On the complete ordered field

Christian d'Elbée (8731086)

09/05/2013

Contents

Acknowledgements

I would like to thank my supervisor Alex Wilkie for overlooking my work and advising me on the contents. I am very grateful to Norbert A'Campo for all his help and direction in reference to the construction via slopes (2.4). I am thankful to Mike Prest for advising me on the whereabouts of some articles.

Introduction

The purpose of this report is the study of the real number system. We will present some constructions of it, but to do that we need to understand what the set of real numbers is and how it is characterised. To this end we will write down the properties that we expect the real number system to have and construct a system for which these properties are true. Model theoretically speaking this corresponds to construct a model of the set of axioms translating these properties. By system we mean algebraic structure, as the real numbers are not only a set but a set in which we can add, multiply, compare. These basic properties are all put together in the notion of *ordered field*, an algebraic structure that we will define more formally in the first section of the first chapter. It is easy to see that the real number system is an ordered field as is the rational number system $\mathbb{Q}$. However we expect the real number system to have some properties that $\mathbb{Q}$ does not have. For instance, we can see the real numbers as a line without discontinuity whereas $\mathbb{Q}$ is full of discontinuity, since the irrational numbers (such as $\sqrt{2}$ or π) are not included. So we want our real number system to be "complete" and, as we will see, the completeness axiom will enable us to fill the void. So we will see that the real number system as we see it is an *ordered field*, with the completeness property, so we will understand that what we want to construct is a *complete ordered field*. The formal study of ordered field shall led us to the notion of *archimedeanness* in an ordered field, a property which we expect the real numbers to have. However we shall see a short introduction to non-archimedean ordered fields and see an example of such an odd object.
In the second chapter some constructions of the reals shall be presented, some of them classical (like from $\mathbb{Q}$-Cauchy sequences, and Dedekind cuts) and some less well-known.

1 Ordered fields

1.1 Basics about ordered fields

In this section we will define the notion of ordered field, which is simply a field in the algebraic sens together with a total order which has a compatible behavior with the operations of the field.

Definition 1.1.1 (**Field**). *A* field *is a set* $\mathbb{k}$ *together with two binary operations* $+$ *(addition),* $\cdot$ *(product) which satisfy the following axioms :*

Fields axioms

(Φ_1) $\forall x, y \in \mathbb{k}\ x + y \in \mathbb{k}\quad x \cdot y \in \mathbb{k}$ *(*$\mathbb{k}$ *is closed under addition and product)*
(Φ_2) $\forall x, y, z \in \mathbb{k}\ (x + y) + z = x + (y + z)\quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *(the binary operations are associative)*
(Φ_3) $\forall x, y \in \mathbb{k}\ x + y = y + x\quad x \cdot y = y \cdot x$ *(the binary operations are commutative)*
(Φ_4) *There exists a unique element* $0_{\mathbb{k}} \in \mathbb{k}$ *and a unique element* $1_{\mathbb{k}} \in \mathbb{k} \neq 0_{\mathbb{k}}$ *such that*

$$\forall x \in \mathbb{k}\ 0_{\mathbb{k}} + x = x \quad 1_{\mathbb{k}} \cdot x = x$$

(existence of neutral elements)
(Φ_5) $\forall x \in \mathbb{k}\ \exists x' \in \mathbb{k}\quad x + x' = 0_{\mathbb{k}}\ \forall x \in \mathbb{k} \setminus \{0_{\mathbb{k}}\}\ \exists x'' \in \mathbb{k}\quad x \cdot x'' = 1_{\mathbb{k}}$ *(Existence of symmetric elements)*
(Φ_6) $\forall x, y, z \in \mathbb{k}\ x \cdot (y + z) = x \cdot y + x \cdot z$ *(Distributive law)*
We denote a field $\mathbb{k}$ *together with the two binary operations* $+$ *and* $\cdot$ *by* $(\mathbb{k}, +, \cdot)$*.*

The symmetric for $+$ is usually called opposite while the symmetric for $\cdot$ is called inverse. In this report we will denote $-x$ as the opposite of x and x^{-1} the inverse of x. Further $x + (-y)$ will be denoted $x - y$ and will be called subtraction, while $x \cdot y^{-1}$ will be called division. Finally the multiplication $x \cdot y$ may sometimes be denoted by $x \cdot_{\mathbb{k}} y$ to be more precise but can also be simply denoted by juxtaposition xy.

Definition 1.1.2 (**Ordered field**). *An* ordered field *is a field* $(\mathbb{k}, +, \cdot)$ *such that a binary predicate* $<$ *is defined on the set* $\mathbb{k}$*, such that* $<$ *satisfies the following axioms :*

Order axioms

(Ω_1) $\forall x, y \in \mathbb{k}$ *one and only one of the following holds :*

$$x < y \quad x = y \quad y < x$$

(Trichotomy law)
(Ω_2) $\forall x, y, z \in \mathbb{k}\ x < y$ *and* $y < z$ *implies* $x < z$ *(Transitivity of* $<$*)*
(Ω_3) $\forall x, y, z \in \mathbb{k}$

$$x < y \text{ implies } x + z < y + z$$

$$x < y \text{ and } 0 < z \text{ implies } x \cdot z < y \cdot z$$

(Monotonicity of $+$ *and* $\cdot$*)*

A field $(\mathbb{k}, +, \cdot)$ *which is an ordered field for* $<$ *will be noted* $(\mathbb{k}, <, +, \cdot)$.

Remark 1. Notice that $<$ is not an order relation per se, as it is not reflexive, but define $x \leq y \iff (x < y)$ *or* $x = y$ and $(\mathbb{k}, \leq)$ is a totally ordered set, by (Ω_1). Moreover, we will use the symbole $>$ so that $x < y \iff y > x$ (similarly for $\geq$). A basic result using these axioms is that $1 > 0$. Otherwise if $x > 0$, apply (Ω_3) to have $x \cdot 1 < 0$ so $x < 0$ and $x > 0$, contradicting (Ω_1). Furthermore, we have $-x = (-1) \cdot x$ as $(-1) \cdot x + x = (-1) \cdot x + 1 \cdot x = (-1 + 1) \cdot x = 0$ by (Φ_4) and (Φ_6). Finally, if $x > 0$ then $x^{-1} > 0$ otherwise by (Ω_3) $xx^{-1} = 1 < 0$ contradicting our previous result.

Henceforth we are given an ordered field $(\mathbb{k}, <, +, \cdot)$.

Definition 1.1.3. *We say that an element* $x \in \mathbb{k}$ *is a* positive element *if* $x > 0_{\mathbb{k}}$. *We denote* $\mathbb{k}^+$ *the set of all positive element.*

Remark 2. Notice that if $x < 0$ then $x - x < 0 - x$ by (Ω_3) so $-x > 0$, hence, given any element $x \neq 0_{\mathbb{k}}$, either $x > 0$ or $-x > 0$ (but not both) so either $x \in \mathbb{k}^+$ or $-x \in \mathbb{k}^+$. Further if $x < y$ then by (Ω_3) we have $x - x = 0 < y - x$ and applying (Ω_3) a second time, we have $-y < y - x - y = -x$ so if $x < y$ then $-y < -x$.

