition 2.5.1. For the other direction, suppose that both $A$ and $B$ are not in $U$. Since $U$ is an ultrafilter, $I \setminus A$ and $I \setminus B$ are in $U$. Then by 2. in Definition 2.5.1, their intersection $(I \setminus A) \cap (I \setminus B)$ is in $U$ as well. However,

$$I \setminus ((I \setminus A) \cap (I \setminus B)) = A \cup B$$

and hence is not in $U$, because *or* in the definition of an ultrafilter is exclusive.

**Definition 2.5.2** Let $(M_i)_{i \in I}$ be a collection of sets. Let $a = (a(i))_{i \in I}, b = (b(i))_{i \in I} \in \prod_{i \in I} M_i$. We write $a =_U b$ if the set $\{i \,:\, a(i) = b(i)\}$ is in $U$.

**Lemma 2.5.2** Let $(M_i)_{i \in I}$ be collection of sets. Then $=_U$ is an equivalence relation on $\prod_{i \in I} M_i$.

*Proof.* Since $I \in U$, we have that $=_U$ is reflexive. Symmetry follows immediately from the definition of $=_U$. Finally we get transitivity from 3. in Definition 2.5.1. ■

Let $\mathscr{L}$ be a language. Let $I$ be a nonempty set and let $U$ be an ultrafilter on $I$. Let $(\mathscr{M}_i)_{i\in I}$ be a family of $\mathscr{L}$-structures. For each $i \in I$, we denote the domain of $\mathscr{M}_i$ by $M_i$. Finally, set $M$ to be the set of $=_U$-equivalence classes of $\prod_{i\in I} M_i$.

**Definition 2.5.3** We now define an $\mathscr{L}$-structure $\mathscr{M}$ on $M$ (called the **ultraproduct**, sometimes written as $\prod_U \mathscr{M}_i$), as follows:

1. for each constant symbol $c$ of $\mathscr{L}$, set
$$c^{\mathscr{M}} = (c^{\mathscr{M}_i})_{i\in I}/ =_U$$
2. for each $n$ and each $n$-ary function symbol $F$ of $\mathscr{L}$, set $F^{\mathscr{M}} : M^n \to M$ be the function given by
$$F^{\mathscr{M}}([a_1],\dots,[a_n]) = \left(F^{\mathscr{M}_i}(a_1(i),\dots,a_n(i))\right)_{i\in I}/ =_U$$
3. for each $n$ and each $n$-ary relation symbol $P$ of $\mathscr{L}$, set $P^{\mathscr{M}}$
$$\{(a_1,\dots,a_n) \in M^n \ : \ \{i \in I : (a_1(i),\dots,a_n(i)) \in P^{\mathscr{M}_i}\} \in U\}/ =_U \ .$$

Let $F$ be a function symbol of $\mathscr{L}$. We still need to argue why $F^{\mathscr{M}}$ is well-defined, that is the definition does not depend on the choice of the representatives of the equivalence class. To do so, let $a = (a_1,\dots,a_n), b = (b_1,\dots,b_n) \in (\prod_{i\in I} M_i)^n$ such that $a_j =_U b_j$ for $j = 1,\dots,n$. Observe that
$$\bigcap_{j=1}^{n} \{i \in I : a_j(i) = b_j(i)\} \in U.$$

This is contained in $\{i \in I \ : \ F^{\mathscr{M}_i}(a_1(i),\dots,a_n(i)) = F^{\mathscr{M}_i}(b_1(i),\dots,b_n(i))\}$. Hence the later set is in $U$ as well.

**Lemma 2.5.3** Let $t(x_1,\dots,x_n)$ be an $\mathscr{L}$-term and let $[a_1],\dots,[a_n] \in M$. Then
$$t^{\mathscr{M}}([a_1],\dots,[a_n]) = \left(t^{\mathscr{M}_i}(a_1(i),\dots,a_n(i))\right)_{i\in I}/ =_U \ .$$

*Proof.* We prove this by induction on terms. The cases of variables and constant symbols is immediate from the definitions. Now suppose that $t$ is of the form $f(t_1,\dots,t_m)$ for some $m$-ary function symbol in $\mathscr{L}$ and $\mathscr{L}$-terms $t_1,\dots,t_m$. By induction hypothesis, $\left(t_j^{\mathscr{M}_i}(a_1(i),\dots,a_n(i))\right)_{i\in I}$ is a representative of the equivalence class $t_j^{\mathscr{M}}([a_1],\dots,[a_n])$ for each $j = 1,\dots,n$. Thus
$$\begin{aligned} t^{\mathscr{M}}([a_1],\dots,[a_n]) &= f^{\mathscr{M}}(t_1^{\mathscr{M}}([a_1],\dots,[a_n]),\dots,t_m^{\mathscr{M}}([a_1],\dots,[a_n])) \\ &= \left(f^{\mathscr{M}_i}\left(t_1^{\mathscr{M}_i}(a_1(i),\dots,a_n(i)),\dots,t_m^{\mathscr{M}_i}(a_1(i),\dots,a_n(i))\right)\right)_{i\in I}/{=_U} \\ &= \left(t^{\mathscr{M}}(a_1(i),\dots,a_n(i))\right)_{i\in I}/{=_U} \end{aligned}$$

■

**Exercise 2.10** Let $P$ be an $n$-ary relation symbol in $\mathscr{L}$, and let $[a_1],\dots,[a_n] \in M$. Show

$$([a_1],\dots,[a_n]) \in P^{\mathscr{M}} \text{ if and only if } \{i \in I \;:\; (a_1(i),\dots,a_n(i)) \in P^{\mathscr{M}_i}\} \in U.$$

**Theorem 2.5.4 — Łos.** Let $\varphi(x_1,\dots,x_m)$ be an $\mathscr{L}$-formula and let $[a_1],\dots,[a_m] \in M$. Then

$$\mathscr{M} \models \varphi([a_1],\dots,[a_m]) \text{ if and only if } \{i \in I \;:\; \mathscr{M}_i \models \varphi(a_1(i),\dots,a_m(i))\} \in U.$$

*Proof.* We proceed by induction on formulas. First consider the base that $\varphi$ is of the form $R(t_1,\dots,t_n)$ for an $n$-ary $\mathscr{L}$-relation symbol $R$ and $\mathscr{L}$-terms $t_1,\dots,t_n$. Let $[a_1],\dots,[a_m] \in M$. For $j = 1,\dots,m$, let $b_j \in \prod_{i\in I} M_i$ be such that $b_j(i) = t_j^{\mathscr{M}_i}(a_1(i),\dots,a_n(i))$ for all $i \in I$. By Lemma 2.5.3, $[b_j] = t_j^{\mathscr{M}}([a_1],\dots,[a_m])$. Then using Exercise 2.10

$$\begin{aligned}
&\mathscr{M} \models R(t_1,\dots,t_n)([a_1],\dots,[a_m])\\
&\quad\text{if and only if } (t_1^{\mathscr{M}}([a_1],\dots,[a_m]),\dots,t_n^{\mathscr{M}}([a_1],\dots,[a_m])) \in R^{\mathscr{M}}\\
&\quad\text{if and only if } \{i \in I \;:\; (b_1(i),\dots,b_n(i)) \in R^{\mathscr{M}_i}\} \in U\\
&\quad\text{if and only if } \{i \in I \;:\; (t_1^{\mathscr{M}_i}(a_1(i),\dots,a_m(i)),\dots,t_n^{\mathscr{M}_i}(a_1(i),\dots,a_m(i))) \in R^{\mathscr{M}_i}\} \in U\\
&\quad\text{if and only if } \{i \in I \;:\; \mathscr{M}_i \models R(t_1,\dots,t_n)(a_1(i),\dots,a_m(i))\} \in U
\end{aligned}$$

The case that $\varphi$ is of the form $t_1 = t_2$ for some $\mathscr{L}$-terms $t_1, t_2$, can be handled the same way. For the first induction step, suppose that $\varphi$ is of the form $\neg\psi$ for some $\mathscr{L}$-formula $\psi$. Then using the induction hypothesis and the fact that $U$ is an ultrafilter, we get

$$\begin{aligned}
&\mathscr{M} \models \neg\psi([a_1],\dots,[a_m])\\
&\quad\text{if and only if } \mathscr{M} \not\models \psi([a_1],\dots,[a_m])\\
&\quad\text{if and only if } \{i \in I : \mathscr{M}_i \models \psi(a_1(i),\dots,a_m(i))\} \notin U\\
&\quad\text{if and only if } \{i \in I : \mathscr{M}_i \not\models \psi(a_1(i),\dots,a_m(i))\} \in U\\
&\quad\text{if and only if } \{i \in I : \mathscr{M}_i \models \neg\psi(a_1(i),\dots,a_m(i))\} \in U.
\end{aligned}$$

Now suppose that $\varphi$ is of the form $(\psi \vee \chi)$ for some $\mathscr{L}$-formula $\psi$ and $\chi$. Using the fact that $A \cup B \in U$ if and only if either $A \in U$ or $B \in U$, we obtain

$$\begin{aligned}
&\mathscr{M} \models (\psi \vee \chi)([a_1],\dots,[a_m])\\
&\quad\text{if and only if } \{i \in I : \mathscr{M}_i \models \psi(a_1(i),\dots,a_m(i))\} \in U \text{ or } \{i \in I : \mathscr{M}_i \models \chi(a_1(i),\dots,a_m(i))\} \in U\\
&\quad\text{if and only if } \{i \in I : \mathscr{M}_i \models \psi(a_1(i),\dots,a_m(i))\} \cup \{i \in I : \mathscr{M}_i \models \chi(a_1(i),\dots,a_m(i))\} \in U\\
&\quad\text{if and only if } \{i \in I : \mathscr{M}_i \models (\psi \vee \chi)(a_1(i),\dots,a_m(i))\} \in U
\end{aligned}$$

Finally consider the case that $\varphi$ is of the form $\exists y\ \psi$ for some $\mathscr{L}$-formula $\psi$. We have

$$\begin{aligned}
&\mathscr{M} \models (\exists y\ \psi)([a_1],\dots,[a_m])\\
&\quad\text{if and only if there is } b \in \prod_{i\in I} M_i \text{ s.t. } \mathscr{M} \models \psi([b],[a_1],\dots,[a_m])\\
&\quad\text{if and only if there is } b \in \prod_{i\in I} M_i \text{ s.t. } \{i \in I \;:\; \mathscr{M}_i \models \psi(b(i),a_1(i),\dots,a_m(i))\} \in U\\
&\quad\text{if and only if } \{i \in I \;:\; \mathscr{M}_i \models (\exists y\psi)(a_1(i),\dots,a_m(i))\} \in U
\end{aligned}$$

■

**Corollary 2.5.5** Let $\sigma$ be an $\mathscr{L}$-sentence. Then

$$\mathscr{M} \models \sigma \text{ if and only if } \{i \in I \ : \ \mathscr{M}_i \models \sigma\} \in U.$$

**Definition 2.5.4** Let $S$ be a subset of $\mathscr{P}(I)$. If $A_1 \cap \cdots \cap A_n \neq \emptyset$ for all $A_1, \ldots, A_n \in S$, we say that $S$ has **the finite intersection property**.

**Lemma 2.5.6** Let $S$ be a subset of $\mathscr{P}(I)$ with the finite intersection property. Then there is an ultrafilter $U$ on $I$ such that $S \subseteq U$.

*Proof.* We first show that every set $S$ with the finite intersection property can be extended to a filter. Set

$$\mathscr{F}(S) := \{A \subseteq I \ : \ \text{there is } n \in \mathbb{N},\ A_1, \ldots, A_n \in S \text{ s.t. } \bigcap_{i=1}^{n} A_i \subseteq A\}.$$

It is easy to check that $\mathscr{F}(S)$ is a filter, since $S$ has the finite intersection property.

We now extended this filter to an ultrafilter. By Zorn's lemma, there is a maximal filter $U$ extending $S$. This filter is an ultrafilter. Indeed, suppose there is $A \subseteq I$ such that $A \notin U$. Since $U$ is maximal, $U \cup \{A\}$ does not have finite intersection, because otherwise it could be extended to a filter extending $U$. Thus there are $B_1, \ldots, B_n \in U$ such that $B_1 \cap \cdots \cap B_n \cap A = \emptyset$. Hence $\bigcap_{i=1}^{n} B_i \subseteq I \setminus A$. These filters are closed under finite intersection, we have $I \setminus A \in U$. ■

**Exercise 2.11** Let $I$ be an infinite set. Let $F(I) = \{A \subseteq I : I \setminus A \text{ is finite}\}$. Show that $F(I)$ is a filter on $I$. This filter $F(I)$ is called the **Fréchet filter** on $I$. Note that by Lemma 2.5.6 this filter can be extended to an ultrafilter. Let $U$ be a nonprincipal ultrafilter on $I$. Show that $U$ contains $F(I)$.

We are now ready to prove the compactness theorem.

*Proof of Theorem 2.5.1.* Let $I$ be the set of all finite subsets of $T$. We construct an ultrafilter $U$ on $I$. For $\sigma \in T$, set $I_\sigma = \{i \in I : \sigma \in i\}$. Let $S$ be the subset of $\mathscr{P}(I)$ given by $\{I_\sigma \ : \ \sigma \in T\}$. Observe that for $\sigma_1, \ldots, \sigma_n \in T$

$$\{\sigma_1, \ldots, \sigma_n\} \in I_{\sigma_1} \cap \cdots \cap I_{\sigma_n},$$

Thus $S$ has the finite intersection property and hence there is an ultrafilter $U$ on $I$ such that $S \subseteq U$.

Now for each $i \in I$, pick an $\mathscr{L}$-structure $\mathscr{M}_i$ such that $\mathscr{M}_i \models i$. We will show that the ultraproduct $\prod_U \mathscr{M}_i$ is a model of $T$. Let $\sigma \in T$. Observe that $\mathscr{M}_i \models \sigma$ whenever $\sigma \in i$. Since $U$ is an ultrafilter and

$$\{i \in I \ : \mathscr{M}_i \models \sigma\} \supseteq I_\sigma,$$

we have that $\{i \in I \ : \mathscr{M}_i \models \sigma\} \in U$. Thus $\prod_U \mathscr{M}_i \models \sigma$ by Theorem 2.5.4. ■

### 2.5.1 Applications of Compactness

**Corollary 2.5.7** Let $T$ be an $\mathscr{L}$-theory and let $\sigma$ be an $\mathscr{L}$-sentence such that $T \models \sigma$. Then there is a finite subset $T_0 \subseteq T$ such that $T_0 \models \sigma$.

*Proof.* Observe that $T \cup \{\neg\sigma\}$ is not satisfiable. Hence by Theorem 2.5.1 there is finite subset $T_0 \subseteq T$ such that $T_0 \cup \{\neg\sigma\}$ is also not satisfiable. Thus $T_0 \models \sigma$. ■

**Corollary 2.5.8** Let $T$ be an $\mathscr{L}$-theory, let $\varphi(x_1,\ldots,x_n)$ be an $\mathscr{L}$-formula, and let $\Sigma$ be a collection of $\mathscr{L}$-formulas in variables $x_1,\ldots,x_n$ such that for every $\mathscr{M} \models T$ and $a \in M^n$ with $\mathscr{M} \models \varphi(a)$ there is $\psi \in \Sigma$ such that $\mathscr{M} \models \psi(a)$. Then there are finitely many $\psi_1 \ldots, \psi_m \in \Sigma$ such that

$$T \models \varphi \to (\bigvee_{i=1}^{m} \psi_i).$$

*Proof.* Define $\mathscr{L}_c$ to be the language $\mathscr{L}$ together with new constant symbols $c_1,\ldots,c_n$. Let $T_\varphi$ be the $\mathscr{L}_c$-theory defined as

$$T \cup \{\varphi(c_1,\ldots,c_n)\}.$$

Let $\Delta(c)$ be the set of all $\mathscr{L}_c$-formula $\{\neg\psi(c_1,\ldots,c_n) \,:\, \psi \in \Sigma\}$. By the assumptions on $T$ and $\Sigma$, we see that $T_\varphi \cup \Delta$ is not satisfiable. By Theorem 2.5.1, there is finite subsets $\Delta'$ of $\Delta$ such that $T_\varphi \cup \Delta'$ is not satisfiable. Let $\chi_1,\ldots,\chi_m$ be $\mathscr{L}$-formulas whose free variables are among the $x_1,\ldots,x_n$ such that

$$\Delta' = \{\chi_1(c),\ldots,\chi_m(c)\}.$$

Since $T_\varphi \cup \Delta'$ is not satisfiable,

$$T_\varphi \models \bigvee_{i=1}^{m} \neg\chi_i(c). \tag{2.1}$$

Let $\psi_1,\ldots,\psi_m \in \Sigma$ such that $\neg\psi_i$ is $\chi_i$ for $i = 1,\ldots,m$. It follows easily from (2.1) that $T \models \varphi \to (\bigvee_{i=1}^{m} \psi_i)$. ■

**Corollary 2.5.9** Let $T$ be an $\mathscr{L}$-theory with infinite models and let $\kappa$ be a cardinal. Then there is a model of $T$ of cardinality at least $\kappa$.

*Proof.* Define $\mathscr{L}_\kappa$ to be the language $\mathscr{L}$ together with constant symbols $c_\alpha$ for each $\alpha \in \kappa$. Let $T_\kappa$ be the $\mathscr{L}_\kappa$-theory defined as

$$T \cup \{c_\alpha \neq c_\beta \,:\, \alpha, \beta \in \kappa, \alpha \neq \beta\}.$$

A model of $T_\kappa$ is model of $T$ and has cardinality at least $\kappa$, because the interpretations of the constant symbols are all distinct. Thus it is left to show that $T_\kappa$ is satisfiable. By Theorem 2.5.1 it is sufficient to prove satisfiability of every finite subsets of $T_\kappa$. Let $T'$ be finite subset of $T$. Then we can assume that there are $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n \in \kappa$ such that

$$T' \subseteq T \cup \{c_{\alpha_i} \neq c_{\beta_i} \,:\, i \in \{1,\ldots,n\}\}.$$

Let $\mathscr{M}$ be an infinite model of $T$. Pick $2n+1$ distinct elements $a_1,\dots,a_n,b_1,\dots,b_n,c \in M$, and expand $\mathscr{M}$ to an $\mathscr{L}_\kappa$-structure $\mathscr{M}'$ by interpreting the constant symbols $c_{\alpha_i}$ as $a_i$, the constant symbols $c_{\beta_i}$ as $b_i$, and all other constant symbols as $c$. Since $a_i \neq b_i$ for $i = 1,\dots,n$, the resulting structure $\mathscr{M}'$ is a model of $T'$. ■

**Exercise 2.12** Let $\sigma$ be an $\mathscr{L}_{\mathrm{ab}}$-sentence that holds in all non-trivial torsion-free abelian groups. Show that there exists $N \in \mathbb{N}$ such that $\sigma$ is true in all groups $\mathbb{Z}/p\mathbb{Z}$ where $p$ is a prime number and $p > N$.

**Exercise 2.13** Let $\mathscr{R} = (\mathbb{R},<,+,-,\cdot,0,1)$ be the real ordered field. Show that there exists a non-archimedean ordered field $\mathscr{F}$ that elementarily equivalent to $\mathscr{R}$ (as $\mathscr{L}_{\mathrm{or}}$-structures). Recall that an ordered field $\mathscr{F}$ is **archimedean** if for every $a \in F$ there is $n \in \mathbb{N}$ such that

$$x < \underbrace{1+\cdots+1}_{n\text{-times}}.$$

# 3. Basic building blocks of model theory

## 3.1 Elementary maps

**Definition 3.1.1** Let $\mathscr{M}, \mathscr{N}$ be $\mathscr{L}$-structures, and let $f : A \to N$ be such that $A \subseteq M$. We say $f$ is **elementary** if for every $\mathscr{L}$-formula $\varphi(x_1, \ldots, x_n)$ and every $a_1, \ldots, a_n \in A$

$$\mathscr{M} \models \varphi(a_1, \ldots, a_n) \text{ if and only if } \mathscr{N} \models \varphi(f(a_1), \ldots, f(a_n)).$$

If $A = M$, we call $f$ an **elementary embedding**. If $\mathscr{M}$ is a substructure of $\mathscr{N}$, we say $\mathscr{M}$ is an **elementary substructure of $\mathscr{N}$** (written: $\mathscr{M} \preceq \mathscr{N}$) if the inclusion map is elementary.

It is immediate from the definition, that if $\mathscr{M}$ is an elementary substructure of $\mathscr{N}$, then $\mathscr{M}$ and $\mathscr{N}$ are elementary equivalent.

■ **Example 3.1**

1. $(\mathbb{Q}, <, +, \cdot)$ is a substructure of $(\mathbb{R}, <, +, \cdot)$, but not an elementary substructure. The two structures are not even elementary equivalent.
2. Note that $(\mathbb{N}_{>0}, <)$ is a substructure of $(\mathbb{N}, <)$. More is true: the map $x \mapsto x+1$ is an isomorphism between the two structures, and thus $(\mathbb{N}_{>0}, <)$ and $(\mathbb{N}, <)$ are elementary equivalent. However, consider the formula $\varphi(x)$ given by

$$\exists y\ y < x.$$

Then $(\mathbb{N}, <) \models \varphi(1)$, but $(\mathbb{N}_{>0}, <) \not\models \varphi(1)$.

■

**Definition 3.1.2** Let $\mathscr{L}$ be a language and let $\mathscr{M}$ be an $\mathscr{L}$-structure. For $A \subseteq M$ we denote by $\mathscr{L}(A)$ the language $\mathscr{L} \cup \{c_a \ : \ a \in A\}$; that is the language we obtain from $\mathscr{L}$ by adding constant symbols for every $a \in A$. The **elementary diagram of $A$ in $\mathscr{M}$** (written: $\mathrm{ED}_{\mathscr{M}}(A)$) is the $\mathscr{L}(A)$-theory

$$\{\varphi(c_{a_1}, \ldots, c_{a_n}) \ : \ \mathscr{M} \models \varphi(a_1, \ldots, a_n)\}.$$

If $A = M$, we write $\mathrm{ED}(\mathscr{M})$ for $\mathrm{ED}_{\mathscr{M}}(M)$.

Under the assumption of the previous definition, every $\mathscr{L}$-superstructure $\mathscr{N}$ of $\mathscr{M}$ (including $\mathscr{M}$ itself) can be expanded into $\mathscr{L}(A)$-structure. We can check easily that $\mathrm{ED}_{\mathscr{M}}(A)$ is the $\mathscr{L}(A)$-theory of $\mathscr{M}$. The following lemma follows easily from the definition of an elementary diagram.

**Lemma 3.1.1** Let $\mathscr{M}, \mathscr{N}$ be two $\mathscr{L}$-structures such that $\mathscr{M}$ is a substructure of $\mathscr{N}$. Then $\mathscr{M}$ is an elementary substructure of $\mathscr{N}$ if and only if $\mathscr{N} \models \mathrm{ED}(\mathscr{M})$.

**Theorem 3.1.2 — Tarski-Vaught test.** Let $\mathscr{M}, \mathscr{N}$ be two $\mathscr{L}$-structure such that $\mathscr{M}$ is a substructure of $\mathscr{N}$ and for every $\mathscr{L}$-formula $\varphi(x_0, x_1, \ldots, x_n)$ and every $(a_1, \ldots, a_n) \in M^n$ the following holds:

if there is $c \in N$ such that $\mathscr{N} \models \varphi(c, a_1, \ldots, a_n)$,
then there is $b \in M$ such that $\mathscr{N} \models \varphi(b, a_1, \ldots, a_n)$.

Then $\mathscr{M}$ is an elementary substructure of $\mathscr{N}$.

*Proof.* We use induction on formulas to show that for all $a \in M^n$, $\mathscr{M} \models \varphi(a)$ if and only if $\mathscr{N} \models \varphi(a)$.

We consider the base case that $\varphi$ is of the form $R(t_1, \ldots, t_m)$ for all $m$-ary relation symbol $R$ in $\mathscr{L}$ and $\mathscr{L}$-terms $t_1, \ldots, t_m$. By Lemma 2.2.1, $t^{\mathscr{M}}(a) = t^{\mathscr{N}}(a)$ for all $\mathscr{L}$-terms $t(x_1, \ldots, x_n)$ and $a \in M^n$. Since $\mathscr{M}$ is a substructure of $\mathscr{N}$, we also have that $R^{\mathscr{N}} \cap M^m = R^{\mathscr{M}}$. Thus for

$$\begin{aligned}
\mathscr{M} \models R(t_1, \ldots, t_m)(a) &\text{ if and only if } (t_1^{\mathscr{M}}(a), \ldots, t_m^{\mathscr{M}}(a)) \in R^{\mathscr{M}} \\
&\text{ if and only if } (t_1^{\mathscr{N}}(a), \ldots, t_m^{\mathscr{N}}(a)) \in R^{\mathscr{N}} \\
&\text{ if and only if } \mathscr{N} \models R(t_1, \ldots, t_m)(a)
\end{aligned}$$

The case when $\varphi$ is of the form $t_1 = t_2$ can be handled the same way.

Now we handle the induction case. Suppose that $\varphi$ is of the form $\neg\psi$ and that the induction hypothesis holds for $\psi$. Then

$$\begin{aligned}
\mathscr{M} \models \neg\psi(a) &\text{ if and only if } \mathscr{M} \not\models \psi(a) \\
&\text{ if and only if } \mathscr{N} \not\models \psi(a) \\
&\text{ if and only if } \mathscr{N} \models \neg\psi(a).
\end{aligned}$$

The case when $\varphi$ is of the form $(\psi \vee \chi)$ can be done similarly.

Finally, consider the case that $\varphi$ is of the form $\exists y \psi$. Then by our assumption on $\mathscr{M}$ and $\mathscr{N}$ and using the induction hypothesis on $\psi$, we have that

$$\begin{aligned}
\mathscr{M} \models (\exists y \psi)(a) &\text{ if and only if } \text{ there is } b \in M \text{ s.t. } \mathscr{M} \models \psi(b, a) \\
&\text{ if and only if } \text{ there is } b \in M \text{ s.t. } \mathscr{N} \models \psi(b, a) \\
&\text{ if and only if } \text{ there is } c \in N \text{ s.t. } \mathscr{N} \models \psi(c, a) \\
&\text{ if and only if } \mathscr{N} \models (\exists y \psi)(a).
\end{aligned}$$

■

Looking at the proof, we see that the Tarski-Vaught test states the following: if you want to use induction of formulas to show that a substructures is an elementary substructure, then you really just have to consider the case of a formula of the form $\exists y \psi$ and only one direction of the equivalence.

**Exercise 3.1** Let $\mathscr{M}, \mathscr{N}_1, \mathscr{N}_2$ be $\mathscr{L}$-structures and for $i = 1,2$, let $\iota_i : \mathscr{M} \to \mathscr{N}_i$ be an $\mathscr{L}$-elementary embedding. Then there is an $\mathscr{L}$-structure $\mathscr{N}_3$ and $\mathscr{L}$-elementary embeddings $\mu_1 : \mathscr{N}_1 \to \mathscr{N}_3$ and $\mu_2 : \mathscr{N}_2 \to \mathscr{N}_3$ such that $\mu_1 \circ \iota_1 = \mu_2 \circ \iota_2$.

**Theorem 3.1.3 — Downward Löwenheim-Skolem.** Let $\mathscr{N}$ be an $\mathscr{L}$-structure and let $\kappa$ be an infinite cardinal such that $|\mathscr{L}| \leq \kappa \leq |N|$. Then there is an elementary substructure of $\mathscr{N}$ of cardinality $\kappa$.

*Proof.* Inductively, we define a family $(M_i)_{i\in\mathbb{N}}$ of subsets of $N$ such that $M_i \subseteq M_j$ for $i < j$ as follows: Set $M_0$ to be a subset of $N$ of cardinality $\kappa$. Let $i \in \mathbb{N}$ and suppose that $M_i$ is already defined. For every $\mathscr{L}$-formula $\varphi(y, x_1, \dots, x_n)$ and all $a \in M_i^n$ such that

$$\mathscr{N} \models \exists y \varphi(y, a),$$

pick an element $n_{\varphi,a} \in N$ such that $\mathscr{N} \models \varphi(n_{\varphi,a}, a)$. Define

$$M_{i+1} := \{\varphi(n_{\varphi,a}, a) \ : a \in M_i,\ \varphi(y, x_1, \dots, x_n)\ \mathscr{L}\text{-formula s.t.} \mathscr{N} \models \exists y \varphi(y, a)\}.$$

It is easy to check that each $M_i$ has cardinality $\kappa$, because there are only at most $\kappa$-many $\mathscr{L}$-formulas. Set $M := \bigcup_{i\in\mathbb{N}} M_i$. Now $M$ has cardinality $\kappa$, because each $M_i$ has cardinality $\kappa$. The reader can check that the $M$ is closed under all the interpretations (in $\mathscr{N}$) of function symbols in $\mathscr{L}$. Thus $M$ can interpreted as a substructure $\mathscr{M}$ of $\mathscr{N}$. By our construction, it is clear that $\mathscr{M}$ and $\mathscr{N}$ satisfy the assumptions of the Tarski-Vaught test. Thus $\mathscr{M} \preceq \mathscr{N}$. ■

**Corollary 3.1.4** Let $T$ be an $\mathscr{L}$-theory with infinite models and let $\kappa$ be cardinal such that $\kappa \geq |\mathscr{L}|$. Then there is a model of $T$ are cardinality of $\kappa$.

