Ultraproduct and the Ax-Grothendieck theorem

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This note is about the ultraproduct construction for fields, leading to a proof of a theorem proven independently by Grothendieck and Ax in the 60's.

1 Filtres and Ultrafiltres

Let X be a set. We denote by $\mathscr{P}(X)$ the power set of X, i.e. the set consisting of all subsets of X. We say that $\mathscr{F} \subseteq \mathscr{P}(X)$ is a *filter on* X if

- $X \in \mathscr{F}$ and $\emptyset \notin \mathscr{F}$
- if $A, B \in \mathscr{F}$ then $A \cap B \in \mathscr{F}$
- if $A \in \mathscr{F}$ and $A \subseteq B \subseteq X$ then $B \in \mathscr{F}$

Example 1.1. *1. Let λ be the Lebesgue measure on $\mathbb{R}$. Then $\{X \subseteq \mathbb{R}, \lambda(\mathbb{R} \setminus X) = 0\}$ is a filter on $\mathbb{R}$.*

2. For any set X and any cardinal $\kappa < |X|$ the set $\{A \subseteq X, |X \setminus A| \leq \kappa\}$ is a filter on X. For $\kappa = \aleph_0$ we call this filter the Fréchet filter *on X. It is the set of all cofinite subsets of X.*

3. Let $x \in X$. Then $\mathscr{F}_x := \{A \subseteq X, x \in A\}$ is a filter.

We call *principal filter* any filter $\mathscr{F}$ on X such that there exists $x \in X$ such that $\mathscr{F} = \mathscr{F}_x$.

This kind of filter is a bit different from the two others examples, as for any $A \subset X$ we have either $A \in \mathscr{F}_x$ or $X \setminus A \in \mathscr{F}_x$. We call such a filter an *ultrafilter*.

Remark 1.2. *A filter on X is an ultrafilter if and only if it is maximal as a filter in the poset* $(\mathbb{P}(\mathbb{P}(X)), \subseteq)$.

Remark 1.3. *A ultrafilter on X is principal if and only if one of its element is a finite subset of X.*

Remark 1.4. *A ultrafilter on X is non principal if and only if it contains the Fréchet filter on X.*

The existence of non principal ultrafilter is unclear at first sight. The axiom of choice will give us the existence of many a ultrafilter.

Lemma 1.5. *Every filter on X is included in an ultrafilter on X.*

Proof. Let $\mathscr{F}$ be a filter on X. We apply Zorn's lemma to the poset consisting of the filters on X containing $\mathscr{F}$. □

As the topology on a set come from the need to capture the concept of nearness, an ultrafilter captures the notion of largeness. An ultraproduct $\mathscr{U}$ on a set X give birth to a finitely additive measure, with value in $\{0, 1\}$, by $\mathscr{U}(A) = 1$ if $A \in \mathscr{U}$ and 0 if not. There is in fact a bijection between the set of finitely additive $\{0, 1\}$-valued measures on X and the set of all ultrafilters on X.

2 Ultraproducts of fields and the Ax-Grothendieck theorem

For any field K we say that $f : K^n \to K^n$ is a *polynomial map* if $f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))$ with $f_i \in K[X_1, \dots, X_n]$. We now prove the theorem, due independently to James Ax in 1968 and Alexandre Grothendieck in 1966. There has been other proofs later on, by Borel in 1969 and Rudin in 1995.

Theorem 2.1 (Ax, Grothendieck)**.** *Every injective polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ is surjective.*

Remark that the converse is obviousely untrue, as $x \mapsto x^2$ is surjective in $\mathbb{C}$ but not injective. The proof of this theorem is an example of a Lefshetz principle, a transfert theorem from the positive characteristic context to the zero characteristic one. We first need the result in positive characteristic.

Lemma 2.2. *Every injective polynomial map $f : (\mathbb{F}_p^{alg})^n \to (\mathbb{F}_p^{alg})^n$ is surjective.*

Proof. Let $f = (f_1, \dots, f_n)$ be an injective polynomial map from $(\mathbb{F}_p^{alp})^n$ to itself. Assume that f is not surjective, witnessed by $y = (y_1, \dots, y_n)$. Let S be the (finite) set consisting of all the coefficients (in $\mathbb{F}_p^{alg}$) appearing in the f_i's. Let $K = \mathbb{F}_p(S, y)$, this field is finite. Then it makes sens to consider $f_{\restriction K} : K^n \to K^n$, and this restriction is still injective. Now any injection from a finite set to itself is also surjective, and as $y \in K^n$ we reach to a contradiction. □