Definition 1.1.4. *The following function is called absolute value and can always be defined on an ordered field.*

$$\begin{aligned} |\cdot| : \mathbb{k} &\longrightarrow \mathbb{k}^+ \cup \{0_{\mathbb{k}}\} \\ x &\longmapsto |x| = \begin{cases} max(-x, x) & \text{if } x \neq 0_{\mathbb{k}} \\ 0_{\mathbb{k}} & \text{if } x = 0_{\mathbb{k}} \end{cases} \end{aligned}$$

Notice that this definition makes sense by the remark 2, and $\forall x \; |x| > 0_{\mathbb{k}}$.

Proposition 1.1.5. *The function* $|.|$ *defined above is a* norm*, this means that it satisfies the following properties :*

- $|x| = 0_{\mathbb{k}} \iff x = 0_{\mathbb{k}}$
- $|x \cdot y| = |x| \cdot |y|$
- $|x + y| \leq |x| + |y|$

Proof : The first point is true by definition. The proof of the two last point is by cases. Assume $x > 0$ and $y > 0$ then $x > -x$ and $y > -y$ so $|x| = x$ and $|y| = y$ and $xy > 0$ by (Ω_3) so $|xy| = xy = |x||y|$. Similarly $x + y > 0$ so $|x + y| = x + y = |x| + |y|$. If $x > 0$ and $y < 0$ then $x > -x$ so $|x| = x$ while $y < -y$ so $|y| = -y$. Now as $y < 0$, by (Ω_3), $xy < 0$ since $x > 0$, so $|xy| = -xy = (-1)xy = x(-y) = |x||y|$ by remark 1, commutativity and above. Now, $|x| + |y| = x - y$ and $|x + y| = x + y$ or $-(x + y) = -x - y$. But we have $-y > y$ so $x - y > x + y$ by (Ω_3), and $x > -x$ so $x - y > -x - y$ so in either cases, $|x| + |y| > |x + y|$. Now if $x < 0$ and $y < 0$, we have $|x| = -x$ and $|y| = -y$, so $|x||y| = (-x)(-y) = (- - 1)xy = xy$ $(- - 1 - 1 = -1(-1 + 1) = 0$ so $- - 1 = 1)$. Now, $-x > 0$ so by (Ω_3) $(-x)y < 0$ but $(-x)y = (-1)xy = -xy$ so $xy > 0$, so $|xy| = xy = |x||y|$. We know that $x + y < 0$ (by (Ω_3) and (Φ_4) with $x < 0$ and $y < 0$ so $x + y < 0 + 0 = 0$, so $|x + y| = -(x + y) = -x + -y = |x| + |y|$ (by distributivity (Φ_6)). Finally, if x or y is $0_{\mathbb{k}}$, then the results are trivial. □

We will now investigate the internal structure of an ordered field. We shall see that it always contains subsystems that behave like the natural, integers and rational numbers. To identify these systems inside any ordered field, we make the assumption of their existence. In other words, we assume that we are given the following systems $(\mathbb{N}, <, +, \cdot)$, $(\mathbb{Z}, <, +, \cdot)$, $(\mathbb{Q}, <, +, \cdot)$. The existence of these sets is discussed in 2.1.

Definition 1.1.6. *Let $n \in \mathbb{N}$, we define*

$$\underline{n} = \underbrace{1_{\Bbbk} + 1_{\Bbbk} + \cdots + 1_{\Bbbk}}_{n \text{ times}}$$

and call it the n-th natural element. We denote $\mathcal{N}_{\Bbbk}$ the set of all natural elements. Notice that we consider 0 as an element of $\mathbb{N}$ and set $\underline{0} = 0_{\Bbbk}$.

Remark 3. $1_{\Bbbk} + 1_{\Bbbk} + \cdots + 1_{\Bbbk}$ makes sens by associativity (Φ_2). $\mathcal{N}_{\Bbbk}$ is closed under addition and multiplication. It is easy to see that $n \mapsto \underline{n}$ from $\mathbb{N}$ to $\mathcal{N}_{\Bbbk}$ is a bijective function and preserve the addition, multiplication and order. It can be natural to define the notion of isomorphism for the structure of natural number with addition and multiplication. So it can be say that this function is an isomorphism for this structure. We will not define the notion of isomorphism for this structure but we will define it for the structure of ordered field later. So in every ordered field, we have a copy of the natural numbers. Besides, observe that from (Ω_3) follows that $n < m \iff \underline{n} < \underline{m}$. In particular, this means that $\mathcal{N}_{\Bbbk}$ inherite the basic order property of $\mathbb{N}$, that is that $(\mathcal{N}_{\Bbbk}, <)$ is well-ordered [1].

Definition 1.1.7. *We call any element of the following set an* integer element

$$\mathcal{Z}_{\Bbbk} \triangleq \mathcal{N}_{\Bbbk} \cup \{-x \in \Bbbk \mid x \in \mathcal{N}_{\Bbbk} \setminus \{0_{\Bbbk}\}\}$$

Remark 4. It is less obvious but still true that the map $\mathbb{Z} \to \mathcal{Z}_{\Bbbk}$ defined by $n \mapsto \underline{n} \in \mathcal{N}_{\Bbbk}$ if $n \in \mathbb{Z}^+ \triangleq \{n \in \mathbb{Z} \mid n > 0\}$ and $n \mapsto -\underline{-n} \in \{-x \in \Bbbk \mid x \in \mathcal{N}_{\Bbbk} \setminus \{0_{\Bbbk}\}\}$ if $n \in \mathbb{Z}^- \triangleq \{n \in \mathbb{Z} \mid n < 0\}$ and $0 \mapsto 0_{\Bbbk}$ preserve addition, multiplication and order. So we have a copy of the integers in each ordered field. Further, if $x, y \in \mathbb{Z}$ and $x < y$ then if $x, y > 0$ $\underline{x} < \underline{y}$ by the previous remark. If $x < 0$ and $y > 0$ then clearly $\underline{y} > 0_{\Bbbk}$ and $\underline{-x} > 0_{\Bbbk}$ so $\underline{x} = -\underline{-x} < 0_{\Bbbk}$ by remark 2 and so $\underline{x} < \underline{y}$. Finally if $x, y < 0$ then $0 < -y < -x$ so $0_{\Bbbk} < \underline{-y} < \underline{-x}$ by remark 3 and so $-\underline{-x} = \underline{x} < -\underline{-y} = \underline{y}$ by remark 2. So in fact we have $\forall x, y \in \mathbb{Z}\ x < y \iff \underline{x} < \underline{y}$.