*Proof.* By Corollary 2.5.9, there is a model $\mathscr{M}$ of $T$ of cardinality at least $\kappa$. If $\kappa < |M|$, then we can find an elementary substructure of $\mathscr{M}$ of cardinality $\kappa$ by Theorem 3.1.3. Such an elementary substructure is also a model of $T$. ■

**Theorem 3.1.5 — Upward Löwenheim-Skolem.** Let $\mathscr{M}$ be an infinite $\mathscr{L}$-structure and let $\kappa$ be an infinite cardinal such that $|\mathscr{L}| \leq \kappa$ and $|M| \leq \kappa$. Then there is $\mathscr{L}$-structure $\mathscr{N}$ of cardinality $\kappa$ such that there is an elementary embedding of $\mathscr{M}$ into $\mathscr{N}$.

*Proof.* Consider $\mathscr{M}$ as an $\mathscr{L}(M)$-structure. By Corollary 3.1.4, there is model of $\mathrm{ED}(\mathscr{M})$ of cardinality $\kappa$. Call this model $\mathscr{N}$. Let $\mu : M \to N$ map $a \in M$ to $c_a^{\mathscr{N}}$. Since $\mathscr{N} \models \mathrm{ED}(\mathscr{M})$, it follows immediately that $\mu$ is an elementary embedding. ■

**Definition 3.1.3** Let $T$ be an $\mathscr{L}$-theory. We say $T$ **has built-in Skolem functions** if for all

$\mathscr{L}$-formulas $\varphi(y, x_1, \ldots, x_n)$ there is an $n$-ary function symbol $f$ in $\mathscr{L}$ such that

$$T \models \forall x_1 \ldots \forall x_n \big((\exists y\, \varphi(y, x_1, \ldots, x_n)) \to \varphi(f(x_1, \ldots, x_n), x_1, \ldots, x_n)\big).$$

The following lemma is a direct consequence of the Tarski-Vaught test

**Lemma 3.1.6** Let $T$ be an $\mathscr{L}$-theory with built-in Skolem functions, and let $\mathscr{M}, \mathscr{N}$ be two models of $T$. If $\mathscr{M}$ is a substructure of $\mathscr{N}$, then $\mathscr{M} \preceq \mathscr{N}$.

**Definition 3.1.4** Let $(I, <)$ be a linear order and let $(\mathscr{M}_i \,:\, i \in I)$ be a collection of $\mathscr{L}$-structures. We say $(\mathscr{M}_i \,:\, i \in I)$ is a **chain** if $\mathscr{M}_i$ is a substructure of $\mathscr{M}_j$ whenever $i, j \in I$ and $i < j$. If $\mathscr{M}_i$ is an elementary substructure of $\mathscr{M}_j$ for $i, j \in I$ with $i < j$, we say that chain is an **elementary chain**.

Let $(\mathscr{M}_i \,:\, i \in I)$ be a chain of $\mathscr{L}$-structures. Since every $\mathscr{M}_i$ is a substructure of $\mathscr{M}_j$ for $i < j$, the set $\bigcup_{i \in I} M_i$ can be turned into $\mathscr{L}$-structure $\mathscr{M}$ such that

- $c^{\mathscr{M}} = c^{M_i}$ for all $i \in I$ and constant symbols $c$ in $\mathscr{L}$,
- $f^{\mathscr{M}}(a_1, \ldots, a_n) = f^{\mathscr{M}_i}(a_1, \ldots, a_n)$ for $a_1, \ldots, a_n \in M_i$ and $n$-ary function symbols $f$ in $\mathscr{L}$,
- $R^{\mathscr{M}} \cap M_i^n = R^{\mathscr{M}_i}$ for all $i \in I$ and $n$-ary relation symbols $R$ in $\mathscr{L}$.

We use $\bigcup_{i \in I} \mathscr{M}_i$ to denote this $\mathscr{L}$-structure.

**Proposition 3.1.7** Let $(I, <)$ be a linear order and let $(\mathscr{M}_i \,:\, i \in I)$ be an elementary chain. Then $\mathscr{M}_j$ an elementary substructure of $\bigcup_{i \in I} \mathscr{M}_i$ for $j \in I$.

*Proof.* We apply the Tarski-Vaught test. Let $\varphi(y, x_1, \ldots, x_n)$ be an $\mathscr{L}$-formula, $a \in M_j^n$ and $b \in \bigcup_{i \in I} M_i$ such that $\bigcup_{i \in I} \mathscr{M}_i \models \varphi(b, a_1, \ldots, a_n)$. Then there is $k \in I$ such that $b \in M_k$. We can reduce to the case that $k > j$. Since $\mathscr{M}_j \preceq \mathscr{M}_k$, there is $c \in M_j$ such that $\mathscr{M}_i \models \varphi(c, a_1, \ldots, a_n)$. ■

## 3.2 Categoricity and completeness

**Definition 3.2.1** Let $\kappa$ be an infinite cardinal and let $T$ be an $\mathscr{L}$-theory. We say that $T$ is **$\kappa$-categorical** if it has models of cardinality $\kappa$ and every two models of $T$ of cardinality $\kappa$ are $\mathscr{L}$-isomorphic.

■ **Example 3.2** 1. The theory of infinite sets in the empty language is $\kappa$-categorical for every $\kappa$.
2. Recall from algebra that two algebraically closed fields are $\mathscr{L}_r$-isomorphic if and only if they have the same characteristic and have the same transcendence degree over their prime field. Let $\kappa > \aleph_0$. It is easy to check that if an algebraically closed field has transcendence degree $\kappa$, then it is cardinality is also $\kappa$. Thus $\mathrm{ACF}_0$ and $\mathrm{ACF}_p$ for a prime $p$ are $\kappa$-categorical.

■

**Theorem 3.2.1 — Łos-Vaught test.** Let $T$ be an $\mathscr{L}$-theory with no finite models. If $T$ is $\kappa$-categorical for some infinite cardinal $\kappa$ with $\kappa \geq |\mathscr{L}|$, then $T$ is complete.

*Proof.* Suppose $T$ is not complete. Then there is an $\mathscr{L}$-sentence $\sigma$ such that both $T \cup \{\sigma\}$ and $T \cup \{\neg\sigma\}$ are satisfiable. Since every model of $T$ is infinite, so is every model of $T \cup \{\sigma\}$ and $T \cup \{\neg\sigma\}$. By Corollary 3.1.4, there are $\mathscr{L}$-structure $\mathscr{M}_1$ and $\mathscr{M}_2$ of cardinality $\kappa$ such that $\mathscr{M}_1 \models T \cup \{\sigma\}$ and $\mathscr{M}_2 \models T \cup \{\neg\sigma\}$. Since both $\mathscr{M}_1$ and $\mathscr{M}_2$ are models of $T$ of cardinality $\kappa$, they are $\mathscr{L}$-isomorphic by $\kappa$-categoricity of $T$. Thus by Proposition 2.2.3, $\mathscr{M}_1$ and $\mathscr{M}_2$ are elementary equivalent. This contradicts that $\mathscr{M}_1 \models \varphi$, but $\mathscr{M}_2 \not\models \varphi$. ■

**Corollary 3.2.2** All theories mentioned in Example 3.2 are complete.

The following exercise shows that the nonexistence of finite models is an important precondition for the Łoś-Vaught test.

**Exercise 3.2** Find a theory $T$ (in a suitable language $\mathscr{L}$) with a finite model that is $\kappa$-categorical for some infinite cardinal $\kappa \geq |\mathscr{L}|$, but fails to be complete.

### 3.2.1 An application to algebra

**Theorem 3.2.3 — First-order Lefschetz Principle.** Let $\sigma$ be a sentence in $\mathscr{L}_{\mathrm{r}}$. Then the following are equivalent:

1. $(\mathbb{C},+,-,\cdot,0,1) \models \sigma$,
2. $\mathrm{ACF}_0 \models \sigma$,
3. $\mathrm{ACF}_p \models \sigma$ for all large enough $p$,
4. $(\overline{\mathbb{F}_p},+,-,\cdot,0,1) \models \sigma$ for all large enough $p$.

*Proof.* The equivalence of (1) and (2) follows from the completeness of $\mathrm{ACF}_0$, while the equivalence of (3) and (4) follows from the the completeness of $\mathrm{ACF}_p$ for every prime $p$.We now establish the equivalence of (2) and (4).
Suppose $\mathrm{ACF}_0 \models \sigma$. By Corollary 2.5.7, there is finite subset $T$ of $\mathrm{ACF}_0$ such that $T \models \sigma$. Let $p$ be a prime such that

$$T \subseteq \mathrm{ACF} \cup \{\neg\forall x \underbrace{x+\cdots+x}_{q\text{-times}} = 0 \;:\; q \text{ is prime}, q < p\}.$$

Thus for every prime $r$ with $r \geq p$, we have that $\overline{\mathbb{F}_r} \models \sigma$.
Suppose $\mathrm{ACF}_0 \not\models \sigma$. Since $\mathrm{ACF}_0$ is complete by Corollary 3.2.2, we have that $\mathrm{ACF}_0 \models \neg\sigma$. As above, we can now argue that $\overline{\mathbb{F}_p} \models \neg\sigma$ for all large enough primes $p$. ■

**Theorem 3.2.4 — Ax-Grothendieck.** Every injective polynomial map $p : \mathbb{C}^n \to C^n$ is surjective.

*Proof.* Let $d,n \in \mathbb{N}$. We first observe that there is an $\mathscr{L}_{\mathrm{r}}$-sentence $\sigma_{n,d}$ such that $(\mathbb{C},+,-,\cdot,0,1) \models \sigma_{n,d}$ if and only every polynomial map of degree $d$ in $n$ variables that is injective, is also surjective. Thus by Theorem 3.2.3, it is enough to that $(\overline{\mathbb{F}_q},+,-,\cdot,0,1) \models \sigma_{n,d}$ for every prime $q$.
We first note that if $K$ is a finite field, then every injective map from $K^n$ to $K^n$ is surjective (whether or not it is polynomial). So now $p : \overline{\mathbb{F}_q}^n \to \overline{\mathbb{F}_q}^n$ be polynomial and injective. Suppose $b \in \overline{\mathbb{F}_q}^n \setminus p(\overline{\mathbb{F}_q}^n)$. Let $A$ be a finite subset of $\overline{\mathbb{F}_q}^n$ such that contains all coordinates of $b$ and all coefficients of $p$. Denote by $K$ the subfield of $\overline{\mathbb{F}_q}^n$ generated by $A$. Since finitely generated subfield of $\overline{\mathbb{F}_q}^n$ are finite, we known that $K$ is finite. However, since all coefficients of $p$ are in $K$, we have that $p(K^n) \subseteq K^n$. However, $b \in K^n \setminus f(K^n)$, contradicting the surjectivity of injective maps for finite fields. ■

## 3.3 Decidability

We say that language $\mathscr{L}$ is **computable** if there is an algorithm that decides whether or not a sequences of symbols is an $\mathscr{L}$-formula.

**Definition 3.3.1** Let $T$ be an $\mathscr{L}$-theory. We say $T$ is **decidable** if there is an algorithm that takes as an input an $\mathscr{L}$-sentence $\sigma$ and returns the truth value of $T \models \sigma$. We say $T$ is **computable** if there is an algorithm that takes as an input an $\mathscr{L}$-formula $\sigma$ and returns the truth value of $\sigma \in T$. We say $T$ is **computably enumerable** if there is an algorithm that enumerates of all elements of $T$; that is, there is an algorithm that takes as an input $n \in \mathbb{N}$ produces element of $\sigma_n \in T$, such that $T = \{\sigma_n \ : \ n \in \mathbb{N}\}$.

**Proposition 3.3.1** Let $\mathscr{L}$ be a computable language and let $T$ be a computable $\mathscr{L}$-theory. Then the $\mathscr{L}$-theory $\{\sigma \ : \ T \models \sigma\}$ is computably enumerable.

*Sketch of proof.* Enumerate all possible proofs. ■

**Theorem 3.3.2** Let $\mathscr{L}$ be a computable language and let $T$ be a complete computable $\mathscr{L}$-theory. Then $T$ is decidable.

*Proof.* We can reduce to the case that $T$ is satisfiable. Then the $\mathscr{L}$-theories $T_+ := \{\varphi \ : \ T \models \varphi\}$ and $T_- := \{\varphi \ : \ T \models \neg\varphi\}$ are disjoint. Since $T$ is complete, the union $T_+ \cup T_-$ is the set of all $\mathscr{L}$-sentences. By Proposition 3.3.1, both $T^+$ and $T_-$ are computably enumerable. For let $n \in \mathbb{N}$, let $\sigma_n^+$ be the output of the algorithm enumerating $T_+$ on input $n$, and let $\sigma_n^-$ be the output of the algorithm enumerating $T_-$ on input $n$. We now describe the algorithm to decide whether $T \models \sigma$ for a given $\mathscr{L}$-sentence $\sigma$. So let $\sigma$ be an $\mathscr{L}$-sentence. Since $T_+ \cup T_-$ is the set of all $\mathscr{L}$-sentences, there is $m \in \mathbb{N}$ such that $\sigma = \sigma_m^+$ or $\sigma = \sigma_m^-$. We can find this $m$ by a brute-force search; that is checking the $\sigma_m^+$ and $\sigma_m^-$'s until we find an $m$ with the desired property. If $\sigma = \sigma_m^+$, the algorithm returns *true*, otherwise it returns *false*. ■

**Corollary 3.3.3** All theories mentioned in Example 3.2 are decidable.

## 3.4 Types

**Notation 3.1.** *Let $p$ be a set of $\mathscr{L}$-formulas. When we write $p(x_1,\dots,x_n)$ for this set, we mean that all free variables occurring in the $\mathscr{L}$-formulas in $p$ are among $x_1,\dots,x_n$. We say that $p(x_1,\dots,x_n)$ is **satisfiable** if there is an $\mathscr{L}$-structure $\mathscr{M}$ and $a \in M^n$ such that $\mathscr{M} \models \varphi(a)$ for all $\varphi(x_1,\dots,x_n) \in p$.*

**Definition 3.4.1** Let $\mathscr{M}$ be an $\mathscr{L}$-structure, $A \subseteq M$ and $p(x_1,\dots,x_n)$ be a set of $\mathscr{L}(A)$-formulas. We call $p(x_1,\dots,x_n)$ a **type of $\mathscr{M}$ over $A$** if for all $k \in \mathbb{N}$ and $\varphi_1,\dots,\varphi_k \in p$ there is $a \in M^k$ such that $\mathscr{M} \models \bigwedge_{i=1}^k \varphi_i(a)$. We say a type $p(x_1,\dots,x_n)$ of $\mathscr{M}$ over $A$ is **complete** if for every $\mathscr{L}(A)$-formula $\varphi(x_1,\dots,x_n)$ either $\varphi \in p$ or $\neg\varphi \in p$. For each $n \in \mathbb{N}$, we denote the set of of all complete types $p(x_1,\dots,x_n)$ of $\mathscr{M}$ over $A$ by $S_n^{\mathscr{M}}(A)$.

By the compactness theorem, each type over $\mathscr{M}$ is satisfiable, although not necessarily in $\mathscr{M}$.

**Notation 3.2.** *Sometimes types that are not complete, are referred to as* ***partial types****. We almost always just say $p(x_1,\ldots,x_n)$ is type over $A$ (rather than $p(x_1,\ldots,x_n)$ is a type of $\mathscr{M}$ over $A$), because $\mathscr{M}$ can usually be determined from the context.*

■ **Example 3.3** Let $\mathscr{M} := (\mathbb{N},<,+,1)$ and let $p(x)$ be set of formulas containing for each $n \in \mathbb{N}$ the following formula:

$$\underbrace{1+\cdots+1}_{n} < x$$

Clearly, $p(x)$ is finitely satisfiable and hence a type of $\mathscr{M}$ over $\emptyset$. We will later see that this is a complete type. Let $q(x)$ be set of formulas $\varphi$ such that $\mathscr{M} \models \varphi(5)$ . Obviously, $q(x)$ is type of $\mathscr{M}$ as well. In this case it is clear that $q$ is complete, since either $\mathscr{M} \models \varphi(5)$ or $\mathscr{M} \models (\neg\varphi)(5)$. ■

**Definition 3.4.2** Let $\mathscr{M}$ be an $\mathscr{L}$-structure, $A \subseteq M$ and $p(x_1,\ldots,x_n)$ be type over $A$. We say $a \in M^n$ **realizes** $p(x_1,\ldots,x_n)$ if $\mathscr{M} \models \varphi(a)$ for all $\varphi(x_1,\ldots,x_n) \in p(x_1,\ldots,x_n)$. In this situation, we say that $p$ **is realized in** $\mathscr{M}$. We say $\mathscr{M}$ **omits** $p$ if $p$ is not realized in $\mathscr{M}$.

**Lemma 3.4.1** Let $\mathscr{M}$ be an $\mathscr{L}$-structure, $A \subseteq M$ and $p(x_1,\ldots,x_n)$ be type of $\mathscr{M}$ over $A$. Then there is an $\mathscr{L}$-structure $\mathscr{N}$ such that $\mathscr{M} \preceq \mathscr{N}$ and $p$ is realized in $\mathscr{N}$.

*Proof.* Let $q$ be $p \cup \mathrm{ED}(\mathscr{M})$. It is enough to show that $q$ is satisfiable. By compactness, it is enough to show $q$ is finitely satisfiable. Let $\varphi_1,\ldots,\varphi_k \in p$ and $\psi_1,\ldots,\psi_\ell \in \mathrm{ED}(M)$. Since $p$ is a type of $\mathscr{M}$, there is $a \in M^n$ such that

$$\mathscr{M} \models \bigwedge_{i=1}^{k} \varphi_i(a).$$

Since $\mathscr{M} \models \mathrm{ED}(\mathscr{M})$, we have $\mathscr{M} \models \bigwedge_{i=1}^{\ell} \psi_i \wedge \bigwedge_{i=1}^{k} \varphi_i(a)$. Thus is finitely satisfiable. ■

**Definition 3.4.3** Let $\mathscr{M}$ be an $\mathscr{L}$-structure, $A \subseteq M$, and let $a \in M^n$. The **type of $a$ in $\mathscr{M}$ over $A$** (written: $\mathrm{tp}^{\mathscr{N}}(a|A)$) is defined as

$$\{\varphi(x_1,\ldots,x_n) \ : \ \mathscr{M} \models \varphi(a)\}.$$

**Exercise 3.3** Let $\mathscr{M} \models T_\infty$ and let $A \subseteq M$. Show that $\mathrm{tp}^{\mathscr{M}}(b|A) = \mathrm{tp}^{\mathscr{M}}(c|A)$ for all $b,c \in M \setminus A$.

**Corollary 3.4.2** Let $\mathscr{M}$ be an $\mathscr{L}$-structure, $A \subseteq M$, and let $p(x_1,\ldots,x_n)$ be a set of $\mathscr{L}(A)$-formulas. Then $p$ is a complete type over $A$ if and only if there is elementary extension $\mathscr{N}$ of $\mathscr{M}$ and $a \in N^n$ such that $p = \mathrm{tp}^{\mathscr{N}}(a|A)$.

*Proof.* Suppose $p$ is a complete type. Then by Lemma 3.4.1 there is an elementary extension $\mathscr{N}$ of $\mathscr{M}$ such that $p$ is realized in $\mathscr{N}$. Let $a \in N^n$ be such an realisation. Then $p$ is a subset of $\mathrm{tp}^{\mathscr{N}}(a|A)$. Since $p$ is complete, the two types are equal. ■

**Definition 3.4.4** Let $T$ be an $\mathscr{L}$-theory and $p(x_1,\dots,x_n)$ be a set of $\mathscr{L}$-formulas. We call $p(x_1,\dots,x_n)$ a **$T$-type** if for all $k \in \mathbb{N}$ and $\varphi_1,\dots,\varphi_k \in p$ there is a model of $\mathscr{M}$ of $T$ such that $\mathscr{M} \models \exists x_1 \dots \exists x_n \bigwedge_{i=1}^{k} \varphi_i(x_1,\dots,x_n)$. We say $p(x_1,\dots,x_n)$ is **complete** if for every $\mathscr{L}$-formula $\varphi(x_1,\dots,x_n)$ either $\varphi \in p$ or $\neg\varphi \in p$. For each $n \in \mathbb{N}$, we denote the set of of all complete $T$-types $p(x_1,\dots,x_n)$ by $S_n(T)$.

**Proposition 3.4.3** Let $T$ be a $\mathscr{L}$-theory. Let $x_1,\dots,x_n$ be variables and let $\Sigma$ be a subset of the set of all $\mathscr{L}$-formulas of the form $\varphi(x_1,\dots,x_n)$ such that $\Sigma$ is closed under boolean operations. Then the following are equivalent:

1. for every $\mathscr{L}$-formula $\varphi$ there is $\psi \in \Sigma$ such that $T \models \varphi \leftrightarrow \psi$.
2. for $p,q \in S_n(T)$ with $p \neq q$, there is a formula $\psi \in \Sigma$ such that $\psi \in p$ and $\psi \notin q$.

*Proof.* Suppose (1) holds. Let $p,q \in S_n(T)$ with $p \neq q$. There is a $\mathscr{L}$-formula $\varphi(x_1,\dots,x_n)$ such that $\varphi \in p$ and $\varphi \notin q$. By (1), there is $\mathscr{L}$-formula $\psi(x_1,\dots,x_n) \in \Sigma$ such that $T \models \varphi \leftrightarrow \psi$. Thus $\psi \in p$ and $\psi \notin q$.

Suppose (2) holds. Consider an $\mathscr{L}$-formula $\varphi(x_1,\dots,x_n)$. Let $p \in S_n(T)$ be such that $\varphi \in p$. Then by (2) and Corollary 2.5.8 (applied to $\neg\varphi$ and $\{\neg\psi \ : \ \psi \in \Sigma \cap p\}$), there are $\psi_1,\dots,\psi_m \in p \cap \Sigma$ such that

$$T \models (\neg\varphi) \to (\neg\psi_1 \vee \dots \vee \neg\psi_m).$$

Since $\Sigma$ is closed under conjunction, there is $\chi_p \in p \cap \Sigma$ such that $T \models \neg\varphi \to \neg\chi_p$. Thus for each $p \in S_n(T)$ with $\varphi \in p$, we can find $\chi_p$ such that $T \models \chi_p \to \varphi$.

Let $\Delta$ be the set of all such $\chi_p$; that is

$$\Delta := \{\chi_p \ : \ p \in S_n(T), \varphi \in p\}.$$

By Corollary 2.5.8 (applied to $\varphi$ and $\Delta$), there are $\xi_1,\dots,\xi_\ell \in \Delta$ such that

$$T \models \varphi \to (\bigvee_{i=1}^{\ell} \xi_i).$$

Since $\xi_1,\dots,\xi_\ell \in \Delta$, we have that $T \models \varphi \leftrightarrow (\bigvee_{i=1}^{\ell} \xi_i)$. Because $\Sigma$ is closed under disjunctions, $\bigvee_{i=1}^{\ell} \xi_i \in \Sigma$. ■

## 3.5 Saturated models

**Definition 3.5.1** Let $\kappa$ be an infinite cardinal and let $\mathscr{M}$ be an $\mathscr{L}$-structure. We say that $\mathscr{M}$ is **$\kappa$-saturated** if every type $p(x_1,\dots,x_n)$ over $A$ is realized in $\mathscr{M}$ for all $A \subseteq M$ with $|A| < \kappa$ and every $n \in \mathbb{N}$.

**Exercise 3.4** Let $\kappa$ be cardinal. What are the $\kappa$-saturated models of $T_\infty$?

**Lemma 3.5.1** Let $\kappa$ be an infinite cardinal and let $\mathscr{M}$ be an $\mathscr{L}$-structure. Then there is an elementary extension $\mathscr{N}$ of $\mathscr{M}$ such that for every $A \subseteq M$ with $|A| \leq \kappa$ and every $p(x) \in S_n^{\mathscr{M}}(A)$, the type $p$ is realized in $\mathscr{N}$.

*Proof.* Let $\lambda$ be the cardinality of sets of $p(x) \in S_n^{\mathscr{M}}(A)$, where $|A| \leq \kappa$. Then let $(p_\alpha : \alpha < \lambda)$ be an enumeration of these types. By Lemma 3.4.1 and Proposition 3.1.7, we can construct a elementary chain $(\mathscr{M}_\alpha)_{\alpha \in \lambda}$ of elementary extensions of $\mathscr{M}$ such that

1. $\mathscr{M}_0 := \mathscr{M}$,
2. $\mathscr{M}_{\alpha+1} \succeq \mathscr{M}_\alpha$ and $\mathscr{M}_{\alpha+1}$ realizes $p_\alpha$, and
3. $\mathscr{M}_\alpha = \bigcup_{\beta<\alpha} \mathscr{M}_\beta$ for limit ordinal $\alpha$.

Set $\mathscr{N} := \bigcup_{\alpha<\lambda} \mathscr{M}_\alpha$. By Proposition 3.1.7, we have that $\mathscr{N} \succeq \mathscr{M}_\alpha$ for all $\alpha < \lambda$. Thus $\mathscr{N}$ realizes $p_\alpha$ for all $\alpha < \lambda$. ■

Careful analysis of the proof of Lemma 3.5.1 reveals that if $\kappa \geq |\mathscr{L}|$, then $\mathscr{N}$ can be constructed such that $|N| \leq |M|^\kappa$.

**Theorem 3.5.2** Let $\kappa$ be an uncountable cardinal and let $\mathscr{M}$ be an $\mathscr{L}$-structure. Then there is an $\mathscr{L}$-structure $\mathscr{N}$ such that $\mathscr{M} \preceq \mathscr{N}$ and $\mathscr{N}$ is $\kappa^+$-saturated.

*Proof.* By Lemma 3.5.1 (and Proposition 3.1.7) we can construct an elementary chain $(\mathscr{N}_\alpha)_{\alpha<\kappa^+}$ such that

1. $\mathscr{N}_0 := \mathscr{M}$,
2. $\mathscr{N}_{\alpha+1} \succeq \mathscr{N}_\alpha$, and for all $A \subseteq N_\alpha$ with $|A| \leq \kappa$ every type $p \in S_n^{\mathscr{N}_\alpha}(A)$ is realized in $\mathscr{N}_{\alpha+1}$,
3. $\mathscr{N}_\alpha = \bigcup_{\beta<\alpha} \mathscr{N}_\beta$

Now set $\mathscr{N} := \bigcup_{\alpha<\kappa^+} \mathscr{N}_\alpha$. By Proposition 3.1.7, we know that $\mathscr{N} \succeq \mathscr{N}_\alpha$ for all $\alpha < \kappa$. We now show that $\mathscr{N}$ is $\kappa^+$-saturated. Let $A \subseteq N$ be such that $|A| \leq \kappa$, and consider a type $p \in S_n^{\mathscr{N}}(A)$. Since successor cardinals are regular, there is $\alpha < \kappa^+$ such that $A \subseteq N_\alpha$. By construction $p$ is realized in $\mathscr{N}_{\alpha+1}$. Since $\mathscr{N}_{\alpha+1} \preceq \mathscr{N}$, we have that $p$ is realized in $\mathscr{N}$. ■

**Exercise 3.5** Let $\mathscr{M}$ be a $\kappa$-saturated $\mathscr{L}$-structure. Then every infinite definable subset of $M^n$ has cardinality at least $\kappa$.

**Exercise 3.6** Let $\mathscr{M}$ be an $\omega$-saturated $\mathscr{L}$-structure, and let $a, b \in M^n$ such that $\mathrm{tp}^M(a) = \mathrm{tp}^M(b)$. Let $a_1, \ldots, a_m \in M$. Show that there are $b_1, \ldots, b_m \in M$ such that

$$\mathrm{tp}^{\mathscr{M}}(a, a_1, \ldots, a_m) = \mathrm{tp}^{\mathscr{M}}(b, b_1, \ldots, b_m).$$

# 4. Quantifier elimination

**Definition 4.0.1** Let $T$ be an $\mathscr{L}$-theory. We say $T$ **has quantifier-elimination** (short: has QE) if for every $\mathscr{L}$-formula $\varphi$ there is a quantifier-free $\mathscr{L}$-formula $\psi$ such that

$$T \models \varphi \leftrightarrow \psi.$$

We say an $\mathscr{L}$-structure $\mathscr{M}$ has quantifier-elimination if $\mathrm{Th}(\mathscr{M})$ has quantifier-elimination.

When $\mathscr{L}$ has at least one constant symbol, we can assume that $\varphi$ and $\psi$ have the same free variables. Simply substitute a constant symbol for every free variable that appears in $\psi$, but not in $\varphi$.