Now the idea is to construct a field which is in some sens the limit of the $\mathbb{F}_p^{alg}$. Let $\mathbb{P}$ be the set of prime numbers and let $\mathscr{U}$ be a non principal ultrafilter on the set $\mathbb{P}$. Let $\mathscr{C}$ be the set $\prod_{p \in \mathbb{P}} \mathbb{F}_p^{alg}$, the infinite cartesian product. Note that $\mathscr{C}$ is a ring, for the componentwise addition and multiplication. We denote 0 and 1 the infinite tuples consisting of 0_p and 1_p respectively. Set an equivalence relation on $\mathscr{C}$ in the following way :

$$a \sim_{\mathscr{U}} b \iff \{p \in \mathbb{P}, a_p = b_p\} \in \mathscr{U}$$

Now consider the quotient set $\mathfrak{C} := \mathscr{C}/_{\sim_{\mathscr{U}}}$. Denote by $\bar{a}$ the $\sim_{\mathscr{U}}$-class of $a \in \mathscr{C}$. We define on $\mathfrak{C}$ the following operations for $\bar{a}, \bar{b} \in \mathfrak{C}$:

$$\bar{a} + \bar{b} = \overline{a + b}$$

$$\bar{a} \cdot \bar{b} = \overline{a \cdot b}$$

These two operations are compatible with the equivalence relation and turn $\mathfrak{C}$ into a ring.

Lemma 2.3. *$\mathfrak{C}$ is an algebraically closed field of characteristic 0.*

PROOF. We show that the inverse exists. Take $\bar{a} \in \mathfrak{C} \setminus \{\bar{0}\}$. As $a \nsim 0$ there is $A \in \mathscr{U}$ such that for all $p \in A$ $a_p \neq 0$. Now set $b_p = a_p^{-1}$ for $p \in A$ and $b_p = 0$ for $p \notin A$. Now let $\bar{c} = \bar{a}\bar{b}$. For every $p \in A$ we have $c_p = 1$ and hence $c \sim \bar{1}$. Let $\bar{P}(X) \in \mathfrak{C}[X]$ of degree at least 1, say $\bar{P}(X) = {}_n\bar{a}X^n + \cdots + {}_0\bar{a}$ with $n \geq 1$ and ${}_na \nsim 0$. This polynomial induces polynomials for each p, namely ${}_na_pX^n + \cdots + {}_0a_p \in \mathbb{F}_p^{alg}[X] =: P_p(X)$. As ${}_na \nsim 0$, there is $A \in \mathscr{U}$ such that for all $p \in A$, the polynomial $P_p(X)$ is not constant. Let α_p be a root of the latter for each $p \in A$ and set the infinite tuple $\alpha = (\alpha_p)_{p\in A} \cup (0_p)_{p\notin A}$. Then $P(\alpha) = (P_p(\alpha_p))_{p\in\mathbb{P}}$ and these coordinate are 0 for each $p \in A$, hence $P(\alpha) \sim 0$. So $\mathfrak{C}$ is algebraically closed. Suppose that $s = char\mathfrak{C} > 0$ and let $A = \mathbb{P} \setminus s$. A is cofinite so $A \in \mathscr{U}$. Now $s1 = (s1_p)_{p\in\mathbb{P}}$ has nonzero coordinate for each $p \in A$ so $s1 \nsim 0$, a contradiction. □

The field $\mathfrak{C}$ is called the *ultraproduct of the familly $(\mathbb{F}_p^{alg})_{p\in\mathbb{P}}$ over $\mathscr{U}$.*

Lemma 2.4. *Every injective polynomial map $\bar{f} : \mathfrak{C}^n \to \mathfrak{C}^n$ is surjective.*

PROOF. Let $\bar{f} = ({}_1\bar{f}, \ldots, {}_n\bar{f})$ be such a polynomial map, with ${}_if \in \mathscr{C}[X_1, \ldots, X_n]$ and assume that it is injective. Observe that for each p, ${}_if$ induces a polynomial map say ${}_if_p : (\mathbb{F}_p^{alg})^n \to \mathbb{F}_p^{alg}$, and so $f_p := ({}_1f_p, \ldots, {}_nf_p) : (\mathbb{F}_p^{alg})^n \to (\mathbb{F}_p^{alg})^n$.

<u>Claim :</u> There is $A \in \mathscr{U}$ such that for each $p \in A$, f_p is injective.