Definition 1.1.8. *We call any element of the following set a* rational element

$$\mathcal{Q}_{\Bbbk} \triangleq \left\{x \cdot y^{-1} \in \Bbbk \mid x \in \mathcal{Z}_{\Bbbk},\ y \in \mathcal{Z}_{\Bbbk}^{+} = \mathcal{N}_{\Bbbk} \setminus \{0\}\right\}$$

Remark 5. Again the isomorphism $\mathbb{Z} \to \mathcal{Z}_{\Bbbk}$ extends to an isomorphism $\mathbb{Q} \to \mathcal{Q}_{\Bbbk}$, but we will refine this construction below. We extend the definition of $\underline{\cdot}$ by $\underline{x^{-1}} \triangleq \underline{x}^{-1}\ \forall x \in \mathbb{Q} \setminus \{0\}$. We will see later the purpose of these definitions, even if doing this is very natural.

Lemma 1.1.9. *Let $n, m \in \mathbb{N}$ we have the following*

1. $\forall n, m \in \mathbb{N}$ $\underline{n \cdot m} = \underline{n} \cdot \underline{m}$ *and* $\underline{n + m} = \underline{n} + \underline{m}$
2. $\forall n \in \mathbb{N}$ $\underline{-n} = -\underline{n}$ $\underline{n^{-1}} = \underline{n}^{-1}$

Proof :
Let $n, m \in \mathbb{N}$, by definition, we have the following :

1. In a well-ordered set, every non-empty subset has a minimal element, ie an element in the subset which is lower than or equal to every element in the subset.

$$
\begin{aligned}
\underline{n+m} &= \underbrace{1_{\Bbbk} + 1_{\Bbbk} + \cdots + 1_{\Bbbk}}_{n+m \text{ times}} \\
&= \underbrace{1_{\Bbbk} + \cdots + 1_{\Bbbk}}_{n \text{ times}} + \underbrace{1_{\Bbbk} + \cdots + 1_{\Bbbk}}_{m \text{ times}} \text{ by associativity } (\Phi_2) \\
&= \underline{n} + \underline{m} \\
\underline{n \cdot m} &= \underbrace{1_{\Bbbk} + 1_{\Bbbk} + \cdots + 1_{\Bbbk}}_{nm \text{ times}} \\
&= \underbrace{\underbrace{1_{\Bbbk} + \cdots + 1_{\Bbbk}}_{n \text{ times}} + \underbrace{1_{\Bbbk} + \cdots + 1_{\Bbbk}}_{n} + \cdots + \underbrace{1_{\Bbbk} + \cdots + 1_{\Bbbk}}_{n}}_{m \text{ times}} \text{ by associativity } (\Phi_2) \\
&= \underbrace{\underline{n} + \underline{n} + \cdots + \underline{n}}_{m \text{ times}} \\
&= \underline{n} \cdot (\underbrace{1_{\Bbbk} + 1_{\Bbbk} + \cdots + 1_{\Bbbk}}_{m \text{ times}}) \text{ by distributivity } (\Phi_6) \\
&= \underline{n} \cdot \underline{m}
\end{aligned}
$$

By definition, for $n < 0$, $\underline{n} = -\underline{-n}$ so for $n > 0$ $\underline{-n} = -\underline{n}$. Similarly the other is true by definition.

□

Definition 1.1.10. *Let* $(\Bbbk_1, <_1, +_1, \cdot_1)$, $(\Bbbk_2, <_2, +_2, \cdot_2)$ *be two ordered field and let* $\phi : \Bbbk_1 \to \Bbbk_2$*. We say that* ϕ *is an* isomorphism *between* $\Bbbk_1$ *and* $\Bbbk_2$ *if and only if* ϕ *is surjective, and*

- $\forall x, y \in \Bbbk_1\ \phi(x +_1 y) = \phi(x) +_2 \phi(y)$
- $\forall x, y \in \Bbbk_1\ \phi(x \cdot_1 y) = \phi(x) \cdot_2 \phi(y)$
- $\forall x, y \in \Bbbk_1\ x <_1 y \iff \phi(x) <_2 \phi(y)$

We say that two ordered fields are isomorphic *if there is an isomorphism between them (denoted* $(\Bbbk_1, <_1, +_1, \cdot_1) \cong (\Bbbk_2, <_2, +_2, \cdot_2)$*).*

Remark 6. It doesn't take long to see that $(\mathbb{Q}, <_{\mathbb{Q}}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}})$ is an ordered field, and we will show that any ordered field contains a copy of it, namely $\mathcal{Q}_{\Bbbk}$.

Theorem 1.1.11. *For every ordered field* $(\Bbbk, <, +, \cdot)$, $\mathcal{Q}_{\Bbbk}$ *is isomorphic to* $\mathbb{Q}$*.*

Proof : Recall that $\mathbb{Q}$ is the set of all elements of the form $p \cdot_{\mathbb{Q}} q^{-1}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$. Following the ideas of the remarks, we define the isomorphism

$$
\begin{aligned}
\phi : \mathbb{Q} &\longrightarrow \mathcal{Q}_{\Bbbk} \\
p \cdot_{\mathbb{Q}} q^{-1} &\longmapsto \underline{p} \cdot \underline{q}^{-1}
\end{aligned}
$$

We prove the second point of definition (1.1.10). Let $p_1q_1^{-1}, p_2q_2^{-1} \in \mathbb{Q}$, $\phi(p_1q_1^{-1} \cdot_{\mathbb{Q}} p_2q_2^{-1}) = \phi(p_1p_2 \cdot_{\mathbb{Q}} (q_1q_2)^{-1}) = \underline{p_1p_2} \cdot \underline{(q_1q_2)}^{-1}$. Now if $p_1p_2 > 0$ then $\underline{p_1p_2} = \underline{p_1} \cdot \underline{p_2}$ by lemma (1.1.9) 1. If $p_1p_2 < 0$ $\underline{p_1p_2} = -(\underline{-p_1p_2}) = -(-\underline{p_1} \cdot \underline{p_2}) = \underline{p_1} \cdot \underline{p_2}$ by lemma (1.1.9)1, 2. Besides, for all $x, y \in \Bbbk \setminus \{0\}$, $y^{-1}x^{-1} \cdot xy = 1_{\Bbbk}$ (by associativity (Φ_2)) so $(xy)^{-1} = x^{-1}y^{-1}$ (by commutativity (Φ_3)) and we have $\underline{(q_1q_2)^{-1}} = \underline{q_1}^{-1}\underline{q_2}^{-1}$. So finally, we have

$$\phi(p_1q_1^{-1} \cdot_{\mathbb{Q}} p_2q_2^{-1}) = \underline{p_1} \cdot \underline{p_2} \cdot \underline{q_1}^{-1} \cdot \underline{q_2}^{-1} = \underline{p_1} \cdot \underline{q_1}^{-1} \cdot \underline{p_2} \cdot \underline{q_2}^{-1} = \phi(p_1q_1^{-1})\phi(p_2q_2^{-1})$$