**Exercise 4.1** Let $T$ be an $\mathscr{L}$-theory. Then there exists a language $\mathscr{L}' \supseteq \mathscr{L}$ and an $\mathscr{L}'$-theory $T'$ such that $T'$ is an extension of $T$ by definitions and $T'$ has quantifier-elimination.

**Definition 4.0.2** Let $\varphi(x_1,\ldots,x_n)$ be an $\mathscr{L}$-formula. We say $\varphi(x_1,\ldots,x_n)$ is **existential** if it is the form $\exists y_1 \ldots \exists y_m \psi(y_1,\ldots,y_m,x_1,\ldots,x_n)$ where $\psi$ is quantifier-free. If $\psi$ is a conjunction of atomic formulas and negations of atomic formulas, then we say $\varphi$ is **primitive existential**.

**Lemma 4.0.1** Let $T$ be an $\mathscr{L}$-theory such that for every primitive existential formula $\varphi$ there is a quantifier-free formula $\psi$ such that $T \models \varphi \leftrightarrow \psi$. Then $T$ has quantifier-elimination.

*Proof.* We prove this by induction on $\varphi$. The base case is immediate. For the induction step, suppose that for two $\mathscr{L}$-formulas $\varphi_1$ and $\varphi_2$ there are quantifier-free $\mathscr{L}$-formulas $\psi_1$ and $\psi_2$ such that $T \models \varphi_1 \leftrightarrow \psi_1$ and $T \models \varphi_2 \leftrightarrow \psi_2$. Then

$$T \models (\varphi_1 \vee \varphi_2) \leftrightarrow (\psi_1 \vee \psi_2) \text{ and } T \models (\neg\varphi_1) \leftrightarrow (\neg\psi_1).$$

Thus it is left to consider the case that $\varphi$ is of the form $\exists y \varphi_1$. Then

$$T \models (\exists y \varphi_1) \leftrightarrow (\exists y \psi_1).$$

Hence it is left to find a quantifier-free $\mathscr{L}$-formula $\psi$ such that $T \models (\exists y \psi_1) \leftrightarrow \psi$. Note that $\psi_1$ is logically equivalent to a $\mathscr{L}$-formula of the form $\bigvee_{i \in I} \chi_i$, where $I$ is finite and each $\chi_i$ is conjunction of atomic formulas and their negations. Thus

$$T \models (\exists y \psi_1) \leftrightarrow (\exists y \bigvee_{i \in I} \chi_i).$$

However, note that the $\mathscr{L}$-formulas $\exists y \bigvee_{i \in I} \chi_i$ and $\bigvee_{i \in I} \exists y \chi$ are logically equivalent. By our assumptions there are quantifier-free $\mathscr{L}$-formulas $\theta_i$ for $\in I$ such that $T \models \chi_i \leftrightarrow \theta_i$ for each $i \in I$.

$$T \models (\exists y \psi_1) \leftrightarrow (\bigvee_{i \in I} \theta_i).$$

■

**Proposition 4.0.2** The theory of infinite sets $T_\infty$ has quantifier-elimination.

*Proof.* By Lemma 4.0.1 we can assume that $\varphi(x_1, \ldots, x_n)$ is of the form

$$\exists y \bigwedge_{i \in I} y = x_i \bigwedge_{j \in J} y \neq x_j \wedge \chi(x_1, \ldots, x_n),$$

where $I, J \subseteq \{1, \ldots, n\}$ are finite sets and $\chi$ is a conjunction of atomic formulas and negations of atomic formulas. Suppose that $I$ is non-empty and let $i_0 \in I$. Then

$$T_\infty \models \varphi(x_1, \ldots, x_n) \leftrightarrow \Big( \bigwedge_{i \in I} x_{i_0} = x_i \bigwedge_{j \in J} x_{i_0} \neq x_j \wedge \chi(x_1, \ldots, x_n) \Big).$$

Thus we can assume that $\varphi$ is of the form

$$\exists y \bigwedge_{j \in J} y \neq x_j \wedge \chi(x_1, \ldots, x_n).$$

Since every model of $T_\infty$ is infinite,

$$T_\infty \models \varphi(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n).$$

■

**Definition 4.0.3** Let $\mathscr{M}$ be an $\mathscr{L}$-structure. We say $\mathscr{M}$ is **minimal** if every definable subset of $M$ is either finite or cofinite. A $\mathscr{L}$-theory $T$ is **strongly minimal** if every model of $T$ is minimal. We say $\mathscr{M}$ is **strongly minimal** if its theory is strongly minimal.

**Corollary 4.0.3** The theory of $T_\infty$ is strongly minimal.

*Proof.* Let $\mathscr{M}$ be an infinite set, and let $X \subseteq M$ be definable in $\mathscr{M}$. By Proposition 4.0.2, we can assume that there is a quantifier-free formula $\varphi(x, y_1, \ldots, y_m)$ and $b \in M^m$ such that

$$X = \{a \in M \ : \ \mathscr{M} \models \varphi(a, b)\}.$$

Since the set of finite or cofinite subsets of $M$ is closed under unions and complements, we may assume that $\varphi$ is an atomic formula. Thus we can assume that there is $i \in \{1, \ldots, m\}$ such that either $\varphi$ is $x = y_i$ or $x = x$. In the first case $X$ is a singleton and the second case $X$ is co-finite. ■

**Exercise 4.2** Let $\mathscr{M}$ be an $\mathscr{L}$-structure. A **definable family** $(X_b)_{b\in Y}$ is a family of subsets of $M^n$ such that $Y \subseteq M^m$ and there exists an $\mathscr{L}$-formula $\phi(x_1,\dots,x_n,y_1,\dots,y_m)$ such that

$$X_b = \{a \in M^n : \mathscr{M} \models \phi(a,b)\}.$$

We say that $\mathscr{M}$ is **uniformly finite** if for every definable family $(X_b)_{b\in Y}$ there exists an integer $n$ such that for every $b \in Y$, if $X_b$ is finite, then $|X_b| \le n$. Now suppose that $\mathscr{M}$ is strongly minimal. Show that $\mathscr{M}$ is uniformly finite.

**Exercise 4.3** Use Theorem 4.0.2 to show that that $S_n(T_\infty)$ is finite for each $n$, and for every $\mathscr{M} \models T_\infty$ and every countable $A \subseteq M$, the set $S_n^{\mathscr{M}}(A)$ is countable. A theory satisfying the later statement is called **$\omega$-stable**.

**Exercise 4.4** Let $\kappa$ be cardinal. Which models of $T_\infty$ are $\kappa$-saturated?

Quantifier-elimination does not imply completeness. We will later see that ACF has quantifier-elimination, but we already know it is not complete. However, the additional assumption of the following proposition is often satisfied in practice.

**Proposition 4.0.4** Let $T$ be an $\mathscr{L}$-theory with quantifier-elimination, and let $\mathscr{M} \models T$ such that for every model $\mathscr{N} \models T$ there is a $\mathscr{L}$-embedding of $\mathscr{M}$ into $\mathscr{N}$. Then $T$ is complete.

*Proof.* Let $\mathscr{N} \models T$, and let $\mu : \mathscr{M} \to \mathscr{N}$ be an $\mathscr{L}$-embedding. It is enough to show that $\mathscr{M}$ and $\mathscr{N}$ are elementary equivalent. For this, we prove that $\mu$ elementary. So let $\varphi(x_1,\dots,x_n)$ be an $\mathscr{L}$-formula and $a \in M^n$. Since $T$ has quantifier-elimination, we can assume that $\varphi$ is quantifier-free. By Proposition 2.2.2, we have that $\mathscr{M} \models \varphi(a)$ if and only $\mathscr{N} \models \varphi(a)$. Thus $\mu$ is elementary. ■

■ **Example 4.1** We already know that $T_\infty$ is complete, but now can also deduce this from Proposition 4.0.4. Simply observe that for every infinite set there is an injection of $\mathbb{N}$ into this infinite set. ■

**Proposition 4.0.5** Let $T$ be a decidable $\mathscr{L}$-theory with quantifier-elimination. Then there is an algorithm that given a $\mathscr{L}$-formula $\varphi$ outputs a quantifier-free $\mathscr{L}$-formula $\psi$ such that $T \models \varphi \leftrightarrow \psi$.

*Proof.* Take a computable enumeration $\psi_1, \psi_2, \dots$ of all quantifier-free $\mathscr{L}$-formulas. Since $T$ is decidable, we can check for each $i \in \mathbb{N}$ whether $T \models \varphi \leftrightarrow \psi_i$. By just check this for every $i \in \mathbb{N}$, we will eventually find an $i$ such that this holds, since $T$ has quantifier-elimination. We then return this $\psi_i$. ■

## 4.1 Back and forth

**Definition 4.1.1** Let $\mathscr{M}$ and $\mathscr{N}$ be two $\mathscr{L}$-structures, and let $A \subseteq M$ and $B \subseteq N$. We say a bijection $\iota : A \to B$ is a **partial isomorphism between $\mathscr{M}$ and $\mathscr{N}$** if

1. $c^{\mathscr{M}} = a$ if and only $c^{\mathscr{N}} = \iota(a)$ for each constant symbol $c$ in $\mathscr{L}$ and $a \in A$,
2. $f^{\mathscr{M}}(a_1,\dots,a_n) = a_{n+1}$ if and only $f^{\mathscr{N}}(\iota(a_1),\dots,\iota(a_n)) = \iota(a_{n+1})$ for every function symbol $f$ in $\mathscr{L}$ and $a_1,\dots,a_{n+1} \in A$
3. $(a_1,\dots,a_n) \in R^{\mathscr{M}}$ if and only if $(\iota(a_1),\dots,\iota(a_n)) \in R^{\mathscr{N}}$, for every relation symbol $R$ in $\mathscr{L}$ and every $a_1,\dots,a_n \in A$.

**Definition 4.1.2** Let $\mathscr{M}$ and $\mathscr{N}$ be two $\mathscr{L}$-structures and let $\mathscr{F}$ be a set of partial isomorphisms between $\mathscr{M}$ and $\mathscr{N}$. We say $\mathscr{F}$ is a **back-and-forth system** if

**back**: for every $\iota : A \to B$ in $\mathscr{F}$ and every $b \in N$ there is $\iota' : A' \to B'$ such that $\iota'$ extends $\iota$ and $b \in B'$.

**forth**: for every $\iota : A \to B$ in $\mathscr{F}$ and every $a \in M$ there is $\iota' : A' \to B'$ such that $\iota'$ extends $\iota$ and $a \in A'$.

■ **Example 4.2** Let $\mathscr{L}_\emptyset$ be the empty language and let $\mathscr{M}, \mathscr{N}$ be two infinite $\mathscr{L}$-structures (ie. two infinite sets). Let $\mathscr{F}$ be the set of bijections between finite subsets of $M$ and $N$ is a back-and-forth system. ■

**Lemma 4.1.1** Let $\mathscr{M}$ and $\mathscr{N}$ be $\mathscr{L}$-structures, let $\mathscr{F}$ be an back-and-forth system between $\mathscr{M}$ and $\mathscr{N}$, and let $t(x_1, \ldots, x_n)$ be an $\mathscr{L}$-term. If $\iota : A \to B$ is in $\mathscr{F}$, then for every $a_1, \ldots, a_{n+1} \in A$

$$t^{\mathscr{M}}(a_1, \ldots, a_n) = a_{n+1} \text{ if and only if } t^{\mathscr{N}}(\iota(a_1), \ldots, \iota(a_n)) = \iota(a_{n+1}).$$

*Proof.* The case of $t$ being a constant or being a variable symbol follows immediately from the definition of a partial isomorphism. So now let $t(x_1, \ldots, x_n)$ be of the form $f(t_1, \ldots, t_m)$ of some $m$-ary function symbol $f$ in $\mathscr{L}$ and $\mathscr{L}$-terms $t_1, \ldots, t_m$ for which the conclusion of the lemma holds. Let $a_1, \ldots, a_{n+1} \in A$ such that $t^{\mathscr{M}}(a_1, \ldots, a_n) = a_{n+1}$. Since $\mathscr{F}$ is an back-and-forth system, there is $\iota' : A' \to B'$ in $\mathscr{F}$ extending $\iota$ such that $\{t_1^{\mathscr{M}}(a_1, \ldots, a_n), \ldots, t_m^{\mathscr{M}}(a_1, \ldots, a_n)\} \subseteq A'$. Since $\iota'$ is a partial isomorphism,

$$f^{\mathscr{N}}(\iota'(t_1^{\mathscr{M}}(a_1, \ldots, a_n)), \ldots, \iota'(t_m^{\mathscr{M}}(a_1, \ldots, a_n)) = \iota(a_{n+1}).$$

By our induction hypothesis, we have that for $i \in \{1, \ldots, m\}$

$$t_i^{\mathscr{N}}(\iota(a_1), \ldots, \iota(a_n)) = \iota'(t_i^{\mathscr{M}}(a_1, \ldots, a_n)).$$

Thus

$$t^{\mathscr{N}}(\iota(a_1), \ldots, \iota(a_n)) = f^{\mathscr{N}}(\iota'(t_1^{\mathscr{M}}(a_1, \ldots, a_n)), \ldots, \iota'(t_m^{\mathscr{M}}(a_1, \ldots, a_n))) = \iota(a_{n+1}).$$

The other direction of the "if and only if" statement can be shown similarly. ■

**Theorem 4.1.2** Let $\mathscr{M}$ and $\mathscr{N}$ be $\mathscr{L}$-structures, and let $\mathscr{F}$ be an back-and-forth system between $\mathscr{M}$ and $\mathscr{N}$. Then each $\iota \in \mathscr{F}$ is elementary. In particular, if $\mathscr{F}$ is nonempty, then $\mathscr{M}$ and $\mathscr{N}$ are elementary equivalent.

*Proof.* Let $\varphi(x_1, \ldots, x_n)$ be an $\mathscr{L}$-formula. We show by induction on $\varphi$ that for all $\iota : A \to B$ in $\mathscr{F}$ and all $a_1, \ldots, a_n \in A$

$$\mathscr{M} \models \varphi(a_1, \ldots, a_n) \text{ if and only if } \mathscr{N} \models \varphi(\iota(a_1), \ldots, \iota(a_n)).$$

For the base case, consider that $\varphi$ is of the form $R(t_1, \ldots, t_m)$, where $R$ is an $m$-ary relation symbol in $\mathscr{L}$ and $t_1, \ldots, t_m$ are $\mathscr{L}$-terms. For $i = 1, \ldots, m$, set $c_i := t_i^{\mathscr{M}}(a_1, \ldots, a_m)$. Since $\mathscr{F}$ is a back-and-forth system, we can find $\iota' : A' \to B'$ extending $\iota$ such that $\{c_1, \ldots, c_m\} \subseteq A'$. By Lemma 4.1.1, we have for each $i = 1, \ldots, m$

$$\iota'(c_i) = t_i^{\mathscr{N}}(\iota(a_1), \ldots, \iota(a_n)).$$

Since $\iota'$ is a partial isomorphism, we have

$$R^{\mathscr{M}}(c_1,\dots,c_m) \text{ if and only if } R^{\mathscr{N}}(\iota'(c_1),\dots,\iota'(c_n)).$$

Combining the last two observations, we obtain $\mathscr{M} \models \varphi(a_1,\dots,a_n)$ if and only if $\mathscr{N} \models \varphi(\iota(a_1),\dots,\iota(a_n))$.

The induction steps for $\vee$ and $\neg$ are immediate. So we just consider the case that $\varphi$ is of the form $\exists y\psi$ for some $\mathscr{L}$-formula $\psi(y,x_1,\dots,x_n)$. Let $\iota : A \to B$ in $\mathscr{F}$ and let $a_1,\dots,a_n \in A$. Suppose that $\mathscr{M} \models \varphi(a_1,\dots,a_n)$. Then there is $c \in M$ such that $\mathscr{M} \models \psi(c,a_1,\dots,a_n)$. Using again that $\mathscr{F}$ is a back-and-forth system, we can find $\iota' : A' \to B'$ in $\mathscr{F}$ such that $c \in A'$. By the induction hypothesis, we obtain that

$$\mathscr{M} \models \psi(c,a_1,\dots,a_n) \text{ if and only if } \mathscr{N} \models \psi(\iota'(c),\iota(a_1),\dots,\iota(a_n)).$$

Thus $\mathscr{N} \models \exists y\psi(a_1,\dots,a_n)$. The other direction can be shown similarly. ■

### 4.1.1 An example: dense linear orders

**Corollary 4.1.3** Let $T$ be an $\mathscr{L}$-theory such that for every two models $\mathscr{M}$ and $\mathscr{N}$ of $T$ the set of partial isomorphisms between $\mathscr{M}$ and $\mathscr{N}$ with finite domain is a back-and-forth system. Then $T$ is complete and has quantifier-elimination.

*Proof.* Completeness follows immediately from Theorem 4.1.2. For quantifier-elimination, it is enough to show by Proposition 3.4.3 that for every $n \in \mathbb{N}$ and for every $p,q \in S_n(T)$ with $p \neq q$, there is quantifier-free formula $\psi$ such that $\psi \in p$ and $\psi \notin q$. Let $p,q \in S_n(T)$ and suppose that $\varphi \in p$ if and only $\varphi \in q$ for every quantifier-free $\mathscr{L}$-formula $\varphi(x_1,\dots,x_n)$. We need to show that $p = q$. By Lemma 3.4.1, there are models $\mathscr{M}$ and $\mathscr{N}$ of $T$, $a \in M^n$ and $b \in \mathbb{N}^n$ such that $a$ realizes $p$ and $b$ realizes $q$. Let $\iota : \{a_1,\dots,a_n\} \to \{b_1,\dots,b_n\}$ map $a_i$ to $b_i$ for $i = 1,\dots,n$. Since $a$ and $b$ satisfy the same quantifier-free formulas, $\iota$ is partial isomorphism between $\mathscr{M}$ and $\mathscr{N}$ with finite domain. Since such maps form a back-and-forth system by assumption, $\iota$ is elementary. Thus $\mathrm{tp}^{\mathscr{M}}(a) = \mathrm{tp}^{\mathscr{N}}(b)$ and so $p = q$. ■

■ **Example 4.3** The theory $T_\infty$ clearly satisfies the assumptions of Corollary 4.1.3. Thus we have new proof of Proposition 4.0.2 without eliminating quantifiers by hand. ■

**Exercise 4.5** Let $\mathscr{L}$ be the language consisting of a single unary relation symbol $P$. Let $T_{\infty,2}$ be the $\mathscr{L}$-theory stating that both the interpretation of $P$ and its complement are infinite.
1. Show that $T_{\infty,2}$ has quantifier-elimination.
2. Is $T_{\infty,2}$ strongly minimal?

**Theorem 4.1.4** The theory DLO is complete, and has quantifier-elimination.

*Proof.* Let $\mathscr{M} = (M,<_M), \mathscr{N} = (N,<_N) \models \mathrm{DLO}$. By Theorem 4.1.2 and Corollary 4.1.3, it is enough to show that the set of partial isomorphisms between $\mathscr{M}$ and $\mathscr{N}$ with finite domain is a back-and-forth system. Let $\iota : A \to B$ be an partial isomorphism between $\mathscr{M}$ and $\mathscr{N}$ such that $A$ and $B$ are finite. Let $n \in \mathbb{N}$ such that $|A| = n$. Let $a_1,\dots,a_n \in A$ such that $a_1 <_M a_2 <_M \cdots <_M a_n$ and $A = \{a_1,\dots,a_n\}$. Set $b_i = \iota(a_i)$ for $i = 1,\dots,n$. Since $\iota$ is a partial isomorphism, $b_1 <_N b_2 <_N$

$\cdots <_N b_n$.

We only argue the "forth" case as the "back" follows by the same arguments. Let $a \in M$. Without loss of generality, we can assume that $a \notin A$. Since $\mathcal{N} \models$ DLO, we can find $b \in B$ such that $b <_N b_i$ if and only if $a <_M a_i$. Indeed, if $a < a_1$, we pick some $b < b_1$, and if $a > a_n$, we pick some $b > b_n$. If $a_i < a < a_{i+1}$ for some $i \in \{1, \dots, n-1\}$, we use the density of $<_N$ to find a $b \in N$ with $b_i < b < b_{i+1}$. Now extend $\iota$ to $\iota' : A \cup \{a\} \to B \cup \{b\}$ by mapping $a$ to $b$. ■

**Definition 4.1.3** Let $\mathcal{L}$ be a language containing $\mathcal{L}_<$, and let $\mathcal{M}$ be $\mathcal{L}$-structures such that $\mathcal{M} \models$ DLO. We say that $\mathcal{M}$ is **o-minimal** if every subset of $M$ definable in $\mathcal{M}$ is a finite union of intervals and points. Let $T$ be an $\mathcal{L}$-theory such that DLO $\subseteq T$. We way $T$ is **o-minimal** if every model of $T$ is o-minimal.

Remarkably, if an $\mathcal{L}$-structure $\mathcal{M}$ is o-minimal, then it is $\mathcal{L}$-theory is o-minimal as well. We might give a proof of this non-trivial result later on.

We make a useful observation: let $\mathcal{M}$ be an expansion of a model $(M, <_M)$ of DLO. Then if $X \subseteq M$ is finite union of intervals and points, then so is $M \setminus X$. Thus the collection of sets that are finite unions of intervals and points, is closed under all unions and complements. Thus if $\mathcal{M}$ has quantifier-elimination, then in order to that $\mathcal{M}$ is o-minimal, it is enough to show that every subsets of $\mathcal{M}$ definable by an atomic formula is a finite union of intervals and points.

**Corollary 4.1.5** The theory DLO is o-minimal.

*Proof.* Let $\mathcal{M} = (M, <_M) \models$ DLO and let $X \subseteq M$ be definable in $\mathcal{M}$. By Theorem 4.1.4, there is a quantifier-free $\mathcal{L}_<$-formula $\varphi(x, y_1, \dots, y_m)$ and $b \in M^m$ such that

$$X := \{a \in M \ : \ \mathcal{M} \models \varphi(a, b)\}.$$

By the preceding remark, $\varphi$ is of the form $x_i < y$, $y < x_i$ or $y = x$. In the first two cases $X$ defines an interval and a singleton set in the third case. ■

**Exercise 4.6** Let $\mathcal{M} \models$ DLO and let $A \subseteq M$ be countable. Is $S_n^{\mathcal{M}}(A)$ countable as well?

**Exercise 4.7** Find a countable model of DLO that is $\aleph_0$-saturated. Is $(\mathbb{R}, <)$ $|\mathbb{R}|$-saturated?

**Exercise 4.8** Let $\mathcal{L}_{<,2}$ be the language of consisting of a binary relation symbol $<$ and a unary relation symbol $P$. Let DLO$_2$ be the $\mathcal{L}_{<,2}$-theory containing DLO such that the interpretation of $P$ is a dense and codense subset. Show that DLO$_2$ has quantifier-elimination. Is DLO$_2$ o-minimal?

### 4.1.2 An example: vector spaces

**Theorem 4.1.6** Let $K$ be a field. Then the theory of infinite $K$-vector spaces $T_{\mathrm{VS}}(K)$ is complete, decidable, and has quantifier-elimination.

*Proof.* Let $\mathscr{M}$ and $\mathscr{N}$ be two infinite dimensional $K$-vector space. We will first construct a back-and-forth system between $\mathscr{M}$ and $\mathscr{N}$. Let $\mathscr{F}$ be the set of partial isomorphisms $\iota : V \to W$ such that $V$ is a finite-dimensional $K$-linear subspace of $\mathscr{M}$ and is a finite-dimensional $W$ is $K$-linear subspace. Recall that $K$-linear subspaces of $\mathscr{M}$ are precisely the $\mathscr{L}_K$-substructures of $\mathscr{M}$ (same for $\mathscr{N}$). It is easy to see that all bijective linear maps between a subspace of $\mathscr{M}$ and a subsapce of $\mathscr{N}$ are in $\mathscr{F}$. We now show that $\mathscr{F}$ is a back-and-forth system.

Let $\iota : V \to W$ be in $\mathscr{F}$, and let $V$ and $W$ be finite-dimensional $K$-linear subspaces of $\mathscr{M}$ and $\mathscr{N}$ respectively. Let $a \in M \setminus V$. Since $V$ is a $K$-subspace, we know that $a$ is $K$-linearly independent over $V$. Since $W$ has finite dimension and $\mathscr{N}$ has infinite dimensional as $K$-vector spaces, there $b \in N$ such that $b$ is $K$-linearly independent over $W$. Now take $\iota'$ to be the linear map extending $\iota$ sending $a$ to $b$. This map is a bijection between the $K$-linear subspace of $\mathscr{M}$ generated by $V$ and $a$ and the $K$-linear subspace of $\mathscr{N}$ generated by $W$ and $b$. Thus $\iota'$ is in $\mathscr{F}$.

Completeness of $T_{\mathrm{VS}}(K)$ follows from Theorem 4.1.2. We now conclude that $T_{\mathrm{vs}}(K)$ has quantifier-elimination. Let $p, q \in S_n(T_{\mathrm{vs}}(K))$ and suppose that $\varphi \in p$ if and only if $\varphi \in q$ for every quantifier-free $\mathscr{L}_K$-formula $\varphi(x_1, \dots, x_n)$. By Lemma 3.4.1, there are $\mathscr{M}, \mathscr{N} \models T_{\mathrm{vs}}(K)$, $a = (a_1, \dots, a_n) \in M^n$ and $b = (b_1, \dots, b_n) \in N^n$ such that $a$ realizes $p$ and $b$ realizes $q$. By Theorem 3.5.2 we may assume that both $\mathscr{M}$ and $\mathscr{N}$ have infinite dimension. Let $\ell \in \{1, \dots, n\}$ be the maximal number such that there is a $K$-linear subset of $\{a_i \,:\, i \in \{1, \dots, n\}\}$ of cardinality $\ell$. Without loss of generality, we can assume that $a_1, \dots, a_\ell$ is $K$-linear independent. Thus for $i = l+1, \dots, n$, there is $k_i = (k_{i,1}, \dots, k_{i,\ell}) \in K^\ell$ such that

$$a_i = k_{i,1}a_1 + \cdots + k_{i,\ell}a_\ell. \tag{4.1}$$

Furthermore, for every $k' = (k'_1, \dots, k'_\ell) \in K^\ell$ with $k' \neq (0, \dots, 0)$

$$k'_1 a_1 + \cdots + k'_\ell a_\ell \neq 0. \tag{4.2}$$

Observe that the equalities in (4.1) and the inequalities in 4.2 can be expressed as quantifier-free $\mathscr{L}_K$-formulas. Since $a$ and $b$ satisfy the same quantifier-free $\mathscr{L}_K$-formulas, we have that for $i = \ell+1, \dots, n$

$$b_i = k_{i,1}b_1 + \cdots + k_{i,\ell}b_\ell \tag{4.3}$$

and for every for every $k' = (k'_1, \dots, k'_\ell) \in K^\ell$,

$$k'_1 b_1 + \cdots + k'_\ell b_\ell \neq 0.$$

Thus $b_1, \dots, b_\ell$ are $K$-linear independent. Let $V$ be $K$-linear subspace of $\mathscr{M}$ generated by $a_1, \dots, a_\ell$, and $W$ be the $K$-linear subspace of $\mathscr{N}$. Note the map sending $a_i$ to $b_i$ for $i = 1, \dots, \ell$ extends to a bijective linear map $\iota$ between $V$ and $W$. Since $\iota$ is linear, we get from (4.1) and (4.3) that $\iota(a_i) = b_i$ for $i = \ell+1, \dots, n$. Since $\iota$ is in back-and-forth system between $\mathscr{M}$ and $\mathscr{N}$, it is elementary by Theorem 4.1.2. Thus $\mathrm{tp}^M(a) = \mathrm{tp}^N(b)$, and so $p = q$. ■

**Corollary 4.1.7** Let $K$ be a field. Then $T_{\mathrm{VS}}(K)$ is strongly minimal.

Note that both $(\mathbb{R}, +, 0)$ and $(\mathbb{C}, +, 0)$ be expanding by definitions into $\mathscr{L}_{\mathbb{Q}}$, because divisible torsion-free groups are essential $\mathbb{Q}$-vector space. Thus we obtain the following corollary.

**Corollary 4.1.8** The $\mathscr{L}_{\text{ab}}$-structures $(\mathbb{R},+,0)$ and $(\mathbb{C},+,0)$ are elementarily equivalent, and their theory is decidable.

**Exercise 4.9** Show that for every $\mathscr{M} \models T_{\text{VS}}(K)$ and every countable $A \subseteq M$, the set $S_n^{\mathscr{M}}(A)$ is countable.

**Exercise 4.10** Let $\mathscr{L}_{\text{ab}}(U)$ be the language $\mathscr{L}_{\text{ab}}$ together with a new unary function symbol. Consider the $\mathscr{L}_{\text{ab}}(U)$-structure $(\mathbb{R},+,0,\mathbb{Q})$. Find a simple axiomatization of the theory of this structure. Does this theory have quantifier-elimination?