Let A be the set of prime numbers such that f_p is injective. If A is not in $\mathscr{U}$, then $B := \mathbb{P} \setminus A$ is in $\mathscr{U}$. Hence for each $p \in B$ we can find $\tilde{\alpha}_p, \tilde{\beta}_p \in (\mathbb{F}_p^{alg})^n$ such that $\tilde{\alpha}_p \neq \tilde{\beta}_p$ and $f_p(\tilde{\alpha}_p) = f_p(\tilde{\beta}_p)$. Now complete the tuples $(\tilde{\alpha}_p)_{p\in B}$ and $(\tilde{\beta}_p)_{p\in B}$ in any ways, in two tuples (of n-tuples) say $\tilde{\alpha} = (\tilde{\alpha}_p)_{p\in\mathbb{P}}$ and $\tilde{\beta} = (\tilde{\beta}_p)_{p\in\mathbb{P}}$. Now $\tilde{\alpha} \nsim_{\mathscr{U}} \tilde{\beta}$ as they don't agree on $B \in \mathscr{U}$, but $f(\tilde{\alpha}) \sim_{\mathscr{U}} f(\tilde{\beta})$, and this contradict the injectivity of $\bar{f}$. By the lemma 2.2, for each $p \in A$ f_p is also surjective. Now for any $({}_1b, \ldots, {}_nb) \in \mathscr{C}^n$ we can find for each $p \in A$, $({}_1a_p, \ldots, {}_na_p) \in (\mathbb{F}_p^{alg})^n$ such that $f_p({}_1a_p, \ldots, {}_na_p) = ({}_1b_p, \ldots, {}_nb_p)$. Now complete $({}_ia_p)_{p\in A}$ with any coordinate in an element ${}_1a, \ldots, {}_na \in \mathscr{C}^n$, and $f({}_1a, \ldots, {}_na) \sim_{\mathscr{U}} ({}_1b, \ldots, {}_nb)$. □

Lemma 2.5. *The cardinality of $\mathfrak{C}$ is $2^{\aleph_0}$.*

PROOF. Begin by a claim : There exists a set $E \subseteq \mathbb{N}^{\mathbb{N}}$ of functions such that $|E| = 2^{\aleph_0}$ and for all $f, g \in E$ with $f \neq g$, then $\{i \in \mathbb{N}, f(i) = g(i)\}$ is finite. Indeed, consider for each $\phi \in 2^{\aleph_0}$ the function $f_\phi \in \mathbb{N}^{\mathbb{N}}$ defined by $f_\phi(n) = \sum_{i<n} \phi(i)2^i$, and let $E = \left\{f_\phi, \phi \in 2^{\mathbb{N}}\right\}$.[to be finish] This set can be injected in $\mathfrak{C}$ in the following way : consider $\psi : \mathbb{P} \to \mathbb{N}$ any bijection and $\sigma_p : \mathbb{F}_p^{alg} \to \mathbb{N}$ [to be finish] □

Recall the following classical result of Steinitz :

Fact 2.6 (Steinitz's theorem)**.** *Algebraically closed fields of fixed characteristic are classified up to the cardinality of their transcendance basis.*

We conclud from the lemma that $\mathfrak{C}$ is isomorphic to $\mathbb{C}$ and hence the theorem is proven.

3 Topological contents, and model theory

Filters and ultrafilters can also be used in topology. When some topological space is getting too complicated, for instance if it is not metrisable, filters and ultrafilters can be used to speak of convergence. Let X be a topological space, and $x \in X$. We call $\mathscr{V}(x)$ the set of all neigbourhood of x, i.e. the subsets of X which contains an open containing x. The set $\mathscr{V}(x)$ is a filter on X, we call it the *neigbourhood filter*. Let $\mathscr{U}$ be an ultrafilter on X, we say that $\mathscr{U}$ *converges* to $x \in X$ if $\mathscr{V}(x) \subseteq \mathscr{U}$. Let $f : X \to Y$ any function between two topological spaces. Let $\mathscr{U}$ be an ultrafilter on X, we say that $y \in Y$ *is a limit of f along $\mathscr{U}$* if $f(\mathscr{U})$ converges to y ($f(\mathscr{U})$ is an ultrafilter on Y). We say that $x \in X$ is a limit of $(x_i)_{i\in I}$ along $\mathscr{U}$ if We have the following basics results.