Now we show the first point of definition (1.1.10). Let $p_1q_1^{-1}, p_2q_2^{-1} \in \mathbb{Q}$, we have

$$\begin{aligned}
\phi(p_1q_1^{-1} +_{\mathbb{Q}} p_2q_2^{-1}) &= \phi((p_1q_2 + p_2q_1)\cdot(q_1q_2)^{-1}) \text{ by distributivity } (\Phi_6)\\
&= \underline{p_1q_2 + p_2q_1} \cdot \underline{(q_1q_2)^{-1}}\\
&= (\underline{p_1}\cdot\underline{q_2} + \underline{p_2}\cdot\underline{q_1})\cdot\underline{q_1}^{-1}\underline{q_2}^{-1} \text{ by lemma (1.1.9) and above}\\
&= \underline{p_1}\cdot\underline{q_2}\cdot\underline{q_1}^{-1}\underline{q_2}^{-1} + \underline{p_2}\cdot\underline{q_1}\cdot\underline{q_1}^{-1}\underline{q_2}^{-1} \text{ by distributivity } (\Phi_6)\\
&= \underline{p_1}\cdot\underline{q_1}^{-1} + \underline{p_2}\cdot\underline{q_2}^{-1}\\
&= \phi(p_1q_1^{-1}) + \phi(p_2q_2^{-1})
\end{aligned}$$

It remains to show the third point of the definition (1.1.10). Assume $pq^{-1} <_{\mathbb{Q}} p'q'^{-1}$, this is equivalent to $pq' <_{\mathbb{Q}} p'q$, as $qq' > 0$ and we have $pq', p'q \in \mathbb{Z}$, so by remark 4 we have $\underline{pq'} < \underline{p'q}$ that is equivalent to $\underline{pq'q'^{-1}q^{-1}} < \underline{p'qq'^{-1}q^{-1}}$ since $\underline{q'^{-1}q^{-1}} > 0_{\Bbbk}$ (see remark 1). But now, as previously seen, $\underline{pq'q'^{-1}q^{-1}} = \underline{pq^{-1}}$ and $\underline{p'qq'^{-1}q^{-1}} = \underline{p'q'^{-1}}$ so we have the desired result. By construction of $\mathcal{Q}_{\Bbbk}$ ϕ is surjective, and since $x < y \iff \phi(x) < \phi(y)$ it is also injective. So ϕ is an isomorphism. □

So we have shown that every ordered field contains at least a copy of the rational numbers. In fact the basic idea of ordered field comes from the field of the rational numbers. However we want more than the rational number, we want to describe an ordered field which behave like the real number system. Clearly the real number system as we conceive it in our mind is an ordered field, but to what extent is it different from $\mathbb{Q}$? The principal weakness of $\mathbb{Q}$ is that it is not "full". This means that we can easily conceive numbers that are not in it, for example the number whose square is 2 or the number π. We can translate this deficiency by saying that it is not *complete*, a notion we will discuss later. We want numbers like $\sqrt{2}$ or π to be in our concept of real numbers, but with the axioms we have it is not generally the case. To this end we will add another axiom called the completeness axiom which, as we will see, will characterise the real number system.

1.2 Complete ordered fields

In this part, we formalise the notion of complete ordered field, by adding another axiom to an ordered field namely the *completeness axiom.* To do that we need some basic definitions. We still consider as given an ordered field $(\Bbbk, <, +, \cdot)$.

Definition 1.2.1 (Upper bound, Least upper bound)**.** *Let $S \subseteq \Bbbk$, we say that $u \in \Bbbk$ is an* upper bound for *S if $\forall x \in S$ $x \leq u$. We say that u is the least upper bound (or* supremum*) if u is an upper bound and for every upper bound z we have $u \leq z$. We denote it (when it exists) by $Sup(S)$. We say that S is* bounded above *if is has an upper bound.*

Axiom 1.2.2 (Completeness)**.** *Every bounded above subset of $(\Bbbk, <, +, \cdot)$ has a least upper bound.*

This axiom is called completeness axiom, or least upper bound property.

Remark 7. From a model theoretic point of view, we observe that the axioms of ordered fields belong to a first order logic, so a model theoretic study of ordered fields is conceivable. However the least upper bound axiom is not in first order logic for the axiom implies a quantifier over all the subsets that are bounded above.

Definition 1.2.3 (Complete ordered field)**.** *We say that $(\Bbbk, <, +, \cdot)$ is* complete *if it satisfies the completeness axiom.*

It is also sometimes called *Dedekind-complete*, to differentiate this property from other equivalent notions of completeness. Now we can convince ourselves that this definition is correct by considering for example the set $\{x \in \mathbb{Q} \mid x \cdot x < 2\}$. Indeed, this set has an upper bound in $\mathbb{Q}$ but does not have a least upper bound (as $\sqrt{2} \notin \mathbb{Q}$). One can say that the set of algebraic numbers contains $\sqrt{2}$ so this property is not only for the reals, but the set $\{3, 3.1, 3.14, 3.141, \dots\}$ has no upper bound in the field of algebraic numbers, whereas there is one (and only one) in the real numbers (namely π). Of course, one should observe that the least upper bound property is symmetric in the sense that it implies what should be called the greatest lower bound property [2].

Two other interesting properties of certain ordered fields are defined below :

Definition 1.2.4 (Archimedean field)**.** *We say that* $(\Bbbk, <, +, \cdot)$ *is* archimedean *if for all* $x \in \Bbbk$ *there is a* $n \in \mathcal{N}_{\Bbbk}$ *such that* $x < n$*. This is equivalent to* $\forall x, y \in \Bbbk^+$ *there is a* $n \in \mathcal{N}_{\Bbbk}$ *such that* $x < n \cdot y$*.*

Informally, in an archimedean field there is no element bigger than any natural element. This is of course a property that we expect the real number system to have, but as we will see, it comes for free with the completeness. The notion of non-archimedean fields is interesting, as it led to the existence of elements that are bigger than any others, and, dually, elements that are smaller than any element. These are called *infinite* and *infinitesimal* element respectively, and provides powerful tools developed in non-standard analysis. In fact the main result of this article can be proved by going through non-archimedean fields. We will come back to non-archimedean ordered fields in section 1.4.

Definition 1.2.5 (Density of $\mathcal{Q}_{\Bbbk}$)**.** *We say that* $\mathcal{Q}_{\Bbbk}$ is dense *in* $(\Bbbk, <, +, \cdot)$ *if for all* $x, y \in \Bbbk$ *such that* $x < y$ *there is a* $q \in \mathcal{Q}_{\Bbbk}$ *such that* $x < q < y$*.*

This notion of density is also very natural, as we expect that if we have two elements, we want to find an element between them (for example the middle of the segment between these two points, if we think of the real numbers as a line).