## 4.2 Embedding tests

**Definition 4.2.1** Let $\mathscr{M}$ be an $\mathscr{L}$-structure and $A \subseteq M$ be nonempty. Let $\langle A \rangle_{\mathscr{M}}$ denote the $\mathscr{L}$-structure whose universe is

$$\{t^{\mathscr{M}}(a_1,\ldots,a_n) \;:\; t(x_1,\ldots,x_n) \text{ is an } \mathscr{L}\text{-term, } a_1,\ldots,a_n \in A\}.$$

and such that
- $c^{\langle A \rangle_{\mathscr{M}}} = c^{\mathscr{M}}$ for all constant symbols in $\mathscr{L}$,
- for all $m$-ary functions symbols $f$ in $\mathscr{L}$, all $\mathscr{L}$-terms $t_1,\ldots,t_m$, and $a_1,\ldots,a_n \in A$

$$f^{\langle A \rangle_{\mathscr{M}}}(t_1^{\mathscr{M}}(a_1,\ldots,a_n),\ldots t_m^{\mathscr{M}}(a_1,\ldots,a_n)) = (f(t_1,\ldots,t_m))^{\mathscr{M}}(a_1,\ldots,a_n),$$

- $P^{\langle A \rangle_{\mathscr{M}}}$ is the restriction of $P^{\mathscr{M}}$ to the universe of $\langle A \rangle_{\mathscr{M}}$ for relation symbol $P$ in $\mathscr{L}$.

It is easy to check that $\langle A \rangle_{\mathscr{M}}$ is indeed an $\mathscr{L}$-structure. From the definition we also directly obtain the following lemma.

**Lemma 4.2.1** Let $\mathscr{M}$ be an $\mathscr{L}$-structure and $A \subseteq M$ be nonempty. Then $\langle A \rangle_{\mathscr{M}}$ is a substructure of $\mathscr{M}$.

Because of this observation, we call $\langle A \rangle_{\mathscr{M}}$ the **substructure of $\mathscr{M}$ generated by $A$**. If $\mathscr{N}$ is a $\mathscr{L}$-substructure of $\mathscr{M}$, we say $\mathscr{N}$ is **finitely generated** if there is a finite subset $A$ of $M$ such that $\mathscr{N} = \langle A \rangle_{\mathscr{M}}$.

**Definition 4.2.2** Let $\mathscr{M}$ and $\mathscr{N}$ be $\mathscr{L}$-structures. Let $\text{Sub}(\mathscr{M},\mathscr{N})$ be the set of all map $\iota : A \to N$ such that $A$ is a substructure of $\mathscr{M}$ and $\iota$ is an embedding of this substructure into $\mathscr{N}$. We let $\text{Sub}_0(\mathscr{M},\mathscr{N})$ be the subset of $\text{Sub}(\mathscr{M},\mathscr{N})$ of all $\iota : A \to N$ such that $A$ is finitely generated.

**Lemma 4.2.2** Let $\mathscr{M}$ and $\mathscr{N}$ be $\mathscr{L}$-structures, and let $a = (a_1,\ldots,a_n) \in M^n$ and $b = (b_1,\ldots,b_n) \in N^n$ such that there is a partial isomorphism $\iota : A \to B$ between $\mathscr{M}$ and $\mathscr{N}$ such that
1. $\langle\{a_1,\ldots,a_n\}\rangle_{\mathscr{M}} \subseteq A$ and $\langle\{b_1,\ldots,b_n\}\rangle_{\mathscr{N}} \subseteq B$, and
2. $\iota(a_i) = b_i$ for $i = 1,\ldots,n$.

Then for every quantifier-free $\mathscr{L}$-formula $\varphi(x_1,\ldots,x_n)$

$$\mathscr{M} \models \varphi(a) \text{ if and only if } \mathscr{N} \models \varphi(b).$$

*Proof.* We first show by induction on $\mathscr{L}$-terms that for all $\mathscr{L}$-term $t(x_1,\dots,x_n)$

$$\iota(t^M(a)) = t^{\mathscr{N}}(b). \tag{4.4}$$

The case of $t$ being a constant or being a variable symbol follows immediately from the definition of a partial isomorphism. So now let $t(x_1,\dots,x_n)$ be of the form $f(t_1,\dots,t_m)$ of some $m$-ary function symbol $f$ in $\mathscr{L}$ and $\mathscr{L}$-terms $t_1,\dots,t_m$ for which the equality in the statement of the lemma holds. Since $\iota$ is a partial isomorphism, we have

$$t^{\mathscr{N}}(b) = f^{\mathscr{N}}(t_1^{\mathscr{N}}(b),\dots,t_m^{\mathscr{N}}(b)) = f^{\mathscr{N}}(\iota(t_1^{\mathscr{M}}(a)),\dots,\iota(t_m^{\mathscr{M}}(a)) = \iota(t^{\mathscr{M}}(a)).$$

We now prove the lemma by induction on $\mathscr{L}$-formulas. For the base case, consider that $\varphi$ is of the form $R(t_1,\dots,t_m)$, where $R$ is an $m$-ary relation symbol in $\mathscr{L}$ and $t_1,\dots,t_m$ are $\mathscr{L}$-terms. By (4.4), we have that $\iota(t_i^{\mathscr{M}}(a)) = t_i^{\mathscr{N}}(b)$ for each $i = 1,\dots,m$. Since $\iota$ is a partial isomorphism, we have

$$R^{\mathscr{M}}(t_1^{\mathscr{M}}(a),\dots,t_m^{\mathscr{M}}(a)) \text{ if and only if } R^{\mathscr{N}}(t_1^{\mathscr{N}}(b),\dots,t_m^{\mathscr{N}}(b)).$$

The induction steps for $\vee$ and $\neg$ are immediate. ■

**Lemma 4.2.3** Let $\mathscr{M}$ and $\mathscr{N}$ be $\mathscr{L}$-structures, and let $a = (a_1,\dots,a_n) \in M^n$ and $b = (b_1,\dots,b_n) \in N^n$ such that for every quantifier-free $\mathscr{L}$-formula $\varphi(x_1,\dots,x_n)$

$$\mathscr{M} \models \varphi(a) \text{ if and only if } \mathscr{N} \models \varphi(b).$$

Then there is a partial isomorphism $\iota : \langle\{a_1,\dots,a_n\}\rangle_{\mathscr{M}} \to \langle\{b_1,\dots,b_n\}\rangle_{\mathscr{N}}$ such that $\iota(a_i) = b_i$ for $i = 1,\dots,n$.

*Proof.* For ease of notation, let $\mathscr{A}$ be the $\langle\{a_1,\dots,a_n\}\rangle_{\mathscr{M}}$ and let $\mathscr{B}$ be $\langle\{b_1,\dots,b_n\}\rangle_{\mathscr{N}}$. We now construct an partial isomorphism $\iota : A \to B$. Note for every element $u \in A$, there is an $\mathscr{L}$-term $t(x_1,\dots,x_n)$ such that $u = t^{\mathscr{M}}(a)$. We now define $\iota$ to be the map taking $t^{\mathscr{M}}(a)$ to $t^{\mathscr{N}}(b)$. We have to show that $\iota$ is well-defined and partial isomorphism.
For well-definedness, let $t(x_1,\dots,x_n), s(x_1,\dots,x_n)$ be two $\mathscr{L}$ such that $u = t^M(a) = s^M(a)$ for some $u \in M$. Since $a$ and $b$ satisfy the same quantifier-free $\mathscr{L}$-formulas, we have that $t^{\mathscr{N}}(b) = s^{\mathscr{N}}(b)$. Thus $\iota$ is well-defined.
We now prove that $\iota$ is a partial isomorphism. Since every element of $B$ is of the form $s^{\mathscr{N}}(b)$ for $\mathscr{L}$-term $s$, we already have that $\iota$ is surjective. For injectivity, suppose there are $u, v \in A$ such that $\iota(u) = \iota(v)$. Let $t(x_1,\dots,x_n), s(x_1,\dots,x_n)$ be two $\mathscr{L}$-terms such that $u = t^{\mathscr{M}}(a)$ and $v = s^{\mathscr{M}}(a)$. Since $\iota(u) = \iota(v)$, we get that $t^{\mathscr{N}}(b) = s^{\mathscr{N}}(b)$. Since $a$ and $b$ satisfy the same quantifier-free $\mathscr{L}$-formulas, we get $u = t^M(a) = s^M(a) = v$. Thus $\iota$ is injective.
Let $\varphi(x_1,\dots,x_m)$ be a quantifier-free $\mathscr{L}$-formula and let $t_1(x_1,\dots,x_n),\dots,t_m(x_1,\dots,x_n)$ be $\mathscr{L}$-terms. Since $a$ and $b$ satisfy the same quantifier-free $\mathscr{L}$-formulas,

$$\mathscr{M} \models \varphi(t_1,\dots,t_m)(a) \text{ if and only if } \mathscr{N} \models \varphi(t_1,\dots,t_m)(b).$$

It follows immediately that $\iota$ is a partial isomorphism. ■

**Proposition 4.2.4** Let $T$ be an $\mathscr{L}$-theory such that $\mathrm{Sub}_0(\mathscr{M},\mathscr{N})$ is empty or a back-and-forth system for every $\omega$-saturated models $\mathscr{M},\mathscr{N}$ of $T$. Then $T$ has quantifier-elimination.

*Proof.* Let $p,q \in S_n(T)$ and suppose that $\varphi \in p$ if and only if $\varphi \in p$ for every quantifier-free $\mathscr{L}$-formula $\varphi(x_1,\ldots,x_n)$. By Lemma 3.4.1, there are $\mathscr{M},\mathscr{N} \models T$, $a = (a_1,\ldots,a_n) \in M^n$ and $b = (b_1,\ldots,b_n) \in N^n$ such that $a$ realizes $p$ and $b$ realizes $q$. By Theorem 3.5.2 we can assume that $\mathscr{M}$ and $\mathscr{N}$ are $\omega$-saturated. By Lemma 4.2.3, there is a partial isomorphism $\iota : \langle\{a_1,\ldots,a_n\}\rangle_{\mathscr{M}} \to \langle\{b_1,\ldots,b_n\}\rangle_{\mathscr{N}}$ such that $\iota(a_i) = b_i$ for $i = 1,\ldots,n$. Since $\iota$ is in a back-and-forth system between $\mathscr{M}$ and $\mathscr{N}$, it is elementary by Theorem 4.1.2. Thus $\mathrm{tp}^M(a) = \mathrm{tp}^N(b)$, and so $p = q$. ■

**Theorem 4.2.5** Let $T$ be an $\mathscr{L}$-theory such that for every two models $\mathscr{M}$ and $\mathscr{N}$ of $T$, every $\iota \in \mathrm{Sub}_0(\mathscr{M},\mathscr{N})$ and every $a \in M$ there exists an elementary extension $\mathscr{N}'$ of $\mathscr{N}$ and $\iota' \in \mathrm{Sub}(\mathscr{M},\mathscr{N}')$ such that $a$ is in the domain of $\iota'$ and $\iota'$ extends $\iota$. Then $T$ has quantifier-elimination.

*Proof.* We show that the assumption of Proposition 4.2.4 is satisfied. Let $\mathscr{M},\mathscr{N}$ be $\omega$-saturated models of $T$. Let $\mathscr{A}$ be a finitely generated substructure of $\mathscr{M}$, let $\mathscr{B}$ be a finitely generated substructure of $\mathscr{N}$ and let $\iota : A \to B$ be a partial isomorphism between $\mathscr{M}$ and $\mathscr{N}$. Let $a_1,\ldots,a_n$ be generators of $\mathscr{A}$, and set $b_i := \iota(a_i)$ for $i = 1,\ldots,n$. Since $\iota$ is a partial isomorphism, $b_1,\ldots,b_n$ generates $\mathscr{B}$. Let $c \in \mathscr{M}$. By assumption there is an elementary extension $\mathscr{N}'$ of $\mathscr{N}$ and $\iota' \in \mathrm{Sub}(\mathscr{M},\mathscr{N}')$ extending $\iota$ such that $c$ in the domain of $\iota'$. By Lemma 4.2.2, for every quantifier-free $\mathscr{L}$-formula $\varphi(x_1,\ldots,x_n)$

$$\mathscr{M} \models \varphi(a_1,\ldots,a_n,c) \text{ if and only if } \mathscr{N}' \models \varphi(\iota(a_1),\ldots,\iota(a_n),\iota'(c)). \tag{4.5}$$

Let $p$ be the $\mathscr{L}$-type $\mathrm{tp}^{\mathscr{N}'}(\iota'(c)|\{b_1,\ldots,b_n\})$. Since $\mathscr{N}$ is $\omega$-saturated, this type is realized in $\mathscr{N}$. Let $d \in N$ be a realization of this type. By (4.5), for every quantifier-free $\mathscr{L}$-formula $\varphi(x_1,\ldots,x_n)$

$$\mathscr{M} \models \varphi(a_1,\ldots,a_n,c) \text{ if and only if } \mathscr{N}' \models \varphi(\iota(a_1),\ldots,\iota(a_n),d).$$

Thus there is a partial isomorphism $\iota''$ from $\langle\{a_1,\ldots,a_n,c\rangle_{\mathscr{M}}$ to $\langle\{b_1,\ldots,b_n,d\rangle_{\mathscr{M}}$ mapping $c$ to $d$ and $a_i$ to $b_i$ for every $i = 1,\ldots,n$. It is easy to check that $\iota''$ extends $\iota$. ■

**Corollary 4.2.6** Let $T$ be a $\mathscr{L}$-theory and let $\kappa$ be a cardinal such that $\kappa \geq |\mathscr{L}|$. Suppose for all models $\mathscr{M},\mathscr{N}$ of $T$ with $|M| \leq \kappa$ and $\mathscr{N}$ $\kappa^+$-saturated, for every $\iota \in \mathrm{Sub}(\mathscr{M},\mathscr{N})$ either the domain of $\iota$ is $M$ or $\iota$ has a proper extension $\iota' \in \mathrm{Sub}(\mathscr{M},\mathscr{N})$. Then $T$ has quantifier-elimination.

*Proof.* We show that the assumptions of Theorem 4.2.5 are satisfied. Let $\mathscr{M},\mathscr{N}$ be two models of $T$, let $\iota \in \mathrm{Sub}_0(\mathscr{M},\mathscr{N})$ and let $a \in M$. Let $\mathscr{A}$ be the finitely generated substructure of $\mathscr{M}$ such that $A$ is the domain of $\iota$. Since $|A| \leq |\mathscr{L}|$, we get by Theorem 3.1.3 an elementary substructure $\mathscr{M}'$ of $\mathscr{M}$ such that $\langle A \cup \{a\}\rangle_{\mathscr{M}} \subseteq M'$ and $|M'| \leq \kappa$. By Theorem 3.5.2 there is an elementary extension $\mathscr{N}'$ of $\mathscr{N}$ such that $\mathscr{N}'$ is $\kappa^+$-saturated. It is enough to show that $\iota$ can be extended to a partial isomorphism $\iota' \in \mathrm{Sub}(\mathscr{M},\mathscr{N})$ whose domain is $M'$.

Let $\mathscr{I}$ be the subset of $\mathrm{Sub}(\mathscr{M},\mathscr{N})$ containing all partial isomorphisms $\lambda$ who extend $\iota$ and whose domain is a substructure of $\mathscr{M}'$. Since $\iota$ is in $\mathscr{I}$, we know that $\mathscr{I}$ is non-empty. We defined partial order on $\mathscr{I}$: for $\lambda,\mu \in \mathscr{I}$, we say $\lambda \preceq \mu$ if $\mu$ is an extension of $g$. If $C \subseteq \mathscr{I}$ is a chain, then the union of $C$ is an element of $\mathscr{I}$. Thus by Zorn's Lemma, there is a maximal element $\lambda \in \mathscr{I}$. By our assumption on $T$, the domain of $\lambda$ has to be $M'$, because otherwise we could extend $\lambda$. Set $\iota'$ to be $\lambda$. ■

**Exercise 4.11** Show that $(\mathbb{Z}, s)$, where $s(x) := x+1$, has quantifier-elimination and is strongly minimal.

**Exercise 4.12** Let $\mathscr{L}$ be the language consisting of a single unary function symbol, and let $T_{\text{invo}}$ be the $\mathscr{L}$-theory extending $T_\infty$ by the following axioms:

$$\forall x \forall y\ (x = y \leftrightarrow f(x) = f(y))$$
$$\forall x\ ff(x) = x$$
$$\forall x\ x \neq f(x).$$

1. Show that $T_{\text{invo}}$ has quantifier-elimination.
2. Determine whether $T_{\text{invo}}$ is strongly minimal.
3. Let $\kappa$ be an infinite cardinal. Determine which models of $T_{\text{invo}}$ are $\omega$-saturated.

### 4.2.1 An example: ordered vector spaces

Let $K$ be a an ordered field whose order we denote by $\lhd$. Let $\mathscr{L}_{\text{oK}}$ be the language $\mathscr{L}_K$ together with unary relation symbol $<$. Define $T_{\text{oVS}}(K)$ be the $\mathscr{L}_{\text{oK}}$-theory containing $T_{\text{VS}}(K)$ and each $k \in K$ with $0 \lhd k$

$$\forall x \forall y \forall z\ (x < y) \to (x + z < y + z)$$
$$\forall x \forall y\ (x < y) \to (kx < ky).$$

Let $\mathscr{M} \models T_{\text{oVS}}(K)$. Note that $\mathscr{L}_{\text{oK}}$-structures are $K$-linear subspaces. Thus for $A \subseteq M$, the subspace $\langle A \rangle_{\mathscr{M}}$ is just the subspace spanned by $A$.

**Theorem 4.2.7** $T_{\text{oVS}}(K)$ has quantifier-elimination.

*Proof.* We show that the assumptions of Corollary 4.2.6 are satisfied. Let $\mathscr{M}, \mathscr{N}$ be models of $T_{\text{oVS}}(K)$ such that $|M| \leq \kappa$ and $\mathscr{N}$ is $\kappa^+$-saturated. Let $\iota : \mathscr{A} \to \mathscr{B} \in \text{Sub}(\mathscr{M}, \mathscr{N})$ and let $a \in M$. We need to find an extension $\iota' \in \text{Sub}(\mathscr{M}, \mathscr{N})$ of $\iota$ in whose domain $a$ is. We immediately reduce to the case that $a \notin A$. Since $\mathscr{A}$ is a $K$-linear subspace of $\mathscr{M}$, we know that $a$ is $K$-linearly independent over $A$. Let $p(x)$ be the set of all $\mathscr{L}_r(B)$-formulas of the form

$$\iota(c) < x \wedge x < \iota(d),$$

where $c, d \in A$ and $c < a < d$. We show that $p(x)$ is realized in $\mathscr{N}$. Since $\mathscr{N}$ is $\kappa^+$-saturated and $|A| \leq \kappa$, it is enough to show that $p(x)$ is finitely satisfiable in $\mathscr{N}$. Let $n \in \mathbb{N}$ and $c_1, \dots, c_n, d_1, \dots, d_n \in A$ such that $c_i < a < d_i$ for all $i = 1, \dots, n$. Note that

$$\max\{c_1, \dots, c_n\} < \min\{d_1, \dots, d_n\}.$$

Since $\iota$ is a partial isomorphisms,

$$\max\{\iota(c_1), \dots, \iota(c_n)\} < \min\{\iota(d_1), \dots, \iota(d_n)\}.$$

Set $b' := \frac{1}{2}(\max\{\iota(c_1),\dots,\iota(c_n)\}+\min\{\iota(d_1),\dots,\iota(d_n)\})$. Then for $i=1,\dots,n$

$$\iota(c_i) < \max\{\iota(c_1),\dots,\iota(c_n)\} < b' < \min\{\iota(d_1),\dots,\iota(d_n)\} < \iota(d_i)$$

Thus $p(x)$ is finitely satisfiable. Let $b\in N$ be a realization of $p(x)$. Since $a\notin A$, it follows immediately that $b\notin B$. Thus $\iota$ can be extended to a $K$-linear isomorphism $\iota' : \langle A\cup\{a\}\rangle_{\mathscr{M}} \to \langle B\cup\{b\}\rangle_{\mathscr{N}}$ mapping $a$ to $b$. It is left to show that $\iota'$ is a partial $\mathscr{L}_{\mathrm{oK}}$-isomorphisms. Let $u,v\in\langle A\cup\{a\}\rangle_{\mathscr{M}}$. We just need to argue that $u<v$ if and only if $\iota(u)<\iota(v)$. We know that there are $c,d\in A$, $k,\ell\in K$ such that $u=ka+c$ and $v=\ell a+d$. Since $\iota'$ is $K$-linear,

$$\iota'(u) = kb+\iota(c) \text{ and } \iota'(v) = \ell b+\iota(d).$$

Thus if $\ell\lhd k$, we deduce from $b\models p(x)$ that

$$\begin{aligned} u<v &\text{ if and only if } ka+c<\ell a+d \text{ if and only if } a<\frac{1}{k-\ell}(d-c)\\ &\text{ if and only if } b<\frac{1}{k-\ell}(\iota(d)-\iota(c)) \text{ if and only if } kb+\iota(c)<\ell b+\iota(d)\\ &\text{ if and only if } \iota'(u)<\iota'(v).\end{aligned}$$

The cases that $k\lhd\ell$ or $k=\ell$ follow similarly. ■

**Corollary 4.2.8** $T_{\mathrm{oVS}}(K)$ is o-minimal.

**Corollary 4.2.9** The theory $T_{\mathrm{oVS}}(K)$ is complete.

*Proof.* Note that $K$ itself can be seen as a model of $T_{\mathrm{oVS}}(K)$. Since every model of $T_{\mathrm{oVS}}(K)$ is positive dimensional as $K$-vector space, we can embed $K$ into every model of $T_{\mathrm{oVS}}(K)$. Thus by Proposition 4.0.4, the theory $T_{\mathrm{oVS}}(K)$ is complete. ■

**Corollary 4.2.10** The theory of $(\mathbb{R},<,+,-,0,(x\mapsto kx)_{k\in\mathbb{Q}})$ is decidable.

*Proof.* By Corollary 4.2.9 the theory $T_{\mathrm{oVS}}(\mathbb{Q})$ is complete. By Theorem 3.3.2, $T_{\mathrm{oVS}}(\mathbb{Q})$ is decidable. Note that

$$(\mathbb{R},<,+,-,0,(x\mapsto kx)_{k\in\mathbb{Q}}) \models T_{\mathrm{oVS}}(\mathbb{Q}).$$

Because this theory is complete, the decidability of the theory of $(\mathbb{R},<,+,-,0,(x\mapsto kx)_{k\in\mathbb{Q}})$ follows. ■

## 4.3 Algebraically closed fields

In this subsection, we prove quantifier elimination for algebraically closed fields in the language of rings $\mathscr{L}_{\mathrm{r}}$. We make a few easy observations. Let $\mathscr{M}\models \mathrm{ACF}$. Then an $\mathscr{L}_{\mathrm{r}}$-substructure of $\mathscr{M}$ is just a subring, and hence for subset $A\subseteq M$ the substructure $\langle A\rangle_{\mathscr{M}}$ is just the subring generated by $A$. Taking the field of fractions of $\langle A\rangle_{\mathscr{M}}$ in $\mathscr{M}$, we obtain the subfield of $\mathscr{M}$ generated by $A$.

**Lemma 4.3.1** Let $\mathcal{M},\mathcal{N}$ be models of ACF, and let $\iota : A \to B$ be in $\mathrm{Sub}(\mathcal{M},\mathcal{N})$. Then there is $\iota' \in \mathrm{Sub}(\mathcal{M},\mathcal{N})$ extending $\iota$ and sending the algebraic closure of $A$ in $\mathcal{M}$ to the algebraic closure of $B$ in $\mathcal{N}$.

*Proof.* We can directly extend $\iota$ to $\iota' : \langle A\rangle_{\mathcal{M}} \to \langle B\rangle_{\mathcal{N}}$ by mapping $\frac{a}{a'}$ to $\frac{b}{b'}$. It is easy to check that this is well-defined and a partial isomorphism. So we can reduce to the case that $A$ is a subfield of $\mathcal{M}$ and $B$ is a subfield of $\mathcal{N}$. Let $\overline{A}$ be the algebraic closure of $A$ in $\mathcal{M}$, and let $a \in A$. Repeating this process for every such $a$ and taking the union of all such extensions, it is enough to show that we can extend $\iota$ to a partial isomorphism with domain $\langle A\cup\{a\}\rangle_{\mathcal{M}}$. Let $p$ be irreducible polynomial in $A[X]$ such that $a$ is root of $p$. Let $q \in B[X]$ be the polynomial obtained by applying $\iota$ to the coefficients of $p$. Since $\iota$ is a partial isomorphism, $q$ is irreducible. Since $\mathcal{N}$ is algebraically closed, there is $b \in N$ such that $q(b) = 0$. It is an easy exercise in algebra that we now can extend $\iota$ to an $\mathscr{L}_{\mathrm{r}}$-isomorphism $\iota' : \langle A\cup\{a\}\rangle_{\mathcal{M}} \to \langle B\cup\{b\}\rangle_{\mathcal{N}}$ mapping $a$ to $b$. ∎

**Theorem 4.3.2** ACF has quantifier-elimination.

*Proof.* Let $\kappa$ be an infinite cardinal, and let $\mathcal{M},\mathcal{N}$ be two models of ACF such that $|M| \leq \kappa$ and $\mathcal{N}$ is $\kappa^+$-saturated. Let $\iota : A \to B$ in $\mathrm{Sub}(\mathcal{M},\mathcal{N})$. Suppose there is $a \in M$ that is not in the domain of $\iota$. By Lemma 4.3.1, we can assume that $A$ is an algebraically closed subfield of $\mathcal{M}$, $B$ is an algebraically closed subfield of $\mathcal{N}$ and $a$ is transcendental over $A$. Since $\mathcal{N}$ is $\kappa^+$-saturated, there is $b \in N$ such that $b$ is transcendental over $B$. We now find $\iota' : \langle A\cup\{a\}\rangle_{\mathcal{M}} \to \langle B\cup\{b\}\rangle_{\mathcal{M}}$ mapping $a$ to $b$. Note that every element of $\langle A\cup\{a\}\rangle_{\mathcal{M}}$ is of the form $c_0 + c_1 a + \cdots + c_n a^n$, where $c_0,\ldots,c_n \in A$. Thus we choose $\iota'$ such that

$$\iota'(c_0 + c_1 a + \cdots + c_n a^n) = \iota(c_0) + \iota(c_1)b + \cdots + \iota(c_n)b^n.$$

It is easy to check that $\iota'$ is well-defined, extends $\iota$ and is in $\mathrm{Sub}(\mathcal{M},\mathcal{N})$. ∎

We collect the following immediate corollary of quantifier-elimination.

**Corollary 4.3.3** Let $\mathcal{K},\mathcal{F} \models \mathrm{ACF}$ be such that $\mathcal{K}$ is a substructure of $\mathcal{F}$. Then $\mathcal{K} \preceq \mathcal{F}$.

*Proof.* Let $\varphi(x_1,\ldots,x_n)$ be an $\mathscr{L}_{\mathrm{r}}$-formula. Since ACF has quantifier-elimination, we can assume that $\varphi$ is quantifier-free. By Proposition 2.2.2, we have for all $a \in K^n$ that $\mathcal{K} \models \varphi(a)$ if and only $\mathcal{F} \models \varphi(a)$. Thus $\mathcal{K} \preceq \mathcal{F}$. ∎

The property that substructure who are models of the same theory, are elementary substructure, is called **model completeness** and is always a consequence of quantifier-elimination and not just for ACF. We will study this in more detail later on.

**Definition 4.3.1** Let $K$ be field. For $S \subseteq K[X_1,\ldots,X_n]$, let $V_K(S)$ be the set

$$\{a \in K^n \ : \ p(a) = 0 \text{ for all } p \in S\}.$$

We say $X \subseteq K^n$ is **Zariski closed** if $X = V_K(S)$ for some $S \subseteq K[X_1,\ldots,X_n]$. We say $X \subseteq K^n$ is **Zariski constructible** if it is a boolean combination of Zariski closed sets.

By Hilbert's basis theorem, we know that for every $S \subseteq K[X_1,\ldots,X_n]$ there is a finite subset $S_0 \subseteq S$ such that $V_K(S_0) = V_K(S)$. Thus every Zariski closed is definable in by a quantifier-free $\mathscr{L}_{\mathrm{r}}$-formula; indeed, by conjunction of atomic $\mathscr{L}_{\mathrm{r}}$-formulas.

**Corollary 4.3.4** Let $\mathscr{K} \models$ ACF. Then every subset of $K^n$ definable in $\mathscr{K}$ is Zariski constructible.

*Proof.* By Theorem 4.3.2 every subset of $K^n$ definable in $\mathscr{K}$ is a boolean combination of sets defined by atomic formula. It is easy to see that such sets are boolean combinations of sets of the form $V_K(S)$ for some finite $S$. ■

**Corollary 4.3.5** ACF is strongly minimal.

*Proof.* Let $\mathscr{K} \models$ ACF. It is enough to show that subsets of $K$ defined by atomic formulas are finite or cofinite. We already observed that such sets of the form $V(S)$ for some finite $S \subseteq K[X]$. It is well-known that such sets are either finite or equal to $K$. ■

**Theorem 4.3.6 — Chevalley's theorem.** Let $\mathscr{K} \models$ ACF, let $X \subseteq K^m$ be Zariski constructible, and let $p : K^m \to K^n$ be a polynomial map over $K$. Then $p(X)$ is a Zariski constructible subset of $K^n$.