Lemma 3.1.
- *X is Haussdorf if and only if each ultrafilter converges to at most one adherent point.*
- *X is compact if and only if each ultrafilter converges to at least one point.*
- *If X is compact Haussdorf, for any set I, for any sequence $(x_i)_{i \in I}$ and for any ultrafilter $\mathscr{U}$ on I, $lim_{\mathscr{U}}(x_i) \in X$.*

Now we are getting a bit informal, this part is intend to give an intuition of the topological contents of the proof. Consider the class $\mathbb{F}$ of all fields in the language of rings, $\mathscr{L}$. We put on $\mathbb{F}$ a topology given by first order sentences in $\mathscr{L}$ in the following sense. For θ a $\mathscr{L}$-sentence consider the class U_θ of all fields K in $\mathbb{F}$ such that $K \models \theta$. Observe that $U_\theta \cap U_\phi = U_{\theta \wedge \phi}$, and $\{U_\theta, \theta \in \mathscr{L}\}$ is the basis of a topology on $\mathbb{F}$. Now Gödel's theorem tells us that with this topology, the class $\mathbb{F}$ is compact. This implies that for instance the infinite set $\left\{\mathbb{F}_p^{alg}, p \in \mathbb{P}\right\}$ has a limit along any nonprincipal ultrafilter over $\mathbb{P}$, this limit is $\mathbb{C}$. The topology defined before is not Haussdorf as since $\mathbb{Q}^{alg} \equiv \mathbb{C}$, there is no way to seperate them with an open in this topology. To do so, we need to identify these two, and consider fields up to elementary equivalence. So we want to quotient $\mathbb{F}$ by the relation $\equiv$. Now what is a $\equiv$-class but a complete theory ?
We now start this formally. We consider $Fields$ the (incomplete) theory of (commutative) fields in $\mathscr{L}$. Now the collection of all completions of $Fields$ will be called $\mathscr{S}$, this is really a set. Now we define a topology on $\mathscr{S}$ as before, so for θ any $\mathscr{L}$-sentence, let $< \theta >$ be the set of all elements T of $\mathscr{S}$ such that $T \models \theta$. The $< \theta >$ for $\mathscr{L}$-sentences θ form a basis of neigbourhood, and the topology spanned by those is called the *Stone topology.*

Theorem 3.2 (Gödel)**.** *The space $\mathscr{S}$ is compact.*

Now observe that ACF_p, ACF_0 are elements of $\mathscr{S}$. Let $\mathscr{U}$ be a non principal ultrafilter on $\mathbb{P}$. By compactness the sequence $(ACF_p)_{p \in \mathbb{P}}$ has a limit along $\mathscr{U}$. The Ax-Grothendieck theorem is a manifestation of the following fact:

$$lim_{\mathscr{U}} ACF_p = ACF_0$$

4 More model theory

From what we have seen, the ultraproduct construction leads to some universal object in some sens, as some properties are conserved and some not. We can see that ultraproduct of algebraically closed fields stays an algebraically closed field. On the other hand, the ultraproduct of positive characteristic fields may not be of positive characteristic. A question arose : what class of property is conserved when passing to the ultraproduct ? The answer is the first order logical properties. Los' theorem says that a first order property is true in the ultraproduct if and only if it is true in almost all the structures. Here are some other transfert principles.

Example 4.1. *Let K be a field. For each n, the property K has a unique algebraic extension of degree n is first order. Notice that it is true in every finite field. This means that a non-principal ultraproduct of all finite fields will also satisfy. This means that $Gal(\prod_{\mathscr{U}} \mathbb{F}_q) \cong \hat{\mathbb{Z}}$*

Example 4.2. *Every finite field $\mathbb{F}_q$ has some Artin-Schreier extension as the application $x \mapsto x^p - x$ is not surjective (it has nontrivial kernel). Having an Artin-Schreier extension is a first order property, hence the ultraproduct of all the finite field is a field which is not Artin-Schreier clos. Together with a result about NIP fields of positive characteristic, this implies that such a field has the independance property.*

Example 4.3. *Let $\mathscr{U}$ be a non principal ultrafilter on $\mathbb{P}$. The Ax-Kochen principle can be stated as follows :*

$$\prod_{\mathscr{U}} \mathbb{Q}_p \equiv \prod_{\mathscr{U}} \mathbb{F}_p((t))$$

This very strong result has some beautiful applications, such as the following result. For all $d \in \mathbb{N}$ there is $N = N(d) \in \mathbb{N}$ such that for all prime numbers $p \geq N$, if $f(X_1, \ldots, X_n) \in \mathbb{Z}_p[X_1, \ldots, X_n]$ is homogeneous of degree d and $n > d^2$, then there exists a nonzero $x \in \mathbb{Z}_p^n$ with $f(x) = 0$.

5 Historical remarks

Filters and ultrafilters have been used by model theorists since Tarski in 1930. They have been used as well in topology, first by Cartan in 1937, and also by Bourbaki in the 70's. The first construction of ultraproduct is due to Los, in 1955, even though an ultraproduct construction of arithmetic was due to Skolem in 1938. The use of ultraproduct in algebra is first due to Kochen and Ax.