These two properties are linked by

Theorem 1.2.6. *In every archimedean ordered field,* $\mathcal{Q}_{\Bbbk}$ *is dense.*

Proof : Assume $(\Bbbk, <, +, \cdot)$ is an archimedean ordered field. Let $x < y \in \Bbbk$. Assume that $x > 0$, then $(y - x) > 0$ so $(y - x)^{-1} > 0$ (by remark 1). Since $(\Bbbk, <, +, \cdot)$ is archimedean, there is a $m \neq 0 \in \mathcal{N}_{\Bbbk}$ such that $(y - x)^{-1} < m$, so $1 < ym - xm$. Now apply the archimedeanness again to get that $\{k \in \mathcal{N}_{\Bbbk} \mid mx < k\}$ is non empty, so by remark 3 there is a minimal, say n ($n \neq 0$ since $mx > 0$). Now by minimality, $n - 1 < mx$ and as $1 < my - mx$, by (Ω_3) we get that $(n-1)+1 < my - mx + (n-1)$, and applying again $n - 1 < mx$ we get $my - mx + (n-1) < my$ so we have $n < my$. Now $mx < n < my$ and as m^{-1} is positive (since m is and by remark 1) we get $x < nm^{-1} < y$.
Now if $x < 0$, then $-x > 0$ and by the archimedean property, there is a $k \in \mathcal{N}_{\Bbbk}$ such that $-x < k$, so $0 < k + x$ (and so $0 < k + x < k + y$) and by previousely, there is a $d \in \mathcal{Q}_{\Bbbk}$ such that $k + x < d < k + y$ and so $x < d - k < y$ (clearly, $d - k \in \mathcal{Q}_{\Bbbk}$). Now if $x = 0$, choose $n \in \mathcal{N}_{\Bbbk}$ such that $n > y^{-1}$ and now $ny > 1$ so $y > n^{-1} > 0 = x$. □

Theorem 1.2.7. *Every complete ordered field is archimedean.*

Proof : Assume $(\Bbbk, <, +, \cdot)$ is complete and non archimedean. Then there is an $x \in \Bbbk$ such that $x > n$ $\forall n \in \mathcal{N}_{\Bbbk}$, so x is an upper bound of $\mathcal{N}_{\Bbbk}$. By completeness, there is a least upper bound, say l, and then $l - 1$ is not an upper bound of $\mathcal{N}_{\Bbbk}$ (as $l - 1 < l$) so there is a $m \in \mathcal{N}_{\Bbbk}$ such

2. Simply observe that if a set is bounded below, then the set of lower bound is bounded above and so has a least upper bound, further, it is easy to prove that the least upper bound of the set of lower bound is the greatest lower bound of the set.

that $l-1 < m$ ie $l < m+1$, but $m+1 \in \mathcal{N}_{\Bbbk}$ so l can't be an upper bound for $\mathcal{N}_{\Bbbk}$, contradiction.□

We have immediately the following corollary.

Corollary 1.2.8. *In every complete ordered field, $\mathcal{Q}_{\Bbbk}$ is dense.*

1.3 Uniqueness of complete ordered fields

In this section, we prove the main theorem of this report, the uniqueness of the complete ordered field.

Theorem 1.3.1. *Assume $(\Bbbk_1, <_1, +_1, \cdot_1)$ and $(\Bbbk_2, <_2, +_2, \cdot_2)$ are two complete ordered fields, then*

$$(\Bbbk_1, <_1, +_1, \cdot_1) \cong (\Bbbk_2, <_2, +_2, \cdot_2)$$

Nota bene 1. We assume that the reader will make the difference between $+$, $\cdot$, and $<$ regarding in which set ($\Bbbk_1$ or $\Bbbk_2$) the elements in question are, so we will not use $+_1$,$\cdot_1$,$<_1$, $+_2$,$\cdot_2$,$<_2$ but $+$, $\cdot$, $<$. Besides, we denote $0_1, 1_1$ and $0_2, 1_2$ the neutral elements of $\Bbbk_1$ and $\Bbbk_2$ respectively.

First of all, we denote $\mathcal{Q}_1 \triangleq \mathcal{Q}_{\Bbbk_1}$ and $\mathcal{Q}_2 \triangleq \mathcal{Q}_{\Bbbk_2}$. Recall from theorem (1.1.11) that we have two isomorphisms $\phi_1 : \mathbb{Q} \to \mathcal{Q}_1$ and $\phi_2 : \mathbb{Q} \to \mathcal{Q}_2$, so it is easy to see that $\phi_2 \circ \phi_1^{-1} : \mathcal{Q}_1 \to \mathcal{Q}_2$ is an isomorphism (in other words, $\cong$ is a transitive relation). So given $q \in \mathcal{Q}_1$ we will denote $\bar{q} \in \mathcal{Q}_2$ the corresponding element. Recall that for $p, q \in \mathcal{Q}_1$

1. $\overline{p+q} = \bar{p} + \bar{q}$
2. $\overline{p \cdot q} = \bar{p} \cdot \bar{q}$
3. $q < p \iff \bar{q} < \bar{p}$

Now we define

$$\begin{aligned} \psi : \Bbbk_1 &\longrightarrow \Bbbk_2 \\ x &\longmapsto \psi(x) = Sup\,\{\bar{q} \in \mathcal{Q}_2 \mid q < x\} \end{aligned}$$

ψ is well defined by completeness of $\Bbbk_2$, as it is easy to check that, when it exists, the supremum is unique. We will prove that ψ is an isomorphism.
First we regard the action of ψ over $\mathcal{Q}_1$. Given $q \in \mathcal{Q}_1$, we have by definition $\psi(q) = Sup\,\{\bar{p} \in \mathcal{Q}_2 \mid p < q\}$. If $p < q$ then $\bar{p} < \bar{q}$ so we get $\psi(q) \le \bar{q}$ (as $\bar{q}$ is an upper bound of $\{\bar{p} \in \mathcal{Q}_2 \mid p < q\}$) . Now assume $\psi(q) < \bar{q}$. As $\Bbbk_2$ is complete, $\mathcal{Q}_2$ is dense in it by corollary (1.2.8), so there exists $\bar{r} \in \mathcal{Q}_2$ such that $\psi(q) < \bar{r} < \bar{q}$ but then $\psi(q)$ can't be the least upper bound. So in fact, $\psi(q) = \bar{q}\ \forall q \in \mathcal{Q}_1$.