*Proof.* Observe that $p(X) = \{a \in K^n \ : \ \text{there is } b \in X \text{ s.t. } p(b) = a\}$, and hence definable in $\mathscr{K}$. Now apply Corollary 4.3.4. ■

**Exercise 4.13** Let $\mathscr{K}$ be an algebraically closed field and let $\mathscr{F}$ be a subfield of $\mathscr{K}$. For each type $p \in S_n(F)$ let $I_p = \{f \in F[X_1,\dots,X_n] : \ulcorner f(x_1,\dots,x_n) = 0 \urcorner \in p\}$. Let $\mathrm{Spec}(F[X_1,\dots,X_n])$ be the set of prime ideals of $F[X_1,\dots,X_n]$. Define the Zariski topology on $\mathrm{Spec}(F[X_1,\dots,X_n])$ by letting $D(f) = \{P : f \notin P\}$ be the basic open sets. Show that the map that sends a type $p$ to $I_p$ is a continuous bijection between $S_n(F)$ and $\mathrm{Spec}(F[X_1,\dots,X_n])$.

### 4.3.1 An application to algebraic geometry

We now prove a weak version of Hilbert's Nullstellensatz, which can be stated easily. The standard version of the Nullstellensatz can be deduced easily using the Rabinowitsch trick.

**Theorem 4.3.7 — Hilbert's (weak) Nullstellensatz.** Let $\mathscr{K}$ be an algebraically closed field and let $S \subseteq K[X_1,\dots,X_n]$ such that for all $p_1,\dots,p_m \in S$ and all $q_1,\dots,q_m \in K[X_1,\dots,X_n]$

$$q_1p_1 + \cdots + q_mp_m \neq 1.$$

Then $V_K(S)$ is nonempty.

*Proof.* By Hilbert's basis theorem, we can assume that there are polynomial $p_1,\dots,p_n \in K[X_1,\dots,X_n]$ such that $S = \{p_1,\dots,p_n\}$. Let $\mathfrak{p}$ be a prime ideal containing $S$. Then $K[X_1,\dots,X_n]/\mathfrak{p}$ is a ian integral domain $\mathscr{K}'$ containing $\mathscr{K}$ such that $V_{K'}(S)$ is nonempty. Let $\mathscr{K}^\dagger$ be an algebraically closed field containing $\mathscr{K}'$. Then $V_{K^\dagger}(S)$ is nonempty. Thus

$$\mathscr{K}^\dagger \models \exists x_1 \dots \exists x_n \ p_1(x_1,\dots,x_n) = 0 \wedge \cdots \wedge p_n(x_1,\dots,x_n) = 0.$$

Note $\mathscr{K}$ is an algebraically closed subfield of $\mathscr{K}^\dagger$. By Corollary 4.3.3,

$$\mathscr{K} \models \exists x_1 \dots \exists x_n \ p_1(x_1,\dots,x_n) = 0 \wedge \cdots \wedge p_n(x_1,\dots,x_n) = 0.$$

Hence $V_K(S)$ is nonempty. ■

## 4.4 Real closed fields

Let RCF denote the $\mathscr{L}_{\mathrm{or}}$-theory containing $T_{\mathrm{ofields}}$ and the following $\mathscr{L}_{\mathrm{or}}$-sentences:

$$\forall x \exists y (0 < x \to x = y^2)$$
$$\forall u_0 \dots \forall u_{2n} \exists x\, x^{2n+1} + u_{2n} x^{2n} + \cdots + u_0 = 0.$$

We call RCF the **theory of real closed fields**. Since polynomials of odd degree have roots in $\mathbb{R}$, we know that $(\mathbb{R}, <, +, -, \cdot, 0, 1) \models \mathrm{RCF}$.

**Lemma 4.4.1** Let $\mathscr{K} \models T_{\mathrm{ofields}}$ and let $\mathscr{R} \models \mathrm{RCF}$ such that $\mathscr{K} \subseteq \mathscr{R}$. Then there is $\mathscr{K}' \models \mathrm{RCF}$ such that $\mathscr{K} \subseteq \mathscr{K}' \subseteq \mathscr{R}$ and every element of $K'$ is algebraic over $K$.

*Proof.* Consider the set $\mathscr{I}$ of all subfields $\mathscr{E}$ of $\mathscr{R}$ such that every element of $E$ is algebraic over $K$. This set is nonempty and the union of chain with respect to inclusion is in the $\mathscr{I}$. Thus by Zorn's Lemma, $\mathscr{I}$ contains a maximal element $\mathscr{K}'$. Maximality and the fact that $\mathscr{R} \models \mathrm{RCF}$, we get that $\mathscr{K}' \models \mathrm{RCF}$. ■

We call a model $\mathscr{K}'$ of RCF that satisfies the conclusion of Lemma 4.4.1 a **real closure** of $\mathscr{K}$ in $\mathscr{R}$. The following theorem gives that the real closure of an ordered field is unique up to $\mathscr{L}_{\mathrm{or}}$-isomorphism.

**Theorem 4.4.2 — Artin-Schreier.** Let $\mathscr{K}$ be models of $T_{\mathrm{ofields}}$, and let $\mathscr{R}_1$ and $\mathscr{R}_2$ be real closures of $\mathscr{K}$. Then there is a unique $\mathscr{L}_{\mathrm{or}}$-isomorphism $\lambda : \mathscr{R}_1 \to \mathscr{R}_2$ that is the identity of $K$.

We easily obtain the following corollary of Theorem 4.4.2.

**Corollary 4.4.3** Let $\mathscr{K}, \mathscr{K}' \models RCF$. Then $\mathscr{K}$ is algebraically closed in $\mathscr{K}'$.

**Theorem 4.4.4** Let $\mathscr{K} \models \mathrm{RCF}$. Then $(K[i], +, -, \cdot, 0, 1) \models \mathrm{ACF}$, where $i^2 = -1$.

**Corollary 4.4.5** Let $\mathscr{K} \models \mathrm{RCF}$ and let $p(X) \in K[X]$ be monic and irreducible. Then either

1. $p(X) = X - a$ for some $a \in K$,
2. $p(X) = (X-a)^2 + b^2$ for some $a, b \in K$ with $b \neq 0$.

*Proof.* Suppose that $p(X)$ has degree larger than 1. Then by Theorem 4.4.4 $P$ is the minimal polynomial over $K$ of some $a + bi$ for some $a, b \in K$ and $b \neq 0$. Thus

$$p(X) = (X - (a+bi)) \cdot (X - (a-bi)) = (X-a)^2 + b^2.$$

■

We use $\mathbb{Q}^{\mathrm{rc}}$ to denote the real closure of $\mathbb{Q}$ in $\mathbb{R}$.

**Theorem 4.4.6** RCF has quantifier-elimination.

*Proof.* Let $\kappa$ be an infinite cardinal, and let $\mathscr{M}, \mathscr{N}$ be two models of RCF such that $|M| \leq \kappa$ and $\mathscr{N}$ is $\kappa^+$-saturated. Let $\iota : A \to B$ in $\mathrm{Sub}(\mathscr{M}, \mathscr{N})$. By Theorem 4.4.2, we can assume that $A$ is an real closed subfield of $\mathscr{M}$. Since $\iota$ is a partial isomorphism, $B$ is also an real closed subfield of $\mathscr{N}$.

Suppose there is $a \in M$ that is not in the domain of $\iota$. Let $p(x)$ be the set of all $\mathscr{L}_{\mathrm{or}}(B)$-formulas of the form

$$\iota(c) < x \wedge x < \iota(d),$$

where $c, d \in A$ and $c < a < d$. We show that $p(x)$ is realized in $\mathscr{N}$. Since $\mathscr{N}$ is $\kappa^+$-saturated and $|A| \le \kappa$, it is enough to show that $p(x)$ is finitely satisfiable in $\mathscr{N}$. Let $n \in \mathbb{N}$ and $c_1, \dots, c_n, d_1, \dots, d_n \in A$ such that $c_i < a < d_i$ for all $i = 1, \dots, n$. Note that

$$\max\{c_1, \dots, c_n\} < \min\{d_1, \dots, d_n\}.$$

Since $\iota$ is a partial isomorphisms,

$$\max\{\iota(c_1), \dots, \iota(c_n)\} < \min\{\iota(d_1), \dots, \iota(d_n)\}.$$

Set $b' := \frac{1}{2}(\max\{\iota(c_1), \dots, \iota(c_n)\} + \min\{\iota(d_1), \dots, \iota(d_n)\})$. Then for $i = 1, \dots, n$

$$\iota(c_i) < \max\{\iota(c_1), \dots, \iota(c_n)\} < b' < \min\{\iota(d_1), \dots, \iota(d_n)\} < \iota(d_i)$$

Thus $p(x)$ is finitely satisfiable. Let $b \in N$ be a realization of $p(x)$. Since $a \notin A$, it follows immediately that $b \notin B$. Since $A$ and $B$ are real closed fields, we have by Corollary 4.4.3 that $a$ is transcendental over $A$ and $b$ is transcendental over $B$. Thus $\iota$ can be extended to an $\mathscr{L}_{\mathrm{r}}$-isomorphism $\iota' : A[a] \to B[b]$ mapping $a$ to $b$. It is left to argue that $\iota'$ is an order-isomorphism. Let $p \in A[X]$ and let $q(X) \in B[X]$ be the polynomial obtained from $p$ by replacing the cofficients by their images under $\iota$. It is now enough to show that $p(a) > 0$ if and only if $q(b) > 0$. We can easily reduce to the case that $p$ is irreducible and monic. By Corrollary 4.4.5, we can assume that there are $c, d \in A$ with $d \neq 0$ such that either

$$p(X) = X - c \text{ or } p(X) = (X - c)^2 + d^2.$$

If $p(X) = (X - c)^2 + d^2$, then both $p(a) > 0$ and $q(b) > 0$. So now suppose that $p(X) = X - c$. Then

$$p(a) > 0 \text{ if and only if } a > c \text{ if and only if } b > \iota(c) \text{ if and only if } q(b) > 0.$$

■

**Corollary 4.4.7** RCF is complete and decidable.

*Proof.* By Theorem 3.3.2 it is enough to show completeness. Note that whenever $\mathscr{K} \models \mathrm{RCF}$, the field $\mathscr{K}$ has characteristic 0. Thus the field of rational numbers embeds into $\mathscr{K}$. This $\mathscr{L}_{\mathrm{or}}$-embedding extends to an $\mathscr{L}_{\mathrm{or}}$- embedding of $\mathbb{Q}^{\mathrm{rc}}$ by Theorem 4.4.2. Thus RCF is complete by Proposition 4.0.4. ■

The decidability of RCF is used in automated verification. However, the decision algorithm we presented here is highly inefficient.

**Corollary 4.4.8** RCF is o-minimal.

*Proof.* Since RCF has quantifier-elimination, it is enough to check that subsets defined by atomic $\mathscr{L}_{\text{or}}$-formulas, are finite union of intervals or points. We reduce to the case that the set is of the form

$$\{a \in R \ : \ p(a) = 0\} \text{ or } \{a \in R \ : \ p(a) > 0\},$$

where $p \in R[X]$. If the set is a zero sets of polynomial, then it is either finite or $R$, and hence finite union of points or a single interval. For the second case, we can reduce to the case that $p$ is irreducible, since the set of finite union of intervals and points is closed under boolean combinations. By Corollary 4.4.5, we can assume that $p$ is of the form $X - b$ for some $b \in R$. But then $\{a \in R \ : \ p(a) > 0\}$ is the interval $(b, \infty)$. ■

**Definition 4.4.1** Let $\mathscr{R}$ be an $\mathscr{L}_{\text{or}}$-structure. We say a subset of $R^n$ is **semialgebraic** if it is a finite union of sets of the form

$$\{a \in R^n \ : \ p(a) = 0, q_1(a) > 0, \ldots, q_m(a) > 0\},$$

where $p, q_1, \ldots, q_n \in R[X_1, \ldots, X_n]$.

We first observe that the boolean combinations of semialgebraic sets are also semialgebraic. Indeed, this follows easily from the following observation:

$$\{a \in R^n \ : \ p(a) = 0\} \cap \{a \in R^n \ : \ q(a) = 0\} = \{a \in R^n \ : \ p(a)^2 + q(a)^2 = 0\}.$$

**Corollary 4.4.9** Let $\mathscr{R} \models \text{RCF}$. Then every subset of $R^n$ definable in $\mathscr{R}$ is semialgebraic.

*Proof.* By Theorem 4.4.6, it is enough to show that every subset defined by a quantifier-free formula is semialgebraic. Note that every subset defined by a atomic formula is semialgebraic. Since semialgebraic sets are closed under boolean combinations, all quantifier-free definable sets are semialgebraic. ■

**Theorem 4.4.10 — Tarski-Seidenberg.** Let $\mathscr{R} \models \text{RCF}$. Then the projection of a semialgebraic subset of $R^n$ is semialgebraic.

*Proof.* This follows immediately from Corollory 4.4.9 and the fact that definable sets are closed under projections. ■

**Exercise 4.14** Let $\mathscr{F} = (F, 0, 1, -, +, \cdot)$ be a field. A **prepositive cone** in $\mathscr{F}$ is a subset $P$ of $F$ such that

(i) $P$ is closed under $+$ and $\cdot$, i.e. $\forall x, y \in P.\ x + y \in P$ and $\forall x, y \in P.\ x \cdot y \in P$;
(ii) $-1 \notin P$;
(iii) $P$ contains all squares, i.e. $\{x^2 : x \in F\} \subseteq P$.

Let $-P = \{-x : x \in P\}$. A **positive cone** is a prepositive cone $P$ such that $P \cup (-P) = F$ and $P \cap (-P) = \{0\}$. Suppose that $\mathscr{F}$ has at least one prepositive cone. By using Zorn's lemma we can obtain a $\subseteq$-maximal prepositive cone $P_*$ in $\mathscr{F}$.

(a) Show that $P_*$ is a positive cone.

(b) Define $x \leq y \longleftrightarrow y - x \in P_*$. Show that $\leq$ is a total order that is compatible with the field structure, making $\mathscr{F}$ an ordered field.

**Exercise 4.15** Let $\mathscr{F}$ be a real closed field. Let $p,q$ be polynomials in variables $x_1,\ldots,x_n$ with coefficients in $\mathscr{F}$. Suppose that the rational function $f = p/q$ is positive semidefinite, in the sense that for all $a \in F^n$ with $q(a) \neq 0$ we have

$$f(a) = p(a)/q(a) \geq 0.$$

Show that $f$ is equal to a sum of finitely many squares of rational functions in the field of rational functions $\mathscr{F}(x_1,\ldots,x_n)$.

**Hint:** Assume the contrary and show that there is a prepositive cone containing $-f$. For that, the following equation might be helpful:

$$(\sum_{i=1}^{m} x_i^2)^{-1} = \sum_{i=1}^{m} (\frac{x_i}{\sum_{j=1}^{m} x_j^2})^2.$$

Then set things up to use quantifier elimination for real-closed fields.

## 4.5 Presburger arithmetic

Let $\mathscr{L}$ be the language $\mathscr{L} = \{<,+,-,0,1\}$. Let $\mathscr{L}_{\mathrm{Pr}}$ be the language $\mathscr{L}$ extended by adding for each $n \in \mathbb{N}_{\geq 2}$ a unary predicate symbol $P_n$. Let Pr be the $\mathscr{L}_{\mathrm{Pr}}$-theory containing $T_{oab}$ and the $\mathscr{L}_{\mathrm{Pr}}$-sentences

$$0 < 1 \tag{Pr 1}$$

$$\forall x(x \leq 0 \vee 1 \leq x) \tag{Pr 2}$$

and for each $n \in \mathbb{N}_{\geq 2}$ the $\mathscr{L}_{\mathrm{Pr}}$-sentences:

$$\forall x\,(P_n(x) \leftrightarrow (\exists y\; nx = y)) \tag{Pr 3}$$

$$\forall x \bigvee_{i=0}^{n-1} \big(P_n(x+i) \wedge \bigwedge_{j\in\{0,\ldots,n-1\}\setminus\{i\}} \neg P_n(x+j)\big). \tag{Pr 4}$$

**Lemma 4.5.1** Let $k,m,n \in \mathbb{N}$ be such that $k < n$. Then

$$\mathrm{Pr} \models \forall x\; (P_n(x) \leftrightarrow P_{mn}(mx))$$

$$\mathrm{Pr} \models \forall x\; (P_n(x) \leftrightarrow \bigvee_{j=0}^{m-1} P_{mn}(x+jn))$$

$$\mathrm{Pr} \models \forall x \forall y\; \Big(P_n(x+k) \to \big(P_n(x+y) \leftrightarrow P_n(y+n-k)\big)\Big)$$

*Proof.* ■

**Theorem 4.5.2** Pr is complete, decidable and has quantifier-elimination.

*Proof.* ■

# 5. Further concepts from model theory

## 5.1 Model completeness

**Definition 5.1.1** Let $\varphi$ be an $\mathscr{L}$-formula. We say $\varphi$ is **universal** if there is a quantifier-free $\mathscr{L}$-formula $\psi$ such that $\varphi$ is of the form $\forall x_1 \dots \forall x_n \psi$. Let $T$ be an $\mathscr{L}$-theory. We say $T$ is **universal** if all $\mathscr{L}$-sentences in $T$ are universal. We define $T_\forall$ to be the set of all universal $\mathscr{L}$-sentences $\sigma$ such that $T \models \sigma$.

**Lemma 5.1.1** Let $\mathscr{M}, \mathscr{N}$ be $\mathscr{L}$-structures such that $\mathscr{M}$ is a substructure of $\mathscr{N}$, let $a \in M^n$, and let $\varphi(x_1, \dots, x_n)$ be a universal $\mathscr{L}$-formula. If $\mathscr{N} \models \varphi(a)$, then $\mathscr{M} \models \varphi(a)$.

*Proof.* Let $\psi$ be a quantifier-free $\mathscr{L}$-formula such that $\varphi$ is of the form $\forall y_1 \dots \forall y_m \psi$. Suppose that $\mathscr{N} \models \varphi(a)$. Then $\mathscr{N} \models \psi(b,a)$ for all $b \in N^m$. By Proposition 2.2.2 we deduce that $\mathscr{M} \models \psi(c,a)$ for all $c \in M^n$. Hence $\mathscr{M} \models \varphi(a)$. ∎

**Proposition 5.1.2** Let $T$ be an $\mathscr{L}$-theory and let $\mathscr{M}$ be an $\mathscr{L}$-structure. Then $\mathscr{M} \models T_\forall$ if and only if $\mathscr{M}$ is a substructure of a model of $T$.

*Proof.* Let $\mathscr{M} \models T_\forall$. We now consider $\mathscr{M}$ as $\mathscr{L}(M)$-structure. Let $S$ be the set of quantifier-free $\mathscr{L}(M)$-sentence that hold in $\mathscr{M}$. Suppose that $T \cup S$ is satisfiable. Let $\mathscr{N}' \models T \cup S$, and let $\mathscr{N}$ be the $\mathscr{L}$-reduct of $\mathscr{N}'$. Obviously, $\mathscr{N} \models T$. Let $\mu : M \to N$ be the function map $a \in M$ to $c_a^{\mathscr{N}'}$, the interpretation of the constant symbol for $a$ in $\mathscr{N}'$. Since $\mathscr{N}' \models S$, we have for every quantifier-free $\mathscr{L}$-formula $\varphi(x_1, \dots, x_n)$ and every $a_1, \dots, a_n \in M$ that $\mathscr{N} \models \varphi(\mu(a_1), \dots, \mu(a_n))$ if and only if $\mathscr{M} \models \varphi(a_1, \dots, a_n)$. Thus $\mu$ is an embedding of $\mathscr{M}$ into $\mathscr{N}$.

It is left to show that $T \cup S$ is satisfiable. Let $S' \subseteq S$ be finite. By Theorem 2.5.1 it is enough to prove that $T \cup S'$ is satisfiable. Suppose not. Let $\sigma$ be the quantifier-free $\mathscr{L}(M)$-sentence $\bigwedge_{\chi \in S} \chi$. Note that $\sigma \in S$. Since $T \cup S'$ is not satsfiable, we have that $T \models \neg\sigma$. Let $\varphi(x_1, \dots, x_n)$ be a quantifier-free $\mathscr{L}$-formula and $a = (a_1, \dots, a_n) \in M^n$ such that $\sigma$ is $\varphi(c_{a_1}, \dots, c_{a_m})$. Since $c_{a_1}, \dots, c_{a_m}$ are not in $\mathscr{L}$, we get that $T \models \forall x_1 \dots \forall x_m\, \neg\varphi$. However, $\forall x_1 \dots \forall x_m\, \neg\varphi$ is a universal $\mathscr{L}$-formula. Since $\mathscr{M} \models T_\forall$, we have that

$$\mathscr{M} \models \forall x_1 \dots \forall x_m\, \neg\varphi.$$

In particular, $\mathscr{M} \models \neg\varphi(a_1,\dots,a_n)$, and hence $\neg\sigma \in S$. This contradicts $\sigma \in S$. ■

**Corollary 5.1.3** Let $T$ be an $\mathscr{L}$-theory and let $\varphi(x_1,\dots,x_n)$ be an $\mathscr{L}$-formula. Then the following are equivalent:

1. for all $\mathscr{M},\mathscr{N} \models T$ with $\mathscr{M} \subseteq \mathscr{N}$ and $a \in M^n$
$$\text{if } \mathscr{N} \models \varphi(a), \text{ then } \mathscr{M} \models \varphi(a).$$
2. there is a universal $\mathscr{L}$-formula $\psi$ such that $T \models \varphi \leftrightarrow \psi$.

*Proof.* Assume 2 holds. Let $\mathscr{M},\mathscr{N} \models T$ be such that $\mathscr{M} \subseteq \mathscr{N}$ and $a \in M^n$. Let $\psi$ be an universal $\mathscr{L}$-formula such that $T \models \varphi \leftrightarrow \psi$. By Lemma 5.1.1, we have that $\mathscr{M} \models \psi(a)$, whenever $\mathscr{N} \models \psi(a)$. Since both $\mathscr{M}$ and $\mathscr{N}$ are models of $T$ and $T \models \varphi \leftrightarrow \psi$, we get that $\mathscr{M} \models \varphi(a)$, whenever $\mathscr{N} \models \varphi(a)$.

Assume 1 holds. Let $\mathscr{L}_c$ be the language $\mathscr{L}$ expanded by constant symbols $c_1,\dots,c_n$. Let $T^c$ be the $\mathscr{L}_c$-theory $T \cup \{\varphi(c_1,\dots,c_n)\}$. We first show that $T^c_\forall \cup T \models \varphi(c)$.
Let $\mathscr{M} \models T^c_\forall \cup T$. By Proposition 5.1.2 there is $\mathscr{N} \models T_c$ such that $\mathscr{M}$ is an $\mathscr{L}_c$-substructure of $\mathscr{N}$. Set $c^{\mathscr{M}} := (c_1^{\mathscr{M}},\dots,c_n^{\mathscr{M}})$. Since $\mathscr{N} \models \varphi(c)$, we also have that $\mathscr{N} \models \varphi(c^{\mathscr{M}})$. By 1, $\mathscr{M} \models \varphi(c^{\mathscr{M}})$. Thus $\mathscr{M} \models \varphi(c)$.
By Corollary 2.5.7 there is a finite subset $S \subseteq T^c_\forall$ such that $S \cup T \models \varphi(c)$. Let $\psi_1,\dots,\psi_m$ be universal $\mathscr{L}$-formulas such that $S = \{\psi_1(c),\dots,\psi_m(c)\}$. Since $S \cup T \models \varphi(c)$,

$$T \models \Big(\bigwedge_{i=1}^{m} \psi_i(c)\Big) \rightarrow \varphi(c).$$

Since $S \subseteq T^c_\forall$, we also have $T \models \varphi(c) \rightarrow \big(\bigwedge_{i=1}^{m} \psi_i(c)\big)$. ■

**Definition 5.1.2** Let $T$ be an $\mathscr{L}$-theory. We say that $T$ is **model complete** if for all $\mathscr{M},\mathscr{N} \models T$, if $\mathscr{M}$ is a substructure of $\mathscr{N}$, then $\mathscr{M}$ is an elementary substructure of $\mathscr{N}$.

**Theorem 5.1.4** Let $T$ be an $\mathscr{L}$-theory. Then the following are equivalent:

1. $T$ is model complete.
2. For every $\mathscr{L}$-formula $\varphi$ there is an universal $\mathscr{L}$-formula $\psi$ such that $T \models \varphi \leftrightarrow \psi$.
3. For every $\mathscr{L}$-formula $\varphi$ there is an existential $\mathscr{L}$-formula $\psi$ such that $T \models \varphi \leftrightarrow \psi$.

*Proof.* Since the negation of an universal formula is an existential formula (and the other way around), statements 2. and 3. are equivalent.

Assume 1. Let $\varphi(x_1,\dots,x_n)$ be an $\mathscr{L}$-formula. Let $\mathscr{M},\mathscr{N} \models T$ with $\mathscr{M} \subseteq \mathscr{N}$ and $a \in M^n$ such that $\mathscr{N} \models \varphi(a)$. Since $T$ is model complete, $\mathscr{M} \models \varphi(a)$. Thus by Corollary 5.1.3 there is a universal

$\mathscr{L}$-formula $\psi$ such that $T \models \varphi \leftrightarrow \psi$.

Assume 2. Let $\mathscr{M}, \mathscr{N} \models T$ be such that $\mathscr{M} \subseteq \mathscr{N}$. By Corollary 5.1.3, we have for every $\mathscr{L}$-formula $\varphi$ and every $a \in M^n$ that if $\mathscr{N} \models \varphi(a)$, then $\mathscr{M} \models \varphi(a)$. Thus by Theorem 3.1.2, $\mathscr{M}$ is an elementary substructure of $\mathscr{N}$. ■

**Corollary 5.1.5** Let $T$ be an $\mathscr{L}$-theory. If $T$ has quantifier-elimination, then $T$ is model complete.

■ **Example 5.1** The converse of Corollary 5.1.5 is false. Using quantifier-elimination for RCF, one can check that the theory of the $\mathscr{L}_{\mathrm{r}}$-structure $(\mathbb{R}, +, -, \cdot, 0, 1)$ is model complete. However, the sets defined by quantifier-free $\mathscr{L}_{\mathrm{r}}$-formulas are finite or co-finite. ■

**Lemma 5.1.6** Let $T$ be an $\mathscr{L}$-theory such that for every universal $\mathscr{L}$-formula $\varphi$ there is an existential $\mathscr{L}$-formula $\psi$ such that $T \models \varphi \leftrightarrow \psi$. Then $T$ is model complete.

*Proof.* We show by induction on the $\mathscr{L}$-formula $\varphi$ that condition 3. of Theorem 5.1.4 is satisfied. If $\varphi$ is quantifier-free, this follows immediately from our assumption on $T$. The induction step when $\varphi$ is of the form $\exists x \psi$, can be handled easily. So now consider the induction step that $\varphi$ is of the form $\neg\chi$. By induction, there is an existential $\mathscr{L}$-formula $\theta$ such that $T \models \chi \leftrightarrow \theta$. Note that $\neg\theta$ is universal. Thus by our assumption there is an existential $\mathscr{L}$-formula $\psi$ such that $T \models (\neg\theta) \leftrightarrow \psi$. It follows that $T \models \varphi \leftrightarrow \psi$. ■

**Theorem 5.1.7** Let $T$ be an $\mathscr{L}$-theory such that for all $\mathscr{M}, \mathscr{N} \models T$ such that $\mathscr{M}$ is a substructure of $\mathscr{N}$, and for all elementary extensions $\mathscr{M}^*$ of $\mathscr{M}$ that is $\kappa$-saturated for some $\kappa > |N|$, there is an $\mathscr{L}$-embedding of $\mathscr{N}$ to $\mathscr{M}^*$ extending the inclusion embedding of $\mathscr{M}$ into $\mathscr{N}$. Then $T$ is model complete.

*Proof.* Let $\varphi(x_1, \ldots, x_n)$ be a universal $\mathscr{L}$-formula. Let $\mathscr{M}, \mathscr{N} \models T$ be such that $\mathscr{M}$ is a substructure of $\mathscr{N}$. Let $a \in M^n$ such that $\mathscr{N} \models \neg\varphi(a)$.

Suppose towards a contradiction that $\mathscr{M} \models \varphi(a)$. Let $\mathscr{M}^*$ be an elementary extension of $\mathscr{M}$ and let $\mu : \mathscr{N} \to \mathscr{M}^*$ be an $\mathscr{L}$-embedding fixing $M$. Since $\mathscr{M}^*$ is an elementary extension of $\mathscr{M}$, we have that $\mathscr{M}^* \models \varphi(a)$. However, $\varphi$ is universal. Hence, by Lemma 5.1.1, $\mathscr{N} \models \varphi(a)$. This contradicts $\mathscr{N} \models \neg\varphi(a)$. Thus $\mathscr{M} \models \neg\varphi(a)$.

By Corollary 5.1.3, there is an universal $\mathscr{L}$-formula $\psi$ such that $T \models (\neg\varphi) \leftrightarrow \psi$. Note that $\neg\psi$ is existential and $T \models \varphi \leftrightarrow (\neg\psi)$. Thus by Lemma 5.1.6 we have that $T$ is model complete. ■

**Exercise 5.1** Let $\mathscr{L}_{\mathrm{gr}}$ be the languague consisting of single binary relation symbol $P$. A graph $(V, E)$ is **acyclic** if there is no sequence of vertices $v_1, \ldots, v_k \in V$ such that $v_1 E v_2 E \cdots E v_k E v_1$. A graph is **2-regular** if every vertex has exactly 2 neighbours. Show that the $\mathscr{L}_{\mathrm{gr}}$-theory of acyclic 2-regular graphs is model-complete, but does not have quantifier-elimination.

**Exercise 5.2** Let $T$ be a universal model complete $\mathscr{L}$-theory. Show that $T$ has quantifier-elimination.

**Proposition 5.1.8** Let $T$ be a universal $\mathscr{L}$-theory with quantifier-elimination. Then for every $\mathscr{L}$-formula $\varphi(x_1,\ldots,x_n,y)$ there are $\mathscr{L}$-terms $t_1(x_1,\ldots,x_n),\ldots,t_m(x_1,\ldots,x_n)$ such that

$$T \models (\exists y \varphi(x,y) \to (\varphi(x,t_1(x)) \vee \cdots \vee \varphi(x,t_m(x)),$$

where $x = (x_1,\ldots,x_n)$.

*Proof.* ■

## 5.2 Model companions

**Definition 5.2.1** Let $\mathscr{M},\mathscr{N}$ be $\mathscr{L}$-structures. We say $\mathscr{M}$ is **existentially closed in $\mathscr{N}$** (written: $\mathscr{M} \preceq_\exists \mathscr{N}$) if for for every existential $\mathscr{L}$-formula $\varphi(x_1,\ldots,x_n)$ and every $a \in M^n$

$$\mathscr{M} \models \varphi(a) \text{ if and only if } \mathscr{N} \models \varphi(a).$$

Let $T$ be an $\mathscr{L}$-theory and let $\mathscr{M} \models T$. We say $\mathscr{M}$ is an **existentially closed model of $T$** if $\mathscr{M}$ is existentially closed in every model $\mathscr{N}$ of $T$ with $\mathscr{M} \subseteq \mathscr{N}$.

Note that if $T$ is model complete, then every model of $T$ is existentially closed. The converse is true, too.

**Proposition 5.2.1** Let $T$ be a $\mathscr{L}$-theory such that every of $T$ is an existentially closed model of $T$. Then $T$ is model complete.

*Proof.* Let $\varphi(x_1,\ldots,x_n)$ be a universal $\mathscr{L}$-formula. By Lemma 5.1.6, it is enough to find an existential $\mathscr{L}$-formula $\psi$ such that $T \models \varphi \leftrightarrow \psi$. Note that $\neg\varphi$ is an existential formula. Thus by Corollary 5.1.3 and the fact every model of $T$ is existentially closed, there is universal $\mathscr{L}$-formula $\theta$ such that $T \models (\neg\varphi) \leftrightarrow \theta$. Thus $T \models \varphi \leftrightarrow (\neg\theta)$. Since $\theta$ is universal, $\neg\varphi$ is equivalent to an existential $\mathscr{L}$-formula. ■

**Lemma 5.2.2** Let $\mathscr{M},\mathscr{N}$ be $\mathscr{L}$-structures such that $\mathscr{M} \subseteq \mathscr{N}$, let $\varphi(x_1,\ldots,x_n)$ be an existential $\mathscr{L}$-formula, and let $a \in M^n$. If $\mathscr{M} \models \varphi(a)$, then $\mathscr{N} \models \varphi(a)$.

*Proof.* Let $\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)$ be a quantifier-free $\mathscr{L}$-formula such that $\varphi$ is $\exists y_1 \ldots \exists y_m \psi$. Let $b \in M^m$ such that $\mathscr{M} \models \psi(a,b)$. By Proposition 2.2.2, we have $\mathscr{N} \models \psi(a,b)$. Thus $\mathscr{N} \models \varphi(a)$. ■

**Definition 5.2.2** Let $T$ be an $\mathscr{L}$-theory. We say $T$ is **inductive** if every union of an increasing chain of models of $T$ is also a model of $T$.

**Lemma 5.2.3** Let $T$ be an inductive theory and let $\mathscr{M} \models T$. Then there is an extension $\mathscr{M}^*$ of $\mathscr{M}$ that is an existentially closed model of $T$.

*Proof.* Let $\mathscr{M} \models T$ and consider this structure as $\mathscr{L}(M)$-structure. We first construct an $\mathscr{L}$-structure $\mathscr{M}^\dagger$ such that for every $\mathscr{L}$-formula $\varphi(x_1,\dots,x_n)$, every $a \in M^n$, if there is an extension $\mathscr{N}$ of $\mathscr{M}^*$ with $\mathscr{N} \models \varphi(a)$, then $\mathscr{M}^* \models \varphi(a)$.

Let $(\varphi_\lambda)_{\lambda<\kappa}$ be an enumeration of all existential $\mathscr{L}(M)$-sentences. We construct a family $(\mathscr{M}_\lambda)_{\lambda<\kappa}$ of models of $T$. Set $\mathscr{M}_0 := \mathscr{M}$. Suppose that $\lambda = \mu + 1$ and $\mathscr{M}_\mu$ is already defined. If there is an extension of $\mathscr{M}_\mu$ such that $\mathscr{M} \models T$ and $\mathscr{M}_\mu \models \varphi_\lambda$, then set $\mathscr{M}_\lambda$ to be this model. If no such extension exists, set $\mathscr{M}_\lambda := \mathscr{M}_\mu$. If $\lambda$ is a limit ordinal, then $\mathscr{M}_\lambda := \bigcup_{\mu<\lambda} \mathscr{M}_\mu$. Since $T$ is inductive, $\mathscr{M}_\lambda$ is a model of $T$. Finally, set $\mathscr{M}^\dagger = \bigcup_{\lambda<\kappa} \mathscr{M}_\lambda$. Again, since $T$ is inductive, $\mathscr{M}^* \models T$.

We now show that $\mathscr{M}^\dagger$ has the desired property. Let $\varphi(x_1,\dots,x_n)$ be an existential $\mathscr{L}$-formula, let $a \in M^n$, and let $\mathscr{N} \models T$ be such that $\mathscr{M}^\dagger \subseteq \mathscr{N}$ and $\mathscr{N} \models \varphi(a)$. We now show that $\mathscr{M}^* \models \varphi(a)$. Let $\lambda < \kappa$ be such that $\varphi_\lambda$ is $\varphi(c_{a_1},\dots,c_{a_n})$. Since $\mathscr{M}_\lambda \subseteq M^\dagger \subseteq \mathscr{N}$, we have by construction that $\mathscr{M}_{\lambda+1} \models \varphi_\lambda$. Hence $\mathscr{M}_{\lambda+1} \models \varphi(a)$. By Lemma 5.2.2, $M^\dagger \models \varphi(a)$.

We now define a family $(\mathscr{N}_n)_{n\in\mathbb{N}}$ of models of $T$ as follows. Set $\mathscr{N}_0 := \mathscr{M}$, and let $\mathscr{N}_{n+1} := \mathscr{N}_n^\dagger$. Then set $\mathscr{M}^*$ to be $\bigcup_{n\in\mathbb{N}} \mathscr{N}_n$. Since $T$ is inductive, $\mathscr{M}^* \models T$. From the construction, it is clear that $\mathscr{M}^*$ is existential closed and $\mathscr{M} \subseteq \mathscr{M}^*$. ■

**Definition 5.2.3** Let $T$ be an $\mathscr{L}$-theory, and let $T^*$ be model complete $\mathscr{L}$-theory with $T \subseteq T^*$. We say $T^*$ is a **model companion of $T$** if every model of $T$ embeds into a model of $T^*$. We say $T$ is **companionable** if it has a model companion.

▪ **Example 5.2** We have seen several examples of model companions already:

1. The $\mathscr{L}_<$-theory DLO is a model companion of the $\mathscr{L}_<$-theory $T_{\mathrm{lo}}$.
2. The $\mathscr{L}_{\mathrm{r}}$-theory ACF is a model companion of the $\mathscr{L}_r$-theory $T_{\mathrm{fields}}$.
3. The $\mathscr{L}_{\mathrm{or}}$-theory RCF is a model companion of the $\mathscr{L}_{or}$-theory $T_{\mathrm{ofields}}$.

▪

**Theorem 5.2.4** Let $T$ be an inductive theory and let $T^*$ be an $\mathscr{L}$-theory. Then $T^*$ is a model companion of $T$ if and only if the models of $T^*$ are the existentially closed models of $T$.

*Proof.* Suppose that $T^*$ is a model companion of $T$.

Let $\mathscr{M}^* \models T^*$. We first show that $\mathscr{M}^*$ is an existentially closed model of $T$. Let $\mathscr{N} \models T$ be such that $\mathscr{M}^* \subseteq \mathscr{N}$. Since $T^*$ is a model companion of $T$, there is a model $\mathscr{N}^* \models T^*$ such that $\mathscr{N} \subseteq \mathscr{N}^*$. By model completeness of $T$, we have $\mathscr{M}^* \preceq \mathscr{N}^*$. Using Lemma 5.2.2, we easily see that $\mathscr{M}^* \preceq_\exists \mathscr{N}$. Thus $\mathscr{M}^*$ is an existentially closed model of $T$.

Now let $\mathscr{M}$ be an existentially closed model of $T$. We need to show that $\mathscr{M} \models T^*$. Let $\mathscr{M}^* \models T^*$ be such that $\mathscr{M} \subseteq \mathscr{M}^*$. We now show that $\mathscr{M} \preceq \mathscr{M}^*$ and hence $\mathscr{M} \models T^*$. Let $\varphi(x_1,\dots,x_n,y_1,\dots,y_m)$ be $\mathscr{L}$-formula and $a \in M^n$. By Theorem 3.1.2 it is enough to show that whenever there is $b \in (M^*)^m$ with $\mathscr{M}^* \models \varphi(a,b)$, there is $c \in M^m$ with $M^* \models \varphi(a,c)$. Since $T^*$ is model-complete, there is an existential $\mathscr{L}$-formula $\psi$ such that $T^* \models \varphi \leftrightarrow \psi$. Since $\mathscr{M}$ is existential closed, there is $c \in M^m$ such that $\mathscr{M} \models \psi(a,c)$. By Lemma 5.2.2, $\mathscr{M}^* \models \psi(a,c)$, and hence $\mathscr{M}^* \models \varphi(a,c)$.

Suppose the models of $T^*$ are the existentially closed models of $T$. By Lemma 5.2.3 it is enough to show that $T^*$ is model complete. However, every model of $T^*$ is obviously an existential closed of $T^*$. Thus $T^*$ is model complete by Proposition 5.2.1. ■

**Corollary 5.2.5** Let $T$ be an inductive theory. Then $T$ is companionable if and only if the existentially closed models of $T$ form an elementary class.

## 5.3 Algebraic and definable closure

**Definition 5.3.1** Let $\mathscr{M}$ be an $\mathscr{L}$-structure and $A \subseteq M$. We say that $b \in M$ is **algebraic over $A$ in $\mathscr{M}$** if there is an $\mathscr{L}$-formula $\varphi(x, y_1, \ldots, y_n)$ and $a_1, \ldots, a_n \in A$ such that

1. $\mathscr{M} \models \varphi(b, a_1, \ldots, a_n)$, and
2. $\{c \in M : \mathscr{M} \models \varphi(c, a_1, \ldots, a_n)\}$ is finite.

We write $\mathrm{acl}_{\mathscr{M}}(A)$ for the set of all $b \in M$ that are algebraic over $A$ in $\mathscr{M}$. This set is called the **algebraic closure of $A$ in $\mathscr{M}$**. We say $A$ is **algebraically closed in $\mathscr{M}$** if $\mathrm{acl}_{\mathscr{M}}(A) = A$.

**Definition 5.3.2** Let $\mathscr{M}$ be an $\mathscr{L}$-structure. We say that $b \in M$ is **definable over $A$ in $\mathscr{M}$** if the set $\{b\}$ is $\mathscr{L}(A)$-definable in $\mathscr{M}$. We write $\mathrm{dcl}_{\mathscr{M}}(A)$ for the set of all $b \in M$ that are definable over $A$ in $\mathscr{M}$. This set is called the **definable closure of $A$ in $\mathscr{M}$**. We say $A$ is **definably closed in $\mathscr{M}$** if $\mathrm{dcl}_{\mathscr{M}}(A) = A$.

Obviously, $\mathrm{dcl}_{\mathscr{M}}(A) \subseteq \mathrm{acl}_{\mathscr{M}}(A)$ for all $\mathscr{L}$-structures $\mathscr{M}$ and $A \subseteq M$.

■ **Example 5.3** Our quantifier-elimination results allow us to easily compute the algebraic closure in several structures:

1. Let $X$ be an infinite set considered as $\mathscr{L}_\emptyset$-structure. It follows easily from Proposition 4.0.2 that $\mathrm{acl}_X(\{a\}) = \{a\}$ for all $a \in X$.
2. Let $\mathscr{M} \models T_{\mathrm{VS}}(K)$ and let $A \subseteq M$. Then it can be deduced from Theorem 4.2.7 that $\mathrm{dcl}(A) = \mathrm{acl}(A)$ is equal to the $K$-linear span of $A$.
3. Let $\mathscr{K} \models \mathrm{ACF}$ and $A \subseteq K$. By Theorem 4.3.2, $a \in \mathrm{acl}_{\mathscr{K}}(A)$ if and only if $a$ is algebraic over $A$.

■

**Exercise 5.3** Let $\mathscr{M}$ be an expansion of a linear order $(M, <)$. Show that $\mathrm{acl}_{\mathscr{M}}(A) = \mathrm{dcl}_{\mathscr{M}}(A)$ for all $A \subseteq M$.

**Exercise 5.4** Let $\mathscr{M} \models \mathrm{RCF}$, let $A \subseteq M$ and let $a \in M$. Show that $a \in \mathrm{dcl}_{\mathscr{M}}(A)$ if only if $a$ is algebraic over $A$.

**Proposition 5.3.1** Let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $A, B \subseteq M$ and $a \in M$. Then

1. $A \subseteq \mathrm{acl}_{\mathscr{M}}(A)$ and $\mathrm{acl}_{\mathscr{M}}(\mathrm{acl}_{\mathscr{M}}(A)) = \mathrm{acl}_{\mathscr{M}}(A)$,
2. if $A \subseteq B$, then $\mathrm{acl}_{\mathscr{M}}(A) \subseteq \mathrm{acl}_{\mathscr{M}}(B)$,
3. if $a \in \mathrm{acl}_{\mathscr{M}}(A)$, then there is a finite set $F \subseteq A$ such that $a \in \mathrm{acl}_{\mathscr{M}}(F)$.

*Proof.* Statements 2. and 3. are immediate from the definition of the algebraic closure. Note that for all $a \in M$, we have $\{a\} = \{b \in M : \mathscr{M} \models (x = y)(a, b)\}$. So if $a \in A$, then $a \in \mathrm{acl}(A)$.

Let $a \in \operatorname{acl}(\operatorname{acl}(A))$. Then there is $\mathscr{L}$-formula $\varphi(x, y_1, \ldots, y_n)$ and $b_1, \ldots, b_n \in \operatorname{acl}(A)$ such that $\mathscr{M} \models \varphi(a, b_1, \ldots, b_n)$ and $\{c \in M : \mathscr{M} \models \varphi(c, b_1, \ldots, b_n)\}$ is finite. Let $N$ be the cardinality of this set. Let $\theta(y_1, \ldots, y_n)$ be an $\mathscr{L}$-formula such that for all $e_1, \ldots, e_n \in M$

$$|\{c \in M : \mathscr{M} \models \varphi(c, e_1, \ldots, e_n)\}| \leq N \text{ if and only if } \mathscr{M} \models \theta(e_1, \ldots, e_n). \tag{5.1}$$

For each $i \in \{1, \ldots, n\}$, there is an $\mathscr{L}$-formulas $\psi_i(y, z_{i,1}, \ldots, z_{i,m_i})$ and $d_{i,1}, \ldots, d_{i,m_i} \in A$ such that $\mathscr{M} \models \psi(b_i, d_{i,1}, \ldots, d_{i,m_i})$ and

$$\{c \in M : \mathscr{M} \models \psi(c, d_{i,1}, \ldots, d_{i,m_i})\} \text{ is finite.} \tag{5.2}$$

Now let $\xi(x, z_{1,1}, \ldots, z_{n,m_n})$ be the $\mathscr{L}$-formula

$$\exists y_1 \ldots \exists y_n\ \theta(y_1, \ldots, y_n) \wedge \varphi(x, y_1, \ldots, y_n) \wedge \bigwedge_{i=1}^{n} \psi_i(y, z_{i,1}, \ldots, z_{i,m_i})$$

Note that $\mathscr{M} \models \xi(a, d_{1,1}, \ldots, d_{n,m_n})$. By (5.1) and (5.2) the set $\{c \in M : \mathscr{M} \models \psi(c, d_{1,1}, \ldots, d_{n,m_n})\}$ is finite. ■

**Lemma 5.3.2** Let $\mathscr{M}$ be a $\mathscr{L}$-structure and let $A \subseteq M$ and $b \in M$. Then $b \in \operatorname{dcl}_{\mathscr{M}}(A)$ if and only if there is $f : X \subseteq M^n \to M$ definable without parameters in $\mathscr{M}$ and $a \in A^n$ such that $f(a) = b$.

*Proof.* The backward implication is immediate. For the other direction, suppose that $b \in \operatorname{dcl}_{\mathscr{M}}(A)$. Then there is an $\emptyset$-definable set $Z \subseteq M^n \times M$ and $a \in M^n$ such that $(a, b) \in Z$ and $b = b'$ for all $b' \in B$ with $(a, b') \in Z$. Set

$$X := \{c \in M : |\{d \in M : (c, d) \in Z\}| = 1\}.$$

Note that $X$ is $\emptyset$-definable in $\mathscr{M}$, since $Z$ is. Let $f : X \to M$ be the function that maps $c \in X$ to the unique $d \in M$ such that $(c, d) \in Z$. Clearly, $f(a) = b$, and $f$ is $\emptyset$-definable, since $Z$ is. ■

Since the set of $\emptyset$-definable functions is closed under compositions, we easily obtain the analog of Proposition 5.3.1 for dcl.

**Corollary 5.3.3** Let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $A, B \subseteq M$ and $a \in M$. Then

1. $A \subseteq \operatorname{dcl}_{\mathscr{M}}(A)$ and $\operatorname{dcl}_{\mathscr{M}}(\operatorname{dcl}_{\mathscr{M}}(A)) = \operatorname{dcl}_{\mathscr{M}}(A)$,
2. if $A \subseteq B$, then $\operatorname{dcl}_{\mathscr{M}}(A) \subseteq \operatorname{dcl}_{\mathscr{M}}(B)$,
3. if $a \in \operatorname{dcl}_{\mathscr{M}}(A)$, then there is a finite set $F \subseteq A$ such that $a \in \operatorname{dcl}_{\mathscr{M}}(F)$.

**Proposition 5.3.4** Let $T$ be a universal model complete theory, let $\mathscr{M} \models T$, and let $A \subseteq M$. Then $\langle A \rangle_{\mathscr{M}} = \operatorname{dcl}_{\mathscr{M}}(A)$.

*Proof.* It is clear that $\langle A \rangle_{\mathscr{M}} \subseteq \operatorname{dcl}_{\mathscr{M}}(A)$. Since $T$ is universal theory, $\langle A \rangle_{\mathscr{M}}$ is a model of $T$. Since $T$ is model complete, $\langle A \rangle_{\mathscr{M}} \preceq \mathscr{M}$. Thus for every function $f : M^n \to M$ that is $\emptyset$-definable in $\mathscr{M}$, $f(\langle A \rangle^n_{\mathscr{M}}) \subseteq \langle A \rangle_{\mathscr{M}}$. Hence by Lemma 5.3.2,

$$\operatorname{dcl}_{\mathscr{M}}(A) \subseteq \operatorname{dcl}_{\mathscr{M}}(\langle A \rangle_{\mathscr{M}}) \subseteq \langle A \rangle_{\mathscr{M}}.$$

■

### 5.3.1 Pregeometries

**Definition 5.3.3** Let $X$ be a set and let $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$. The pair $(X, \mathrm{cl})$ is a **pregeometry** (or: **combinatorical pregeometry**, or: **finitary matroid**) if for all $A, B \subseteq X$ and for all $a, b \in X$

1. $A \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
2. if $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$,
3. if $a \in \mathrm{cl}(A)$, then there is a finite set $F \subseteq A$ such that $a \in \mathrm{cl}(F)$,
4. if $b \in \mathrm{cl}(A \cup \{a\}) \setminus \mathrm{cl}(A)$, then $a \in \mathrm{cl}(A \cup \{b\})$.

A pregeometry $(X, \mathrm{cl})$ is a **geometry** if $\mathrm{cl}(\emptyset) = \emptyset$ and $\mathrm{cl}(\{a\}) = \{a\}$ for all $a \in X$.

**Definition 5.3.4** Let $(X, \mathrm{cl})$ be a pregeometry, and let $A, B \subseteq X$. We say $A$ is **independent** (or: **cl-independent**) if $a \notin \mathrm{cl}(A \setminus \{a\})$ for every $a \in A$. We say $A$ is a **basis** (or: **cl-basis**) of $B$ if $A \subseteq B$, $A$ is independent, and $B \subseteq \mathrm{cl}(A)$.

**Lemma 5.3.5** Let $(X, \mathrm{cl})$ be a pregeometry and $A$ is independent. If $b \in X \setminus \mathrm{cl}(A)$, then $A \cup \{b\}$ is independent.

*Proof.* Let $a \in A$ and suppose towards a contradiction that $a \in \mathrm{cl}((A \setminus \{a\}) \cup \{b\})$. Since $A$ is independent and $a \notin \mathrm{cl}(A \setminus \{a\})$, we have $b \in \mathrm{cl}(A)$. A contradiction. ■

**Theorem 5.3.6** Let $(X, \mathrm{cl})$ be a pregeometry and $A, B, Y \subseteq X$. If both $A$ and $B$ are a basis of $Y$, then $|A| = |B|$.

*Proof.* Let $B$ be a basis of $Y$. We show that whenever $A \subseteq \mathrm{cl}(B)$ is independent, then $|A| \leq |B|$.

Suppose $B$ is finite. Let $n \in \mathbb{N}$ be such that $|B| = n$. Towards a contradiction, suppose that there are distinct $a_1, \ldots, a_{n+1} \in A$. Note that $a_1 \notin \mathrm{cl}(\{a_2, \ldots, a_{n+1}\})$, and hence $\mathrm{cl}(\{a_2, \ldots, a_{n+1}\}) \neq Y$. Since $B$ is a basis and by property 2 of pregeometries, there is $b_1 \in B$ such that $b_1 \notin \mathrm{cl}(\{a_2, \ldots, a_{n+1}\})$. By property of pregeometries,

$$\mathrm{cl}(\{b_1, a_2, \ldots, a_{n+1}\}) \supseteq \mathrm{cl}(\{a_1, \ldots, a_{n+1}\}) = Y.$$

Furthermore, $\{b_1, a_2, \ldots, a_{n+1}\}$ is independent by Lemma 5.3.5. Inductively, we can replace $a_2, \ldots, a_n$ by $b_2, \ldots, b_n \in B$ such that $\{b_1, \ldots, b_n, a_{n+1}\}$ is independent. However, $B = \{b_1, \ldots, b_n\}$ and $\mathrm{cl}(B) = Y$. Hence $a_{n+1} \in \mathrm{cl}(\{b_1, \ldots, b_n\})$, contradicting independence.

Suppose that $B$ is infinite. We show that $|A| \leq |B|$. Note that for every finite subset $B_0 \subseteq B$ the set $A \cap \mathrm{cl}(B_0)$ is finite by the above argument. Since $B$ is infinite, we have

$$|A| \leq |\bigcup_{B_0 \subseteq B \text{ finite}} A \cap \mathrm{cl}(B_0)| = |B|.$$

■

The **dimension** of $Y$ with respect to cl (written: $\dim_{\mathrm{cl}}(Y)$ or just $\dim(Y)$) is the cardinality of a basis of $Y$.

**Definition 5.3.5** Let $(X,\mathrm{cl})$ be a pregeometry, and let $A \subseteq X$. We call $\mathrm{cl}_A : \mathscr{P}(X) \to \mathscr{P}(X)$ mapping $B \subseteq X$ to $\mathrm{cl}(A \cup B)$ the **localization of $\mathrm{cl}$ to $A$**.

It is easy to check that if $(X,\mathrm{cl})$ is a pregeometry, then every localization is a pregeometry as well. For $A, B \subseteq X$, the **dimension of $B$ over $A$** (written: $\dim_{\mathrm{cl}}(B/A)$, or just: $\dim(B/A)$)) is just $\dim_{\mathrm{cl}_A}(B)$. Sometimes we will use the following abbreviation: if $b = (b_1, \ldots, b_n) \in X^n$, then we write $\dim(b/A)$ for $\dim(\{b_1, \ldots, b_n\}/A)$.

**Theorem 5.3.7** Let $\mathscr{M}$ be a minimal $\mathscr{L}$-structure. Then $\mathrm{acl}_{\mathscr{M}}$ is a pregeometry.

*Proof.* We just need to show that property 4 of a pregeometry is satisfied. Let $A \subseteq M$ and let $a, b \in M$. Suppose that $b \in \mathrm{acl}_{\mathscr{M}}(A \cup \{a\}) \setminus \mathrm{acl}_{\mathscr{M}}(A)$. Let $\varphi(x, y, z_1, \ldots, z_m)$ be an $\mathscr{L}$-formula and let $c = (c_1, \ldots, c_m) \in A^m$ be such that $\mathscr{M} \models \varphi(a, b, c)$ and $\{d \in M \ : \ \mathscr{M} \models \varphi(a, d, c)\}$ is finite. Let $n$ be the cardinality of this set, and let $\psi(x, z_1, \ldots, z_m)$ be the $\mathscr{L}$-formula such that for all $u \in M$ and $v \in M^m$

$$\mathscr{M} \models \psi(u, v) \text{ if and only if } |\{d \in M \ : \ \mathscr{M} \models \varphi(u, d, v)\}| = n.$$

We now show that $a \in \mathrm{acl}_M(A \cup \{b\})$. If

$$\{u \in M \ : \ \mathscr{M} \models \varphi(u, b, c) \wedge \psi(u, c)\}$$

is finite, we are done. Towards a contradiction, assume that this set is infinite, and hence cofinite. Let $\ell \in \mathbb{N}$ be such that

$$\ell = |M \setminus \{u \in M \ : \ \mathscr{M} \models \varphi(u, b, c) \wedge \psi(u, c)\}|.$$

Let $\chi(y, z_1, \ldots, z_m)$ be an $\mathscr{L}$-formula such that for all $v \in \mathscr{M}^m$ and $w \in \mathscr{M}$,

$$\mathscr{M} \models \chi(w, v) \text{ if and only if } |M \setminus \{u \in M \ : \ \mathscr{M} \models \varphi(u, w, v) \wedge \psi(u, v)\}| = \ell.$$

Note that $\mathscr{M} \models \chi(b, c)$. Since $b \notin \mathrm{acl}_{\mathscr{M}}(A)$, the set $\{w \in M \ : \ \mathscr{M} \models \chi(w, c)\}$ has to be infinite. Let $b_1, \ldots, b_{n+1} \in M$ be distinct such that $\mathscr{M} \models \chi(b_i, c)$ for $i = 1, \ldots, n+1$. For $i = 1, \ldots, n$, set

$$B_i := \{w \in M \ : \ \mathscr{M} \models \varphi(w, b_i, c) \wedge \psi(w, c)\}.$$

Since $\mathscr{M} \models \chi(b_i, c)$, we know that $B_i$ is cofinite. Thus the intersection $\bigcap_{i=1}^{n+1} B_i$ is non-empty. Let $e$ be in this intersection. Then $\mathscr{M} \models \varphi(e, b_i, c)$ for $i = 1, \ldots, n+1$ and thus

$$|\{d \in M : \mathscr{M} \models \varphi(d, e, c)\}| > n.$$

This contradicts $\mathscr{M} \models \psi(e, c)$. ∎

**Corollary 5.3.8** Let $T$ be strongly minimal. Then $\mathrm{acl}_{\mathscr{M}}$ is a pregeometry for every $\mathscr{M} \models T$.