Lemma 1.3.2. *For all $x, y \in \Bbbk_1$, we have :*

i $x < y \iff \psi(x) < \psi(y)$

ii For $p, q \in \mathcal{Q}_1$

$$Sup\,\{\bar{q} + \bar{p} \mid q < x\} = Sup\,\{\bar{q} \mid q < x\} + \bar{p}$$

and for $x \in \Bbbk_1^+$, $p \in \mathcal{Q}_1^+$ *and* $q \in \mathcal{Q}_1$

$$Sup\,\{\bar{q} \cdot \bar{p} \mid q < x\} = Sup\,\{\bar{q} \mid q < x\} \cdot \bar{p}$$

iii $\psi(x+y) = \psi(x) + \psi(y)$

iv $\psi(x \cdot y) = \psi(x) \cdot \psi(y)$

Proof :

(i)

Assume $x < y \in \Bbbk_1$. By density, there are $p, q \in \mathcal{Q}_1$ such that $x < p < q < y$. Now clearly, $\bar{p}$ is an upper bound of $\{\bar{r} \mid r < x\}$ so $\psi(x) \leq \bar{p}$. Similarly, $\bar{q} \leq \psi(y)$. Further, $\bar{p} < \bar{q}$ since $p < q$, so $\psi(x) < \psi(y)$. Now assume $\psi(x) < \psi(y)$ then assume $y \leq x$, by above, $\psi(y) \leq \psi(x)$, contradicting the trichotomy law. So we have $x < y \iff \psi(x) < \psi(y)$.

(ii)

If $q < x$ then trivially $\bar{q} < Sup\{\bar{q} \mid q < x\}$, and then $\bar{q} + \bar{p} < Sup\{\bar{q} \mid q < x\} + \bar{p}$ so

$$Sup\{\bar{q} + \bar{p} \mid q < x\} \leq Sup\{\bar{q} \mid q < x\} + \bar{p}$$

Now assume that this inequality is strict, by density there is a $\bar{r} \in \mathcal{Q}_2$ such that $Sup\{\bar{q} + \bar{p} \mid q < x\} < \bar{r} < Sup\{\bar{q} \mid q < x\} + \bar{p}$. In particular, $\bar{r} - \bar{p} = \overline{r - p} < Sup\{\bar{q} \mid q < x\}$ so $r - p < x$ so $\overline{r - p} + \bar{p} \in \{\bar{q} + \bar{p} \mid q < x\}$ and then $\bar{r} = \overline{r - p} + \bar{p} \leq Sup\{\bar{q} + \bar{p} \mid q < x\} < \bar{r}$ so $\bar{r} < \bar{r}$ contradicting (Ω_1) (trichotomy) as $\bar{r} = \bar{r}$ is true.

For the second equality, the exact same proof works, we do it for the sake of completeness. Let x, p, q be as in the hypothesis, then for $q < x$ we get $\bar{q} < Sup\{\bar{q} \mid q < x\}$ so $\bar{q}\bar{p} < Sup\{\bar{q} \mid q < x\}\,\bar{p}$ and so $Sup\{\bar{q} \cdot \bar{p} \mid q < x\} \leq Sup\{\bar{q} \mid q < x\} \cdot \bar{p}$. Now if the inequality is strict, by density we have a $\bar{r} \in \mathcal{Q}_2$ such that $Sup\{\bar{q} \cdot \bar{p} \mid q < x\} < \bar{r} < Sup\{\bar{q} \mid q < x\} \cdot \bar{p}$, so in particular $\overline{rp^{-1}} < Sup\{\bar{q} \mid q < x\}$ (this holds as $\bar{p}^{-1}$ is positive as $\bar{p}$ is by remark 1). Besides, $\bar{p}\overline{p^{-}1} = \overline{pp^{-1}} = 1_2$ so $\bar{p}^{-1} = \overline{p^{-1}}$ and so we have $\overline{rp^{-1}} < Sup\{\bar{q} \mid q < x\}$ so $rp^{-1} < x$ so $\overline{rp^{-1}}\bar{p} \in \{\bar{q}\bar{p} \mid q < x\}$ so $\bar{r} = \overline{rp^{-1}}\bar{p} \leq Sup\{\bar{q}\bar{p} \mid q < x\} < \bar{r}$ contradiction.

(iii)

Assume $x, y \in \Bbbk_1$, then $\psi(x + y) = sup\{\bar{r} \mid r < x + y\}$. Now observe that if $y = p \in \mathcal{Q}_1$, we have $\{\bar{q} \mid q < x + p\} = \{\overline{q + p} \mid q < x\} = \{\bar{q} + \bar{p} \mid q < x\}$. Now by ii $Sup\{\bar{q} + \bar{p} \mid q < x\} = Sup\{\bar{q} \mid q < x\} + \bar{p}$ so $\psi(x + p) = \psi(x) + \bar{p}$. Now we do the general case. Assume $p \in \mathcal{Q}_1$ $p < y$, now $x + p < x + y$ so by i we have $\psi(x + p) < \psi(x + y)$, so $\psi(x) + \bar{p} < \psi(x + y)$ and then for $p < y$ we have $\bar{p} < \psi(x + y) - \psi(x)$, so finally, $\psi(y) = Sup\{\bar{p} \mid p < y\} \leq \psi(x + y) - \psi(x)$. Now assume $\psi(y) < \psi(x + y) - \psi(x)$ now by density there is $\bar{r} \in \mathcal{Q}_2$ such that $\psi(y) < \bar{r} < \psi(x + y) - \psi(x)$ and by (i) $y < r$ so $\psi(x + y) - \psi(x) < \psi(x + r) - \psi(x) = \psi(x) + \bar{r} - \psi(x) = \bar{r}$ so $\bar{r} < \bar{r}$. So by contradiction, $\psi(y) = \psi(x + y) - \psi(x)$.

(iv)