■ **Example 5.4** Let $T$ be a strongly minimal theory, let $\mathscr{M} \models T$ and let $A \subseteq M$. Then

1. If $T = \mathrm{ACF}$, then the $\mathrm{acl}_{\mathscr{M}}$-dimension is the transcendence degree of $A$.
2. If $T = T_{\mathrm{VS}}(K)$ for some field $K$, then the $\mathrm{acl}_{\mathscr{M}}$-dimension is the $K$-linear dimension of $A$.
3. If $T = T_\infty$, then the $\mathrm{acl}_{\mathscr{M}}$-dimension of $A$ is the cardinality of $A$.

■

**Exercise 5.5** Recall from Exercise 4.11 that the theory of $(\mathbb{Z}, s)$, where $s(x) = x+1$, is strongly minimal. Let $\mathscr{M}$ be a model of the theory and $A \subseteq M$. Describe $\mathrm{acl}_{\mathscr{M}}(A)$ and $\dim_{\mathscr{M}}(A)$.

**Exercise 5.6** Let $\mathscr{M} \models \mathrm{DLO}$. Show that $\mathrm{acl}_{\mathscr{M}}$ is a pregeometry.

**Exercise 5.7** Consider the structure $\mathscr{M} := (\mathbb{R}^2, f)$, where $f : \mathbb{R}^2 \to \mathbb{R}^2$ maps $(m,n)$ to $(m,0)$. Show that $\mathrm{acl}_{\mathscr{M}}$ is not a pregeometry.

**Proposition 5.3.9** Let $T$ be a strongly minimal theory, and let $\mathscr{M}, \mathscr{N}_1, \mathscr{N}_2 \models T$ such that $\mathscr{M} \preceq \mathscr{N}_1$ and $\mathscr{M} \preceq \mathscr{N}_2$. Let $A \subseteq M$ and let $b = (b_1, \ldots, b_n) \in N_1^n$ and $c = (c_1, \ldots, c_n) \in N_2^n$. If $\{b_1, \ldots, b_n\}$ is $\mathrm{acl}_{\mathscr{N}_1}$-independent over $A$ and $\{c_1, \ldots, c_n\}$ is $\mathrm{acl}_{\mathscr{N}_2}$-independent over $A$, then $\mathrm{tp}^{\mathscr{N}_1}(b|A) = \mathrm{tp}^{\mathscr{N}_2}(c|A)$.

*Proof.* We proceed by induction on $n$. First consider the case that $n = 1$. Let $\varphi(x_1)$ be an $\mathscr{L}(A)$-formula such that $\mathscr{N}_1 \models \varphi(b)$. Since $b \notin \mathrm{acl}_{\mathscr{N}_1}(A)$, we have that $\varphi(\mathscr{N}_1)$ is infinite. Since $T$ is strongly minimal, $N_1 \setminus \varphi(\mathscr{N}_1)$ is finite. Since both $\mathscr{N}_1$ and $\mathscr{N}_2$ are elementary extensions of $\mathscr{M}$, we get that $N_2 \setminus \varphi(\mathscr{N}_2)$ is finite as well. Because $c \notin \mathrm{acl}_{\mathscr{N}_2}(A)$, we have that $\mathscr{N}_2 \models \varphi(c)$.

Let $n > 1$. By induction, we have

$$\mathrm{tp}^{\mathscr{N}_1}(b_2, \ldots, b_{n-1}) = \mathrm{tp}^{\mathscr{N}_2}(c_2, \ldots, c_{n-1}). \tag{5.3}$$

Let $\varphi(x_1, \ldots, x_n)$ be an $\mathscr{L}(A)$-formula such that $\mathscr{N}_1 \models \varphi(b)$. Since $b_1 \notin \mathrm{acl}_{\mathscr{N}_1}(A \cup \{b_2, \ldots, b_n\})$, the set $\varphi(\mathscr{N}_1, b_2, \ldots, b_n)$ is infinite. Thus, since $T$ is strongly minimal, $N_1 \setminus \varphi(\mathscr{N}_1, b_2, \ldots, b_n)$ is finite. By (5.3), $N_2 \setminus \varphi(\mathscr{N}_2, c_2, \ldots, c_n)$ is finite. Since $c_1 \notin \mathrm{acl}_{\mathscr{N}_2}(A \cup \{c_2, \ldots, c_n\})$, we get $\mathscr{N}_2 \models \varphi(c)$. ■

**Exercise 5.8** Let $T$ be a countable strongly minimal $\mathscr{L}$-theory, let $\mathscr{M} \models T$ and let $A \subseteq M$ be countable. Show that $S_n^{\mathscr{M}}(A)$ is countable.

## 5.4 Morley rank

We introduce a new notation that will be convenient for this section. If $\mathscr{M}$ is an $\mathscr{L}$-structure and $\varphi(x_1, \ldots, x_n)$ is a $\mathscr{L}(M)$-formula, then we set

$$\varphi(\mathscr{M}) := \{a \in M^n \ : \ \mathscr{M} \models \varphi(a)\}.$$

**Definition 5.4.1** Let $\mathscr{M}$ be an $\mathscr{L}$-structure, let $\varphi(x)$ be a $\mathscr{L}(M)$-formula and let $\alpha$ be an ordinal. We define $\mathrm{RM}^{\mathscr{M}}(\varphi) \geq \alpha$ recursively as follows:

1. $\mathrm{RM}^{\mathscr{M}}(\varphi) \geq 0$ if and only if $\varphi(\mathscr{M})$ is non-empty.
2. if $\alpha = \beta + 1$ for some ordinal $\beta$:$\mathrm{RM}^{\mathscr{M}}(\varphi) \geq \alpha$ if and only if there is family $(\psi_i(x))_{i \in \mathbb{N}}$ of $\mathscr{L}(M)$-formulas such that for all $i, j \in \mathbb{N}$
   (a) $\mathrm{RM}^{\mathscr{M}}(\psi_i) \geq \beta$,
   (b) $\psi_i(\mathscr{M}) \cap \psi_j(\mathscr{M}) = \emptyset$ whenever $i \neq j$,
   (c) $\psi_i(\mathscr{M}) \subseteq \varphi(\mathscr{M})$.

3. if $\alpha$ is a limit ordinal:

$$\mathrm{RM}^{\mathscr{M}}(\varphi) \geq \alpha \text{ if and only if } \mathrm{RM}^{\mathscr{M}}(\varphi) \geq \beta \text{ for all ordinals } \beta < \alpha.$$

If $\varphi(\mathscr{M})$ is non-empty, then we define the **Morley rank of $\varphi$ in $\mathscr{M}$** (written: $\mathrm{RM}^{\mathscr{M}}(\varphi)$) is the maximal ordinal $\alpha$ such that $\mathrm{RM}^{\mathscr{M}}(\varphi) \geq \alpha$, if such exists, and $\infty$ otherwise. If $\varphi(\mathscr{M})$ is empty, we set $\mathrm{RM}^{\mathscr{M}}(\varphi)$ to be $-1$.

**Lemma 5.4.1** Let $\mathscr{M}$ be an $\aleph_0$-saturated $\mathscr{L}$-structure, let $a,b \in M^n$, and let $\varphi(x,y)$ be $\mathscr{L}$-formula where $x = (x_1,\dots,x_m)$ and $y = (y_1,\dots,y_n)$. If $\mathrm{tp}^{\mathscr{M}}(a) = \mathrm{tp}^{\mathscr{M}}(b)$, then

$$\mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) = \mathrm{RM}^{\mathscr{M}}(\varphi(x,b)).$$

*Proof.* Let $\alpha$ be an ordinal. We show by induction on $\alpha$ that for all $a,b \in M^n$ with $\mathrm{tp}^{\mathscr{M}}(a) = \mathrm{tp}^{\mathscr{M}}(b)$

$$\mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) \geq \alpha \text{ if and only if } \mathrm{RM}^{\mathscr{M}}(\varphi(x,b)) \geq \alpha.$$

Observe that if $a,b \in M^n$ with $\mathrm{tp}^{\mathscr{M}}(a) = \mathrm{tp}^{\mathscr{M}}(b)$, then $\varphi(\mathscr{M},a)$ is non-empty if and only $\varphi(\mathscr{M},b)$ is non-empty. Thus the desired statement holds for $\alpha = 0$.

Now let $\alpha$ be a limit ordinal and suppose that the statement holds for all ordinals $\beta < \alpha$. Let $a,b \in M^n$ be such that $\mathrm{tp}^{\mathscr{M}}(a) = \mathrm{tp}^{\mathscr{M}}(b)$. Then

$$\begin{aligned}\mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) \geq \alpha &\text{ if and only if } \mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) \geq \beta \text{for all ordinals } \beta < \alpha\\ &\text{ if and only if } \mathrm{RM}^{\mathscr{M}}(\varphi(x,b)) \geq \beta \text{for all ordinals } \beta < \alpha\\ &\text{ if and only if } \mathrm{RM}^{\mathscr{M}}(\varphi(x,b)) \geq \alpha.\end{aligned}$$

Let $\alpha$ be a successor ordinal and $\beta$ be an ordinal such that $\alpha = \beta + 1$. Let $a,b \in M^n$ be such that $\mathrm{tp}^{\mathscr{M}}(a) = \mathrm{tp}^{\mathscr{M}}(b)$. Suppose that $\mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) \geq \alpha$. By symmetry, it is enough to show that $\mathrm{RM}^{\mathscr{M}}(\varphi(x,b)) \geq \alpha$. Let $(\psi_i(x))_{i\in\mathbb{N}}$ be a family of $\mathscr{L}(M)$-formulas of Morley rank at least $\beta$ in $\mathscr{M}$ that witness $\mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) \geq \alpha$. Let $(a_i)_{i\in\mathbb{N}}$ be a family of tuples of elements of $M$ and $(\theta_i(x,y_i))_{i\in\mathbb{N}}$ be a family of $\mathscr{L}$-formulas such that $\psi_i(x)$ is $\theta_i(x,a_i)$. Thus $\psi_i(\mathscr{M}) = \theta_i(\mathscr{M},a_i)$ for each $i \in \mathbb{N}$. Since $\mathscr{M}$ is $\aleph_0$-saturated, there is a family $(b_i)_{i\in\mathbb{N}}$ of tuples of elements of $M$ such that for each $i \in \mathbb{N}$

$$\mathrm{tp}^{\mathscr{M}}(a,a_1,\dots,a_i) = \mathrm{tp}^{\mathscr{M}}(b,b_1,\dots,b_i). \tag{$*$}$$

By our induction hypothesis and $(*)$, we have for each $i \in \mathbb{N}$ that $\mathrm{RM}(\theta_i(\mathscr{M}),b_i) \geq \beta$. Since $(\psi_i(x))_{i\in\mathbb{N}}$ that witness $\mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) \geq \alpha$, it follows easily from $(*)$ that $((\theta_i(\mathscr{M}),b_i))_{i\in\mathbb{N}}$ witnesses $\mathrm{RM}^{\mathscr{M}}(\varphi(x,b)) \geq \alpha$. ∎

**Lemma 5.4.2** Let $T$ be a complete $\mathscr{L}$-theory with infinite models, let $\mathscr{M},\mathscr{N} \models T$ be $\aleph_0$-saturated with $\mathscr{M} \preceq \mathscr{N}$, and let $\varphi(x_1,\dots,x_n)$ be an $\mathscr{L}(M)$-formula. Then

$$\mathrm{RM}^{\mathscr{M}}(\varphi) = \mathrm{RM}^{\mathscr{N}}(\varphi).$$

*Proof.* Let $\alpha$ be an ordinal. We show by induction on $\alpha$ that for all $a \in M^n$ and all $\mathscr{L}$-formulas $\varphi(x,y)$

$$\mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) \geq \alpha \text{ if and only if } \mathrm{RM}^{\mathscr{N}}(\varphi(x,a)) \geq \alpha.$$

First consider the case $\alpha = 0$. Since $\mathscr{M} \preceq \mathscr{N}$, we know that $\varphi(\mathscr{M},a)$ is non-empty if and only if $\varphi(\mathscr{N},a)$. Thus $\mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) \geq 0$ if and only if $\mathrm{RM}^{\mathscr{N}}(\varphi(x,a)) \geq 0$.
When $\alpha$ is a limit ordinal, the statement follows by induction as in the proof of Lemma 5.4.1. So now let $\alpha$ be a successor ordinal such that $\alpha = \beta + 1$ and the desired statement holds for $\beta$. Let $a \in \mathscr{M}^n$ and let $\varphi(x,y)$ be an $\mathscr{L}$-formula.
Suppose that $\mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) \geq \alpha$. Then there is a family $(\psi_i(x))_{i\in\mathbb{N}}$ of $\mathscr{L}(M)$-formulas of Morley rank at least $\beta$ in $\mathscr{M}$ that witness $\mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) \geq \alpha$. By the induction hypothesis for $\beta$, we have that $\mathrm{RM}^{\mathscr{N}}(\psi_i(x,a)) \geq \beta$ for $i \in \mathbb{N}$. Since $\mathscr{M} \preceq \mathscr{N}$, it follows easily that $(\psi_i(x))_{i\in\mathbb{N}}$ witnesses $\mathrm{RM}^{\mathscr{N}}(\varphi(x,a)) \geq \alpha$.
Suppose that $\mathrm{RM}^{\mathscr{N}}(\varphi(x,a)) \geq \alpha$. Then there is a family $(\psi_i(x))_{i\in\mathbb{N}}$ of $\mathscr{L}(M)$-formulas of Morley rank at least $\beta$ in $\mathscr{M}$ that witness $\mathrm{RM}^{\mathscr{M}}(\varphi(x,a)) \geq \alpha$. Let $(a_i)_{i\in\mathbb{N}}$ be a family of tuples of elements of $N$ and $(\theta_i(x,y_i))_{i\in\mathbb{N}}$ be a family of $\mathscr{L}$-formulas such that $\psi_i(x)$ is $\theta_i(x,a_i)$. Thus $\psi_i(\mathscr{N}) = \theta_i(\mathscr{N},a_i)$ for each $i \in \mathbb{N}$. Since $\mathscr{M}$ is $\aleph_0$-saturated, there is a family of tuples of elements of $M$ such that for every $i \in \mathbb{N}$

$$\mathrm{tp}^{\mathscr{N}}(a,a_1,\ldots,a_i) = \mathrm{tp}^{\mathscr{M}}(a,b_1,\ldots,b_i). \tag{5.4}$$

By Lemma 5.4.1, we know that $\mathrm{RM}^{\mathscr{N}}(\theta_i(x,b_i)) \geq \beta$. By our induction hypothesis, $\mathrm{RM}^{\mathscr{M}}(\theta_i(x,b_i)) \geq \beta$. It follows from (5.4) that the $(\theta_i(x,b))_{i\in\mathbb{N}}$ witnesses $\mathrm{RM}^{M}(\varphi(x,a)) \geq \alpha$. ■

**Corollary 5.4.3** Let $\mathscr{M}$ be an $\mathscr{L}$-structure, let $\mathscr{N}_0, \mathscr{N}_1$ be $\aleph_0$-saturated elementary extensions of $\mathscr{M}$, and let $\varphi(x_1\ldots,x_n)$ be an $\mathscr{L}(M)$-formula. Then

$$\mathrm{RM}^{\mathscr{N}_0}(\varphi) = \mathrm{RM}^{\mathscr{N}_1}(\varphi).$$

*Proof.* Let $\mathscr{N}_2$ be a elementary expansion of both $\mathscr{N}_0$ and $\mathscr{N}_1$. Set $\mathscr{N}_3$ be an $\aleph_0$-saturated elementary extensions of $\mathscr{N}_2$. Then by Lemma 5.4.2,

$$\mathrm{RM}^{\mathscr{N}_0}(\varphi) = \mathrm{RM}^{\mathscr{N}_3}(\varphi) = \mathrm{RM}^{\mathscr{N}_1}(\varphi).$$

■

**Definition 5.4.2** Let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $\varphi$ be an $\mathscr{L}(M)$-formula. The **Morley rank of $\varphi$** (written: $\mathrm{RM}(\varphi)$) is $\mathrm{RM}^{\mathscr{N}}(\varphi)$, where $\mathscr{N}$ is some $\omega$-saturated elementary extension of $\mathscr{M}$. Let $T$ be an complete theory with $\mathscr{M} \models T$, and let $X \subseteq M^n$ be definable by some $\mathscr{L}(M)$-formula $\varphi$. The **Morley rank of $X$** (written: $\mathrm{RM}(X)$) is $\mathrm{RM}(\varphi)$.

**Proposition 5.4.4** Let $\mathscr{M}$ be an $\mathscr{L}$-structure, and let $X,Y \subseteq M^n$ be definable in $\mathscr{M}$. Then
1. If $X \subseteq Y$, then $\mathrm{RM}(X) \leq \mathrm{RM}(Y)$.
2. $\mathrm{RM}(X \cup Y) = \max\{\mathrm{RM}(X),\mathrm{RM}(Y)\}$.
3. If $X \neq \emptyset$, then $\mathrm{RM}(X) = 0$ if and only if $X$ is finite.

*Proof.* Statement 1. follows directly from the definition of Morley rank. Now consider Statement 2. First note by 1. that $\mathrm{RM}(X) \leq \mathrm{RM}(X \cup Y)$ and $\mathrm{RM}(Y) \leq \mathrm{RM}(X \cup Y)$. Thus it is only left to show that $\mathrm{RM}(X \cup Y) \leq \max\{\mathrm{RM}(X), \mathrm{RM}(Y)\}$. We now prove by induction that for every ordinal $\alpha$ for all set $X, Y \subseteq M^n$ definable in $\mathscr{M}$

if $\mathrm{RM}(X \cup Y) \geq \alpha$, then $\mathrm{RM}(X) \geq \alpha$ or $\mathrm{RM}(Y) \geq \alpha$.

The case of $\alpha = 0$ can be checked easily. Now suppose that $\alpha = \beta + 1$ and that the induction hypothesis holds for $\beta$. Assume that $\mathrm{RM}(X \cup Y) \geq \alpha$. Then there is a family $(\varphi_i(x))_{i\in\mathbb{N}}$ of $\mathscr{L}(M)$-formulas of Morley rank at least $\beta$ in $\mathscr{M}$ that witness $\mathrm{RM}^{\mathscr{M}}(X \cup Y) \geq \alpha$. Note that

$$(\varphi_i(\mathscr{M}) \cap X) \cup (\varphi_i(\mathscr{M}) \cap Y) = \varphi_i(\mathscr{M})$$

Thus either $\mathrm{RM}(\varphi_i(\mathscr{M}) \cap X) \geq \beta$ or $\mathrm{RM}(\varphi_i(\mathscr{M}) \cap Y)$. Hence there is an infinite subset $I \subseteq \mathbb{N}$ such that either

- $\mathrm{RM}(\varphi_i(\mathscr{M}) \cap X) \geq \beta$ for every $i \in I$ or
- $\mathrm{RM}(\varphi_i(\mathscr{M}) \cap Y) \geq \beta$ for every $i \in I$.

It follows that either $\mathrm{RM}(X) \geq \alpha$ or $\mathrm{RM}(Y) \geq \alpha$. The case when $\alpha$ is a limit ordinal can be handled similarly.

Now consider Statement 3. Suppose that $X \neq \emptyset$. Let $\varphi(x)$ be an $\mathscr{L}(M)$-formula such that $X = \varphi(\mathscr{M})$. If $\mathrm{RM}(X) \geq 1$, then there is an elementary extensions $\mathscr{N}$ of $\mathscr{M}$ such that $\varphi(\mathscr{N})$ is infinite. Since $\mathscr{M} \preceq \mathscr{N}$, we also get that $\varphi(\mathscr{M})$ is infinite. Now assume that $X$ is infinite. Let $(a_i)_{i\in\mathbb{N}}$ be an infinite family of distinct elements of $X$ and let $\psi(x,y)$ be the $\mathscr{L}$-formula $x = y$. Let $\mathscr{N}$ be elementary extension of $\mathscr{M}$. Then $(\psi(x, a_i))_{i\in\mathbb{N}}$ is a family of $\mathscr{L}(N)$-formulas of Morley rank at least 0 in $\mathscr{N}$ that witnesses $\mathrm{RM}^{\mathscr{N}}(\varphi(x)) \geq 1$. Thus $\mathrm{RM}(X) \geq 1$. ■

**Corollary 5.4.5** Let $T$ be strongly minimal and let $\mathscr{M} \models T$. Then $\mathrm{RM}(M) = 1$.

*Proof.* Since $\mathscr{M}$ is infinite, we have by Proposition 5.4.4(3) that $\mathrm{RM}(M) \geq 1$. Now suppose there two infinite subsets $X, Y$ of $M$ definable in $\mathscr{M}$. Since $\mathscr{M}$ is strongly minimal, they can not be disjoint. Thus, using again Proposition 5.4.4(3), we obtain $\mathrm{RM}(M) \leq 1$. ■

**Exercise 5.9** Let $T_{\infty,2}$ be the theory defined in Exercise 4.5, and let $\mathscr{M} \models T_{\infty,2}$. What is $\mathrm{RM}(M)$?

**Exercise 5.10** Let $\mathscr{L}$ be the language consisting of all single binary relation symbol $E$. Let $T_{\mathrm{equiv},\infty,\infty}$ be the $\mathscr{L}$-theory stating that $E$ is an equivalence relation with infinitely many classes each of which is infinite. Let $\mathscr{M} \models T_{\mathrm{equiv},\infty,\infty}$. Show that $\mathrm{RM}(M) = 2$.

**Definition 5.4.3** Let $T$ be a complete $\mathscr{L}$-theory. We say $T$ is **totally transcendental** if $\mathrm{RM}(\varphi) < \infty$ for every $\mathscr{M} \models T$ and every $\mathscr{L}(M)$-formula $\varphi$.

■ **Example 5.5** DLO is not totally transcendental. ■

■ **Example 5.6** Let $\mathrm{DCF}_0$ be the model companion of the theory of differential fields of characteristic 0. This theory is totally transcendental, but not strongly minimal. ■

**Definition 5.4.4** Let $\mathscr{M}$ be an $\mathscr{L}$-structure, and let $p(x) \in S_n^{\mathscr{M}}(A)$. The **Morley rank of $p$** (written: $\mathrm{RM}(p(x))$) is defined as $\inf\{\mathrm{RM}(\varphi(x)) \ : \ \varphi \in p\}$. For $a \in M^n$, we set $\mathrm{RM}(a/A)$ to be $\mathrm{RM}(\mathrm{tp}^{\mathscr{M}}(a|A))$.

**Exercise 5.11** Let $\mathscr{M}$ be an $\aleph_0$-saturated $\mathscr{L}$-structure and let $\mathscr{N}$ be an $|M|^+$-saturated elementary extension of $\mathscr{M}$, let $A \subseteq M^n$ and let $X \subseteq M^n$ be definable using the $\mathscr{L}(A)$-formula $\varphi$. Show that

$$\mathrm{RM}(X) = \sup\{\mathrm{RM}(a/A) \ : \ a \in N^n, \mathscr{N} \models \varphi(a)\}$$

**Lemma 5.4.6** Let $\mathscr{M}$ be an $\mathscr{L}$-structure, let $A \subseteq M$, let $a \in M^n$ and let $b \in M$. If $b \in \mathrm{acl}_{\mathscr{M}}(A \cup \{a_1, \ldots, a_n\})$, then $\mathrm{RM}(a,b/A) = \mathrm{RM}(a/A)$.

*Proof.* Without loss of generality, we can assume that $\mathscr{M}$ is $|A|$-saturated. We show by induction on $\alpha$: suppose $b \in \mathrm{acl}_{\mathscr{M}}(A \cup \{a_1, \ldots, a_n\})$. Then:

$$\text{if } \mathrm{RM}(a,b/A) \geq \alpha, \text{ then } \mathrm{RM}(a/A) \geq \alpha.$$

It is easy to check $\mathrm{RM}(a/A) \geq 0$. Thus the statement holds for $\alpha = 0$. When $\alpha$ is a limit ordinal, then statement follows immediately from the induction hypothesis and the definition of Morley rank. So let $\alpha = \beta + 1$ and suppose the statement holds for $\beta$. Assume that $\mathrm{RM}(a,b/A) \geq \alpha$. By induction, $\mathrm{RM}(a/A) \geq \beta$. Towards a contradiction, suppose that $\mathrm{RM}(a/A) = \beta$. Then we can find a $\mathscr{L}(A)$-formula $\varphi(x_1, \ldots, x_n)$ such that

1. $\mathscr{M} \models \varphi(a)$,
2. $\mathrm{RM}(\varphi) = \beta$, and
3. there is no $\mathscr{L}(A)$-formula $\psi(x_1, \ldots, x_n)$ such that $\mathrm{RM}(\varphi \wedge \psi) = \mathrm{RM}(\varphi \wedge \neg\psi) = \beta$.

Since $b \in \mathrm{acl}_{\mathscr{M}}(A \cup \{a_1, \ldots, a_n\})$, there is a $\mathscr{L}(A)$-formula $\theta(x_1, \ldots, x_n, y)$ and $\ell \in \mathbb{N}$ such that $\mathscr{M} \models \theta(a,b)$ and

$$|\{c \in M \ : \ \mathscr{M} \models \theta(a,c)\}| = \ell.$$

Let $\xi(x_1, \ldots, x_n, y)$ be the $\mathscr{L}(A)$-formula such that for all $d \in M^n, e \in M$, $\mathscr{M} \models \xi(c,d)$ if and only if

- $\mathscr{M} \models \varphi(d) \wedge \theta(d,e)$, and
- $|\{c \in M \ : \ \mathscr{M} \models \theta(d,c)\}| = \ell$.

Since $\mathscr{M} \models \xi(a,b)$ and $\mathrm{RM}(a,b) > \beta$, we have that $\mathrm{RM}(\chi) > \beta$. Let $(\chi_i)_{i \in \mathbb{N}}$ be family of $\mathscr{L}(M)$-formulas of Morley rank at least $\beta$ that witness that $\mathrm{RM}(\xi) \geq \alpha$. Now let $\lambda_i$ be the $\mathscr{L}(M)$-formula $\exists y \chi_i$.

We now show that for $i \in \mathbb{N}$

$$\mathrm{RM}(\lambda_1 \wedge \cdots \wedge \lambda_i) \geq \beta. \tag{5.5}$$

Because $\mathrm{RM}(\chi_i) \geq \alpha$, there is $d \in M^n$ and $e \in M$ such that $\mathscr{M} \models \chi_i(d,e)$ and $\mathrm{RM}(d,e) \geq \alpha$. By induction, $\mathrm{RM}(d) \geq \beta$. Because $\mathscr{M} \models \lambda_i(d)$, we have that $\mathrm{RM}(\lambda_i) \geq \beta$.. Towards a contradiction, suppose that there is $i \in \mathbb{N}$ such that (5.5) fails. Let $i$ be the minimal such element of $\mathbb{N}$. Then

$$\mathrm{RM}(\lambda_1 \wedge \cdots \wedge \lambda_{i-1}) = \mathrm{RM}\left(\lambda_i \wedge \neg(\bigwedge_{j=1}^{i-1} \lambda_i)\right) \geq \beta.$$

This contradicts property 3. of $\varphi$.

Because $\mathcal{M}$ is $|A|$-saturated, there is $d \in M^n$ such that $\mathcal{M} \models \lambda_i(d)$ for all $i \in \mathbb{N}$. For $i \in \mathbb{N}$, let $e_i \in M$ be such that $\mathcal{M} \models \xi_i(d, e_i)$. Since $\chi_i(\mathcal{M}) \cap \chi_j(\mathcal{M}) = \emptyset$ for $i \neq j$, we have that $e_i \neq e_j$ whenever $i \neq j$. Thus the set $\{c \in M \ : \ \mathcal{M} \models \theta(d,c\}$ is infinite. This contradicts $\mathcal{M} \models \xi(d, e_1)$.

■

**Theorem 5.4.7** Let $T$ be strongly minimal, let $\mathcal{M} \models T$, let $A \subseteq M$ and $a \in M^n$. Then $\dim(a/A) = \mathrm{RM}(a/A)$.

*Proof.* Let $a = (a_1, \dots, a_k)$ be such that $\dim(\{a_1, \dots, a_k\}/A) = k$. Let $\mathcal{N}$ be an $|M|^+$-saturated elementary extension of $\mathcal{M}$. Let $d_1, \dots, d_k \in N$ be $\mathrm{acl}_{\mathcal{N}}$-independent over $M$. By Proposition 5.3.9, $\mathrm{tp}^{\mathcal{N}}(a|A) = \mathrm{tp}^{\mathcal{N}}(d|A)$. Thus it is enough to show for all $B \subseteq M$ that $\mathrm{RM}(d/B) = k$. We do by induction on $k$.

Suppose $k = 1$ and let $\varphi(x) \in \mathrm{tp}^{\mathcal{N}}(d|B)$. Since $d_1 \notin \mathrm{acl}_{\mathcal{N}}(B)$, we have to $\varphi(\mathcal{N})$ is infinite. Thus by Proposition 5.4.4(3), $\mathrm{RM}(\varphi(x)) \geq 1$. Since $T$ is strongly minimal, we can deduce $\mathrm{RM}(\varphi(x)) \leq 1$ from Corollary 5.4.5 and Proposition 5.4.4(1).

Let $k > 1$. We first show that $\mathrm{RM}(d|B) \geq k$. Let $\varphi(x) \in \mathrm{tp}^{\mathcal{N}}(d|B)$ be such that $\mathrm{RM}(\varphi(x)) = \mathrm{RM}(\mathrm{tp}^{\mathcal{N}}(d|B))$. Let $(b_i)_{i \in \mathbb{N}}$ be a family distinct elements in $M \setminus \mathrm{acl}_{\mathcal{N}}(B)$, let $(\psi_i(x))_{i \in \mathbb{N}}$ be the family of $\mathcal{L}(M)$-formula such that $\psi_i(x)$ is $\varphi(x) \wedge x_1 = b_i$. Clearly, $\psi_i(\mathcal{N}) \cap \psi_j(\mathcal{N}) = \emptyset$ for all $i, j \in \mathbb{N}$ with $i \neq j$. For each $i \in \mathbb{N}$, let $c_i = (c_{i,1}, \dots, c_{i,k-1}) \in N^{k-1}$ such that $\{c_{i,1}, \dots, c_{i,k-1}\}$ is $\mathrm{acl}_{\mathcal{N}}$-independent over $B \cup \{b_i\}$. By Proposition 5.3.9

$$\mathrm{tp}^{\mathcal{N}}((b_i, c_i)|B) = \mathrm{tp}^{\mathcal{N}}(d|B).$$

Hence $\mathcal{M} \models \varphi(b_i, c_i)$ and thus $\mathcal{M} \models \psi_i(b_i, c_i)$ for each $i \in \mathbb{N}$. Then by induction

$$\mathrm{RM}(\psi_i) \geq \mathrm{RM}((b_i, c_i)/B) \geq \mathrm{RM}(c_i/B) = k - 1.$$

Thus the family $(\psi_i(x))_{i \in \mathbb{N}}$ witnesses that $\mathrm{RM}(\varphi(x)) \geq k$.

We show that $\mathrm{RM}(d|B) \leq k$. It is enough to show for all $\mathcal{L}(M)$-formula $\theta(x)$ that if $\mathcal{N} \models \neg\theta(d)$, then $\mathrm{RM}(\theta(x)) < k$. Let $e = (e_1, \dots, e_k) \in N^k$ be such that $\mathcal{N} \models \theta(e)$. Since $\mathcal{N}$ is $|M|$-saturated, then can take $e$ such that $\mathrm{RM}(e|M) = \mathrm{RM}(\theta(x))$. Thus $\mathrm{tp}^{\mathcal{N}}(d) \neq \mathrm{tp}^{\mathcal{N}}(e)$. By Proposition 5.3.9, $e_1, \dots, e_k$ have to be $\mathrm{acl}_{\mathcal{N}}$-dependent over $M$. Without loss of generality, we can assume that $e_k \in \mathrm{acl}_{\mathcal{N}}(M \cup \{e_1, \dots, e_{k-1}\})$. Thus by Lemma 5.4.6

$$\mathrm{RM}(\theta(x)) = \mathrm{RM}(e/M) = \mathrm{RM}(e_1, \dots, e_{k-1}/M) \leq k - 1.$$

■

**Corollary 5.4.8** Let $T$ be strongly minimal. Then $T$ is totally transcendental.

*Proof.* Let $\mathcal{M} \models T$. Then $\mathrm{RM}(M^n) = n$ by Theorem 5.4.7 and Exercise 5.11. The corollary follows from Proposition 5.4.4(1). ■

### 5.4.1 An application to algebraic geometry

**Definition 5.4.5** Let $\mathscr{K} \models T_{\text{fields}}$. Let $V \subseteq K^n$ be an irreducible variety, and let

$$I(V) := \{p \in K[X_1,\ldots,X_n] \ : \ p(a) = 0 \text{ for all } a \in V\}.$$

The **Krull dimension of $V$** is the maximal $m \in \mathbb{N}$ such that there are prime ideals $\mathfrak{p}_0,\ldots,\mathfrak{p}_\mathfrak{m}$ of $K[X_1,\ldots,X_n]$ such that

$$I(V) = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_\mathfrak{m} \subset K[X_1,\ldots,X_n].$$

We will use the fact that the Krull dimension of $V$ is the transcendence degree of the fraction field $K[X_1,\ldots,X_n]/\mathfrak{p}$ over $K$.

**Theorem 5.4.9** Let $K$ be an algebraically closed field, and let $V \subseteq K^n$ be an irreducible variety. Then the Krull dimension of $V$ is equal to $\mathrm{RM}(V)$.

*Proof.* Let $\mathscr{K}$ be an elementary extensions of $K$ that is $|A|$-saturated. Let $L$ be the field generated by $A$. Let $\mathfrak{p}$ be the prime ideal $I(V)$. Then the Krull dimension of $V$ is the transcendence degree of the fraction field $L[X_1,\ldots,X_n]/\mathfrak{p}$ over $L$. By Theorem 5.4.7 we have that $\mathrm{RM}(V) = \max \dim_{\mathrm{acl}_{\mathscr{K}}}(a/A)$. Note that $\dim_{\mathrm{acl}_K}(a/A)$ is just the transcendence degree of the field generated $a$ and $L$ over $L$. Note that $L[X_1,\ldots,X_n]/\mathfrak{p}$ can be embedded into $\mathscr{K}$ such that the image of $(X_1+\mathfrak{p},\ldots,X_n+\mathfrak{p})$ is in $V$. Thus $\mathrm{RM}(V)$ is at least the transcendence degree of the fraction field $L[X_1,\ldots,X_n]/\mathfrak{p}$. Hence $\mathrm{RM}(V)$ is at least the Krull dimension of $V$. Let $a \in V$. There is a surjective $L$-algebra homomorphism $L[X_1,\ldots,X_n]/\mathfrak{p}$ to $L(a_1,\ldots,a_n)$ such that

$$(X_1+\mathfrak{p},\ldots,X_n+\mathfrak{p}) \mapsto a.$$

Then the transcendence degree of $L(a)$ over $L$ is less than or equal to the transcendence degree of $L[X_1,\ldots,X_n]/\mathfrak{p}$ over $L$. ∎

# 6. Tameness and geometric model theory

## 6.1 O-minimality

Let $\mathscr{M}$ be an expansion of $(M,<) \models \mathrm{DLO}$. Recall that $\mathscr{M}$ is o-minimal if every set $X \subseteq M$ definable in $M$ is finite union of intervals and points. If we allow the intervals to be open, closed, and half-open, then the finitely points can be taken to be isolated.

**Lemma 6.1.1** Let $X \subseteq M$ be definable. Then
1. If $X$ is bounded, then both $\inf(X)$ and $\sup(X)$ exist.
2. The boundary of $X$ is finite.

*Proof.* 1. By o-minimality, $X$ is a finite union of intervals and points. Let $A$ be the finite set consisting of the endpoints of these intervals and the finite many isolated points. Take $b$ to be the maximum of $A$ (which exists, since $A$ is finite). Now one can easily check that $b$ is the supremum of $X$.
2. The boundary of a set has empty interior. Thus by o-minimality the boundary of a subset of $M$ has to be finite. ■

**Remark 6.1**
1. It is easy to see that $\mathscr{M}$ is o-minimal if and only every set $X \subseteq M$ definable in $M$ either has interior or is finite.
2. If $M = \mathbb{R}$, the conclusion of Statement 1 of Lemma 6.1.1 holds even without the assumption of o-minimality, because $\mathbb{R}$ is a complete topological space.
3. Let $\mathscr{Q} := (\{0,1\} \times \mathbb{Q} \times, <_{\mathrm{lex}})$, where $<_{\mathrm{lex}}$ is the usual lexicographic order. Note that $\mathscr{Q} \models$ DLO and hence is o-minimal. However, $\{0\} \times \mathbb{Q}$ does not have supremum. Thus the universe of an o-minimal structure does not need to be complete with respect to the order topology.

■

**Definition 6.1.1** Let $X \subseteq M^n$. We say $X$ is **definably connected** if there are no disjoint definable open sets $U_1, U_2 \subseteq M^n$ such that $X \subseteq U_1 \cup U_2$, $U_1 \cap X \neq \emptyset$ and $U_2 \cap X \neq \emptyset$.

**Lemma 6.1.2** Suppose that $\mathscr{M}$ is o-minimal. Let $I \subseteq M$ be an interval. Then $I$ is definably connected.

*Proof.* Let $U_1, U_2$ be open definable sets such that $I \subseteq U_1 \cup U_2$, $U_1 \cap X \neq \emptyset$ and $U_2 \cap X \neq \emptyset$. It is left to show that $U_1 \cap U_2 \neq \emptyset$. Since both $U_1$ and $U_2$ are open, both $U_1 \cap I$ and $U_2 \cap I$ have nonempty interior. Thus there is a maximal open interval $J \subseteq U_1$ such that $J \cap U_1 \neq \emptyset$. Let $a, b \in M$ be such that $J = (a,b)$. Suppose that $b \in I$. Then $b \in I \setminus U_1$ and hence has to be in $U_2$. But $U_2$ is open. Thus there is open interval $J' \subseteq U_2$ containing $b$. Thus this interval $J'$ has nonempty intersection with $J$. Similarly, we can show that if $a \in I$, then $U_1 \cap U_2 \neq \emptyset$. Now we only need to consider the case that both $a$ and $b$ are not in $I$. But then $I \subseteq J$. Hence $U_2 \cap J_2 \neq \emptyset$ and thus $U_1 \cap U_2 \neq \emptyset$. ■

**Lemma 6.1.3** Let $f : X \to M^n$ be definable and continuous and let $X \subseteq M^m$ be definably connected. Then $f(X)$ is definably connected.

*Proof.* Let $V_1, V_2 \subseteq M^n$ be definable open sets such that $f(X) \subseteq V_1 \cup V_2$, $V_1 \cap f(X) \neq \emptyset$ and $V_2 \cap f(X) \neq \emptyset$. Set $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. Since $U_1$ and $U_2$ are definable, $a \in U_1 \cap U_2$. Thus $f(a) \in V_1 \cap V_2$. ■

**Corollary 6.1.4 — Intermediate Value Theorem for o-minimal structures.** Suppose that $\mathscr{M}$ is o-minimal. Let $f : [a,b] \to M$ be definable continuous. Then for every $c \in [f(a), f(b)]$ there is $d \in [a,b]$ such that $f(d) = c$.

*Proof.* By Lemma 6.1.2, the interval $[a,b]$ is definably connected. Thus by Lemma 6.1.3, $f([a,b])$ is definably connected. Thus the disjoint open sets

$$\{d \in [a,b] \ : \ f(d) < c\} \cup \{d \in [a,b] \ \ f(d) > c\}$$

can not cover $[a,b]$. ■

**Exercise 6.1** Suppose that $\mathscr{M}$ satisfies the Statement 1 of Lemma 6.1.1 (but is not necessarily o-minimal). Show that the conclusion of Corollary 6.1.4 holds for $\mathscr{M}$.

**Exercise 6.2** Let $\mathscr{R} = (R, <, +, 0)$ be a densely ordered group; that is $(R, <, +, 0)$ is an ordered group and $(R, <) \models \mathrm{DLO}$. Assume that $\mathscr{R}$ is o-minimal.

(a) Let $X \subseteq R$ be definable in $\mathscr{R}$ such that $(X, +)$ is a subgroup of $(R, +)$. Show that either $X = \{0\}$ or $X = R$. (**Hint:** First show that $X$ is convex)

(b) Conclude that $\mathscr{R}$ is abelian.

### 6.1.1 The monotonicity theorem

Throughout this subsection, assume that $\mathscr{M}$ is o-minimal.

**Theorem 6.1.5 — Monotonicity Theorem.** Let $f : (a,b) \to M$ be definable. Then are $a_1, \ldots, a_k \in (a,b)$ such that for each $j = 0, \ldots, k$ the restriction of $f$ to $(a_j, a_j + 1)$ is continuous and either constant or strictly monotone, where $a_0 := a$ and $a_{k+1} := b$.

**Lemma 6.1.6** Let $f : I \to M$ be definable function and let $I \subseteq M$ be an interval. Then there is a subinterval $J \subseteq I$ such that $f$ is either constant or injective on $J$.

*Proof.* Let $y \in M$. Suppose that $f^{-1}(y)$ is infinite. Then by o-minimality, there is an interval $J \subseteq f^{-1}(y)$. On this interval $J$, $f$ is constant. So we may reduce to the case that $f^{-1}(y)$ is finite for each $y \in M$. Since $I$ is infinite, we get that $f(I)$ has to be infinite as well. Thus by o-minimality, there is an interval $J_0 \subseteq f(I)$. Define $g : J_0 \to I$ by

$$x \mapsto \min\{x \in I \ : \ f(x) = y\}.$$

This follows directly from the definition that $g$ is injective. Thus $g(J_0)$ is infinite. By o-minimality, there is an interval $J \subseteq g(J_0)$. On this interval $f$ is injective. ■

**Lemma 6.1.7** Let $f : I \to M$ be a definable function and let $I \subseteq M$ be an interval. If $f$ is injective, then there is a subinterval $J \subseteq I$ such that $f$ is strictly monotone on $J$.

*Proof.* Let $a, b \in M$ such that $I = (a, b)$. We define

$$\begin{aligned}
X_{++} &:= \{x \in I \ : \ \exists c_1, c_2 \in I\, \forall y \in (c_1, x)\ f(y) > f(x) \wedge \forall y \in (x, c_2)\ f(y) > f(x)\} \\
X_{+-} &:= \{x \in I \ : \ \exists c_1, c_2 \in I\, \forall y \in (c_1, x)\ f(y) > f(x) \wedge \forall y \in (x, c_2)\ f(y) < f(x)\} \\
X_{-+} &:= \{x \in I \ : \ \exists c_1, c_2 \in I\, \forall y \in (c_1, x)\ f(y) < f(x) \wedge \forall y \in (x, c_2)\ f(y) > f(x)\} \\
X_{--} &:= \{x \in I \ : \ \exists c_1, c_2 \in I\, \forall y \in (c_1, x)\ f(y) < f(x) \wedge \forall y \in (x, c_2)\ f(y) < f(x)\}.
\end{aligned}$$

It follows from injective of $f$ and o-minimality of $\mathscr{M}$ that $I = X_{++} \cup X_{+-} \cup X_{-+} \cup X_{--}$. Thus there is a subinterval $J$ of $I$ that is contained in of these four sets. Without loss of generality, we can assume that $I$ itself is contained in of these sets.

First consider the case that $I \subseteq X_{-+}$. For $x \in I$, set

$$s(x) := \sup\{s \in (x, b) \ : \ f(y) > f(x) \text{ for all } y \in (x, s]\}.$$

Let $x \in I$. Suppose towards a contradiction that $s(x) < b$. Then $s(x) \notin X_{-+}$. This contradicts our assumption. Thus $s(x) = b$ for all $x \in I$. Hence $f$ is strictly increasing on $I$.

Now suppose that $I \subseteq X_{++}$. Let $B := \{x \in I : \forall y \in I (y > x \to f(y) > f(x))\}$. If $B$ is infinite, then $B$ contains an interval $J$. It is clear that $f$ is strictly increasing on $J$. So we can reduce to the case that $B$ is finite. Suppose $B$ is non-empty. Let $c \in B$ be maximal. After replacing $I$ by $(c, b)$, we can even assume that $B$ is empty. Thus for all $x \in I$ there is a $y \in I$ such that $x < y$ and $f(y) < f(x)$.

Let $c \in I$. We now show that there is $d \in I$ such that $f(y) < f(c)$ for all $y \in I_{>d}$. Suppose note. Then there is $d \in I$ such that $f(y) > f(c)$ for all $y \in I_{>d}$. Take $d$ to be minimal with this property. Note that $d \in X_{++}$. Thus $f(d) < f(c)$, because of minimality of $d$. However, we know there must be an $e \in I$ such that $d < e$ and $f(e) < f(d)$. But then $f(e) < f(d) < f(c)$, contradicting our choice of $d$.

Note define $g : I \to I$ to map $c \in I$ to the minimal element $g(c)$ of $[c,b)$ such that $f(y) < f(c)$ for all $y \in I$ with $g(c) < y < b$. Since $c$ and $g(c)$ are in $X_{++}$, we have $c < g(c)$ and $f(g(c)) < f(c)$. Moreover, by minimality of $g(c)$, we obtain that $g(c)$ is in the set

$$Y_{+-} := \{x \in I \;:\; \exists c_1, c_2 \in I \; c_1 < x < c_2 \forall d_1, d_2 (c_1 < d_1 < x < d_2 < d_2 \to f(d_1) > f(d_2))\}.$$

Since this is true for every $c \in I$, we get that $Y_{+-}$ contains an interval. Shrinking $I$, we can assume that $I \subseteq Y_{+-}$. Now define

$$Y_{-+} := \{x \in I \;:\; \exists c_1, c_2 \in I \; c_1 < x < c_2 \forall d_1, d_2 (c_1 < d_1 < x < d_2 < d_2 \to f(d_1) < f(d_2))\}.$$

We can use the same argument to show that $Y_{-+}$ contains an interval. This is a contradiction, as $Y_{-+} \cap Y_{+-} = \emptyset$. ■

**Lemma 6.1.8** Let $f : I \to M$ be a definable function and let $I \subseteq M$ be an interval. If $f$ is strictly monotone, then there is a subinterval $J \subseteq I$ such that $f$ is continuous on $J$.

*Proof.* We consider the case that $f$ is strictly increasing. Since $f(I)$ is infinite, there is an interval $J_0 \subseteq f(I)$ by o-minimality. Let $c, d \in J$ such that $c < d$. Let $a, b \in I$ such that $f(a) = c$ and $f(b) = d$. Note that the restriction of $f$ to interval $(a,b)$ is an order-preservering bijection. The continuity of $f$ follows. ■

**Corollary 6.1.9** Let $f : I \to M$ be a definable function and let $I \subseteq M$ be an interval. If $f$ is strictly monotone, then there is finite set $Z \subseteq I$ such that for every interval $J \subseteq I \setminus Z$ the restriction of $f$ to $J$ is continuous.

*Proof.* Let $X \subseteq I$ be set of all points $x \in I$ such that $f$ is continuous on an interval around $x$. Since $f$ is definable, it is easy to see that $X$ is definable. It is enough to show that $I \setminus X$ is finite. Suppose not. Then by o-mnimality there is an interval $J \subseteq I \setminus X$. Since $f$ is strictly monotone on $J$, there is an interval $J' \subseteq J$ such that the restriction of $f$ to $J'$ is continuous by Lemma 6.1.8. But then $J' \subseteq X$, a contradiction. ■

*Proof of Theorem 6.1.5.* Let $f : (a,b) \to M$ be definable. Consider the set

$$X_1 := \{x \in (a,b) \;:\; f \text{ is constant on interval around around } x\}$$
$$X_2 := \{x \in (a,b) \;:\; f \text{ is strictly increasing on interval around around } x\}$$
$$X_3 := \{x \in (a,b) \;:\; f \text{ is decreasing increasing on interval around around } x\}.$$

By Lemma 6.1.6 and Lemma 6.1.7, $I \setminus (X_1 \cup X_2 \cup X_3)$ has empty interior, and hence is finite by o-minimality of $\mathscr{M}$. In particular, $I \setminus (X_1 \cup X_2 \cup X_3)$ is a subset of union of the boundaries of $X_1, X_2$ and $X_3$. Let $Y$ be this union of boundaries. Note that $I \setminus Y$ is finite disjoint union of open interval. Let $J$ be one of these interval. Since $J \subseteq X_1 \cup X_2 \cup X_3$ but doesn't intersect the boundary of the $X_i$'s, there is $i \in \{1,2,3\}$ such that $J \subseteq X_i$.

Suppose that $J \subseteq X_1$. We now prove that $f$ is constant on $J$. Let $x \in J$. We show that $f(x) = f(y)$ for all $y \in J$. We just handle the case that $f(x) = f(y)$ for all $y \in Y$ with $x < y$. The case when $y < x$ follows similarly. Consider the

$$Z_1 := \{z \in J \;:\; z \geq x \wedge f(y) \neq f(x)\}.$$

Suppose towards a contradiction, that $Z_1 \neq \emptyset$. Let $y \in J$ be $\inf Z_1$. Since $x \in X_1$, $y > x$. However, since $y \in X_1$ as well, there is $z \in J$ such that $x < z < y$ and $f(y) = f(z)$. This contradicts the minimality of $y$.

Suppose $J \subseteq X_2$. We establish that $f$ is strictly increasing on $J$. Let $x \in J$. We show that $f(x) < f(y)$ for all $y \in J$.

$$Z_2 := \{z \in J \ : \ z > x \wedge f(y) \geq f(x)\}.$$

Suppose towards a contradiction, that $Z_2 \neq \emptyset$. Let $y \in J$ be $\inf Z_2$. Since $x \in X_2$, $y > x$. However, since $y \in X_2$ as well, there is $z \in J$ such that $x < z < y$ and $f(z) < f(y)$. This contradicts the minimality of $y$.

Finally consider the case that $J \subseteq X_3$. Arguing as in the case of $J \subseteq X_2$, we can show that $f$ is strictly decreasing on $J$. ■

**Exercise 6.3** Let $A \subseteq M$ and let $f : (a,b) \to M$ be a function definable just with parameters from $A$. Show that we can pick $a_1, \dots, a_n$ in the conclusion of Theorem 6.1.5 such that the set $\{a_1, \dots, a_n\}$ is definable with parameters from $A$.

**Exercise 6.4** Let $f : (a,b) \to M$ be definable and let $c \in (a,b)$. Show that the limits $\lim_{x \uparrow c} f(x)$ and $\lim_{x \downarrow c} f(x)$ exist in $M \cup \{\infty\}$.

**Corollary 6.1.10** $\mathrm{dcl}_{\mathscr{M}}$ is pregeometry.

*Proof.* Let $A \subseteq M$, and $a, b \in M$. Let $b \in \mathrm{dcl}_{\mathscr{M}}(A \cup \{a\}) \setminus \mathrm{dcl}_{\mathscr{M}}(A)$. By Lemma ... there is function $f : X \subseteq M^{m+1} \to M$ definable without parameters in $\mathscr{M}$ and $c = (c_1, \dots, c_n) \in A^m$ such that $f(c,a) = b$. Let $g : Z \subseteq M \to M$ be the function mapping $x$ to $f(c,x)$. Without loss generality, we can assume that $Z$ is an interval $(d_0, d_{k+1})$ such that $d_0, d_{k+1} \in \mathrm{dcl}_{\mathscr{M}}(A)$ and $a \in Z$. By Theorem 6.1.5 there are $d_1, \dots, d_k \in Z$ such that for each $j = 0, \dots, k$ the restriction of $g$ to $(d_j, d_j + 1)$ is continuous and either constant or strictly monotone. By Exercise 6.3, we can assume that $\{d_1, \dots, d_k\} \mathrm{dcl}_{\mathscr{M}}(A)$. Since $a \notin \mathrm{dcl}_{\mathscr{M}}(A)$, we can assume that $a \neq d_i$ for all $i \in \{0, \dots, k+1\}$. So let $j \in \{0, \dots, k\}$ be such that $d_j < a < d_{j+1}$. If $g$ is constant on $(d_j, d_{j+1})$, then $b = f(a) \in \mathrm{dcl}(A)$. Thus $g$ has to strictly monotone on this interval. Without loss of generality assume that $g$ is strictly increasing. Then $g$ is injective and $g^{-1}$ is a $A$-definable function mapping $b$ to $a$. Thus $b \in \mathrm{dcl}_{\mathscr{M}}(A \cup \{a\})$. ■

### 6.1.2 Cell decomposition

**Definition 6.1.2** A subset $X$ of $M$ is a **0-cell** if it is of the form $\{a\}$ for some $a \in M$. A subset $X$ of $M$ is a **1-cell** if it is an interval $(a,b)$ for some $a, b \in M \cup_{-\infty,\infty}$.
A subset $X$ of $M^{n+1}$ is a **$(k+1)$-cell** if there is a $k$-cell $C \subseteq M^n$ and one of the following holds:

1. there are definable, continuous functions $f, g : C \to M$ such that $f(x) < g(x)$ for all $x \in C$,
$$X = \{(x,a) \in C \times M \ : \ f(x) < a < g(x)\},$$
2. there is definable continuous function $f : C \to M$ such that one of the following holds:
   (a) $X = \{(x,a) \in C \times M \ : \ f(x) < a\}$,
   (b) $X = \{(x,a) \in C \times M \ : \ a < f(x)\}$,

A subset $X$ of $M^{n+1}$ is a **$k$-cell** if there is a $k$-cell $C \subseteq M^n$ and definable continuous function $f : C \to M$ such that

$$X = \{(x,a) \in C \times M \ : \ f(x) = a\}.$$

**Exercise 6.5** Show that a cell $C \subseteq M^n$ open if and only if it is a $n$-cell. Show that if $k < n$, then a $k$-cell $C \subseteq M^n$ has empty interior and is nowhere dense in $M^n$.

**Exercise 6.6** Let $C \subseteq M^n$ be a $k$-cell. Then there is a coordinate projection $\pi : M^n \to M^k$ such that the restriction of $\pi$ to $C$ is a homeomorphism.

**Definition 6.1.3** A **cell decomposition** of $M$ is a collection of sets of the form

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\},$$

where $a_1 < \cdots < a_k$ are in $M$. A **cell decomposition** of $M^{n+1}$ is a finite partition of $M^{n+1}$ into cells $\{C_1, \ldots, C_k\}$ such that $\{\pi(C_1), \ldots, \pi(C_k)\}$ is a cell decomposition of $M^n$, where $\pi; M^{n+1} \to M^n$ is the coordinate projection onto the first $n$-coordinates.
Let $X \subseteq M^n$. We say a cell decomposition $\mathscr{D}$ **partitions** $X$ if $X \cap C = \emptyset$ or $C \subseteq X$ for every cell $C \in \mathscr{D}$.

Note that $X$ is partitioned by some cell decomposition $\mathscr{D}$, then $X$ is a finite union of cells in $\mathscr{D}$.

**Theorem 6.1.11 — Cell decomposition theorem.** Let $X_1, \ldots, X_m \subseteq M^n$ be definable and let $f : X_1 \to M$ be a definable function. Then

1. there is a cell decomposition $\mathscr{D}$ of $M^n$ such that $\mathscr{D}$ partitions each $X_1, \ldots, X_m$,
2. there is a cell decomposition $\mathscr{E}$ of $M^n$ such that $\mathscr{E}$ partitions $X_1$ and for every $C \in \mathscr{E}$ with $C \subseteq X_1$ the restriction of $f$ to $C$ is continuous.

**Definition 6.1.4** Let $X \subseteq M^n$ be definable. The **geometric dimension** of $X$ (written: $\dim(X)$) is the largest $k \in \mathbb{N}$ such that $X$ contains a $k$-cell.

**Proposition 6.1.12** Let $X \subseteq M^n$ be definable and let $d \in \mathbb{N}$. Then the following are equivalent:

1. $\dim(X) \geq d$,
2. there is a coordinate projection $\pi : M^n \to M^d$ such that $\pi(X)$ has interior.

*Proof.* By Exercise 6.6, Statement 1 implies Statement 2. Now suppose that Statement 2 holds. Let $\pi : M^n \to M^d$ be a coordinate projection such that $\pi(X)$ has interior. By Theorem 6.1.11 and Exercise 6.5 there is a cell $C \subseteq X$ such that $\pi(C)$ has interior. It is an easy exercise to check that $C$ has to be a $k$-cell for some $k \geq d$. ■

# Bibliography


[ADH17] Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven. *Asymptotic differential algebra and model theory of transseries*. Volume 195. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2017, pages xxi+849. ISBN: 978-0-691-17543-0. DOI: 10.1515/9781400885411. URL: https://doi.org/10.1515/9781400885411 (cited on page 5).

[CK73] C. C. Chang and H. J. Keisler. *Model theory*. Studies in Logic and the Foundations of Mathematics, Vol. 73. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973, pages xii+550 (cited on page 5).

[Dri98] Lou van den Dries. *Tame topology and o-minimal structures*. Volume 248. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1998, pages x+180. ISBN: 0-521-59838-9. DOI: 10.1017/CBO9780511525919. URL: https://doi.org/10.1017/CBO9780511525919 (cited on page 5).

[Dri99] Lou van den Dries. “o-minimal structures and real analytic geometry”. In: *Current developments in mathematics, 1998 (Cambridge, MA)*. Int. Press, Somerville, MA, 1999, pages 105–152 (cited on page 6).

[Hod97] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997, pages x+310. ISBN: 0-521-58713-1 (cited on page 6).

[Mar02] David Marker. *Model theory*. Volume 217. Graduate Texts in Mathematics. An introduction. Springer-Verlag, New York, 2002, pages viii+342. ISBN: 0-387-98760-6 (cited on page 5).

[TZ12] Katrin Tent and Martin Ziegler. *A course in model theory*. Volume 40. Lecture Notes in Logic. Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012, pages x+248. ISBN: 978-0-521-76324-0. DOI: 10.1017/CBO9781139015417. URL: https://doi.org/10.1017/CBO9781139015417 (cited on page 5).