Recall that by isomorphy of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ we get $\psi(0_1) = \bar{0}_1 = 0_2$ and $\psi(1_1) = \bar{1}_1 = 1_2$. Now for $x \in \Bbbk_1$, $\psi(x) + \psi(-x) = \psi(x - x) = \psi(0_1) = 0_2$ by iii so $\psi(-x) = -\psi(x)$. Now we first show that ψ is multiplicative for the positive elements. Let $x, y \in \Bbbk_1^+$. From $\psi(0_1) = 0_2$ and i we get that $\psi(x)$ and $\psi(y)$ are both positive. Following the same idea as in the proof of iii, consider that $y = p$ is a rational element. Then $\{\bar{q} \mid q < xp\} = \{\overline{qp} \mid q < x\} = \{\bar{q}\bar{p} \mid q < x\}$, the first equality holds as p is a positive element and the second by isomorphy. Now by ii $Sup\{\bar{q}\bar{p} \mid q < x\} = \bar{p}Sup\{\bar{q} \mid q < x\}$ so $\psi(xp) = \psi(x)\bar{p}$. Now for $x, y \in \Bbbk_1^+$ we have for $q \in \mathcal{Q}_1$, $0_1 < q < y$, $xq < xy$ so by (iii) $\psi(x)\bar{q} = \psi(xq) < \psi(xy)$ (as x, y, q are positive). Therefore $q < y$ implies $\bar{q} < \psi(xy)\psi(x)^{-1}$ (remark that as $\psi(x) > 0_2$ $\psi(x)^{-1} > 0_2$ by remark 1). Now $\psi(y) = Sup\{\bar{q} \mid q < y\} \leq \psi(xy)\psi(x)^{-1}$ and we will show that this is in fact an equality. Assume $\psi(y) < \psi(xy)\psi(x)^{-1}$ then by density there is a $\bar{r} \in \mathcal{Q}_2$ such that $\psi(y) < \bar{r} < \psi(xy)\psi(x)^{-1}$ and by i $y < r$, $\psi(xy)\psi(x)^{-1} < \psi(xr)\psi(x)^{-1} = \bar{r}$ so again, $\bar{r} < \bar{r}$ so by contradiction, $\psi(xy) = \psi(x)\psi(y)$ holds for $x, y \in \Bbbk_1^+$. Now we do the general case. Remark first that for $\epsilon = \pm 1_1$ we have (for $x \in \Bbbk_1^+$) $\psi(\epsilon x) = \bar{\epsilon}\psi(x)$. Now clearly we can always write $x = \epsilon_x |x|$ for $\epsilon_x = \pm 1_1$ for every

$x \in \Bbbk_1$, so for $x, y \in \Bbbk_1$ we have

$$\begin{aligned} \psi(xy) &= \psi(\epsilon_x|x|\epsilon_y|y|) \\ &= \bar{\epsilon}_x\bar{\epsilon}_y\psi(|x||y|) \\ &= \bar{\epsilon}_x\bar{\epsilon}_y\psi(|x|)\psi(|y|) \\ &= \psi(\epsilon_x|x|)\psi(\epsilon_y|y|) \\ &= \psi(x)\psi(y) \end{aligned}$$

□

Observe that the first point entails the injectivity of ψ, so it remains to show the surjectivity of ψ.

Assume $y \in \Bbbk_2$ and consider the sets $\overline{L} \triangleq \{\bar{r} \in \mathcal{Q}_2 \mid \bar{r} < y\}$ and $L \triangleq \{r \in \mathcal{Q}_1 \mid \bar{r} \in \overline{L}\}$. Clearly $y = sup(\overline{L})$ otherwise, $y > sup(\overline{L})$ as y is an upper bound of $\overline{L}$, and by density, there is $r \in \mathcal{Q}_2$ such that $sup(\overline{L}) < r < y$, which is a contradiction. Now L is bounded above, (take $\bar{q}$ as an upper bound of $\overline{L}$ then q is an upper bound of L), by completeness, there is a least upper bound, say x. We claim that $y = \psi(x)$. Take $\bar{r} \in \overline{L}$, then $r \leq x$, so by lemma (1.3.2) *i* $\bar{r} \leq \psi(x)$ and so $\psi(x)$ is an upper bound of $\overline{L}$, so $y \leq \psi(x)$. Now assume $y < \psi(x)$ then there is $\bar{q} \in \mathcal{Q}_2$ such that $y < \bar{q} < \psi(x)$, but then $q < x$ so $q \in L$ and so $\bar{q} \in \overline{L}$, so $\bar{q} < y$ contradiction. So in fact, $y = \psi(x)$, so ψ is surjective. Consequently, ψ is bijective and the theorem is proven. □

We have proved that if a complete ordered field exists, it is unique up to isomorphism. In the second chapter, we shall construct some complete ordered fields by different methods, but by this theorems, they will in fact represent the same mathematical object, the real number system.

1.4 A little more : Non-Archimedean ordered fields

In this section, we shall enounce -without proof since it goes further than the scope of this article, some results about non archimedean ordered fields. Although the reader should recall some basics about abstract algebra.

Recall the archimedean property

Definition 1.4.1 (Archimedean field). *We say that* $(\Bbbk, <, +, \cdot)$ *is* archimedean *if for all* $x \in \Bbbk$ *there is a* $n \in \mathcal{N}_{\Bbbk}$ *such that* $x < n$.

Clearly we expect the real numbers to have the archimedean property. But for the interest, in this section, we shall talk about non-archimedean ordered field. If the archimedean property is not satisfied in an ordered field $(\Bbbk, <, +, \cdot)$, then it ensure the existence of elements $x \in \Bbbk$ such that for every $n \in \mathcal{N}_{\Bbbk}$, $x > n$. We shall call such an element an *infinite element*. Furthermore $x > n$ entails $x^{-1} < n^{-1}$ so $\Bbbk$ also contains elements that have the property $x < n^{-1}$ $\forall n \in \mathcal{N}_{\Bbbk} \setminus \{0\}$.
Henceforth, $(\Bbbk, <, +, \cdot)$ is a non-archimedean field.

Definition 1.4.2. *We call* infinitesimal *any element* $x \in \Bbbk$ *such that* $|x| < n^{-1}$ $\forall n \in \mathcal{N}_{\Bbbk} \setminus \{0\}$. *Let* $\mathcal{I}_{\Bbbk}$ *be the set of all infinitesimals of* $\Bbbk$.

Definition 1.4.3. *We call* finite element *any element* $x \in \Bbbk$ *such that* $|x| < n$ *for some* $n \in \mathcal{N}_{\Bbbk}$. *Let* $\mathcal{F}_{\Bbbk}$ *be the set of all finite elements of* $\Bbbk$.

Remark 8. Notice that any infinitesimal is a finite element, so $\mathcal{I}_{\Bbbk} \subseteq \mathcal{F}_{\Bbbk}$. However, if $x^{-1} \in \mathcal{I}_{\Bbbk}$ then $x \notin \mathcal{F}_{\Bbbk}$. Notice that a field is archimedean if (and only if) $\mathcal{F}_{\Bbbk} = \Bbbk$ or equivalently $\mathcal{I}_{\Bbbk} = \emptyset$.

The following two propositions tell us more about algebraic properties of these two subsets of $\mathbb{k}$.

Proposition 1.4.4. $(\mathcal{F}_{\mathbb{k}}, +, \cdot)$ *is a subring of* $(\mathbb{k}, +, \cdot)$.

Proposition 1.4.5. $(\mathcal{I}_{\mathbb{k}}, +, \cdot)$ *is a maximal ideal of the ring* $(\mathcal{F}_{\mathbb{k}}, +, \cdot)$.

Recall that a maximal ideal of a ring is a subgroup closed under external multiplication which is maximal with respect to inclusion. From basic algebra we know that an ideal of a ring induces an equivalent relation, namely

$$x, y \in \mathcal{F}_{\mathbb{k}} \; x \sim_{\mathcal{I}_{\mathbb{k}}} y \iff x - y \in \mathcal{I}_{\mathbb{k}}$$

And then we get the factor ring

$$\mathcal{F}_{\mathbb{k}} / \mathcal{I}_{\mathbb{k}} = \{equivalent\ classes\ for\ \sim_{\mathcal{I}_{\mathbb{k}}}\}$$

which is a ring for the operations $+,\cdot$ defined to be compatible with the operations on $(\mathcal{F}_{\mathbb{k}}, +, \cdot)$. Further as $\mathcal{I}_{\mathbb{k}}$ is maximal, the factor ring has no non-zero proper ideal, so in fact

$$(\mathcal{F}_{\mathbb{k}} / \mathcal{I}_{\mathbb{k}}, +, \cdot) \; is\ a\ field$$

Moreover, for $[x], [y]$ the equivalent classes of $x, y \in \mathcal{F}_{\mathbb{k}}$, we can order $\mathcal{F}_{\mathbb{k}} / \mathcal{I}_{\mathbb{k}}$ by $[x] < [y] \iff x < y$ *and* $x \not\sim_{\mathcal{I}_{\mathbb{k}}} y$, to finally turn $(\mathcal{F}_{\mathbb{k}} / \mathcal{I}_{\mathbb{k}}, +, \cdot)$ into an *ordered* field $(\mathcal{F}_{\mathbb{k}} / \mathcal{I}_{\mathbb{k}}, <, +, \cdot)$.
Finally it can be proved that

Theorem 1.4.6.

$$(\mathcal{F}_{\mathbb{k}} / \mathcal{I}_{\mathbb{k}}, <, +, \cdot) \; is\ an\ archimedean\ ordered\ field$$

So it turns out that even in non-archimedean field we can get an archimedean ordered field. Now let us see an example of non-archimedean field.

Example 1.4.7 (The rational fractions)**.** *For this example, basic knowledge about polynomials shall be necessary, and no recall of this theory will be done. Let* $(\mathbb{k}, <, +, \cdot)$ *be an archimedean ordered field, and* X *be a transcendental symbol. In particular, we have a field* $(\mathbb{k}, +, \cdot)$ *so we have the set of all polynomials* $\mathbb{k}[X]$ *with coefficient in* $\mathbb{k}$ *with one indeterminate.* $\mathbb{k}[X]$ *is an integral domain, and can be turned into a field by taking the field of fraction*[3]*, we call it* $\mathbb{k}(X)$*. So* $(\mathbb{k}(X), +, \cdot)$ *is a field, we call it the field of* rational fractions of $\mathbb{k}$.
The purpose of this example is to order this field and turn it into an ordered field. Any element $R(X) \in \mathbb{k}(X)$ *can be represented by* $R(X) = \frac{a_m X^m + a_{m-1} X^{m-1} + \cdots + a_1 X + a_0}{b_n X^n + b_{n-1} X^{n-1} + \cdots + b_1 X + b_0}$ *with* $b_n \neq 0$*. Further, since we have a field* $\mathbb{k}$*, we can represent the same* R *by*

$$R(X) = \frac{c_m X^m + c_{m-1} X^{m-1} + \cdots + c_1 X + c_0}{X^n + u_{n-1} X^{n-1} + \cdots + u_1 X + u_0} = \frac{c_m X^m + c_{m-1} X^{m-1} + \cdots + c_1 X + c_0}{r(X)}$$

So every rational fraction can be represented with a monic polynomial as denominator. From that we can define the order :

$$\frac{c_m X^m + c_{m-1} X^{m-1} + \cdots + c_1 X + c_0}{r(X)} > 0 \iff c_m > 0$$

3. It is the same process as constructing the rational $\mathbb{Q}$ from the integers $\mathbb{Z}$. Given an integral domaine R, it consists of taking the factor of $R \times R \setminus \{0\}$ by the equivalent relation $(x_1, y_1), (x_2, y_2) \in R \times R \setminus \{0\}$ $(x_1, y_1) \sim (x_2, y_2) \iff x_1 y_2 = x_2 y_1$. $R \times R \setminus \{0\} / \sim$ turns out to be a field containing a copy or R as subring.

and set $R(X) > S(X) \iff R(X) + -S(X) > 0$ *(where* $-S(X)$ *denotes the opposite of* $S(X)$ *for* $+$*). Clearly* $(\Bbbk(X), <, +, \cdot)$ *satisfies* (Ω_1)*. Now observe that for* $R(X) = \frac{aX^m + r_1(X)}{r_2(X)} > 0$ *and* $S(X) = \frac{bX^n + s_1(X)}{s_2(X)} > 0$*, we have*

$$\begin{aligned} R(X) + S(X) &= \frac{aX^m + r_1(X)}{r_2(X)} + \frac{bX^n + s_1(X)}{s_2(X)} \\ &= \frac{s_2(X)(aX^m + r_1(X)) + r_2(X)(bX^n + s_1(X))}{r_2(X)s_2(X)} \end{aligned}$$

The numerator has leading coefficient the coefficient of $X^{max\{deg(s_2)+m, deg(r_2)+n\}}$ *which is* a*,* b *or* $a + b$ *so is positive. From this if we have* $R(X), S(X), T(X)$ *such that* $R(X) < S(X)$ *and* $S(X) < T(X)$ *we get that* $T(X) - R(X) = (T(X) - S(X)) + (S(X) - R(X))$ *which is the sum of two positive rational fractions, that is positive again, so* $R(X) < T(X)$*, and we have* (Ω_2)*.* (Ω_3) *comes trivially for the multiplication (observing that the leading coefficient of a product of polynomial is the product of the leading coefficient) and for the addition, it comes from the above result. So finally*

$$(\Bbbk(X), <, +, \cdot) \text{ is an ordered field}$$

We now show that it is not archimedean. Observe that $1_{\Bbbk(X)} = 1_\Bbbk$ *so in fact* $\mathcal{N}_{\Bbbk(X)} = \mathcal{N}_\Bbbk$ *and we shall denote the element* $\underbrace{1_\Bbbk + 1_\Bbbk + \cdots + 1_\Bbbk}_{n\ times}$ *by* n*. Now clearly the polynomial* $X - n$ *is positive for all* $n \in \mathcal{N}_\Bbbk$*, so in fact we get*

$$n < X \ \forall n \in \mathcal{N}_\Bbbk$$

meaning X *is an infinite element of* $\Bbbk(X)$*, and by extension, every rational fraction with degree*[4] *strictly positive is an infinite element of* $\Bbbk(X)$*.*
An example of an infinitesimal element of $\Bbbk(X)$ *is* $\frac{1}{X}$ *as for every* $n > 0$ *we have* $\frac{1}{X} - n^{-1} = \frac{-n^{-1}X + 1}{X} < 0$ *so*

$$\frac{1}{X} < n^{-1} \ \forall n \in \mathcal{N}_\Bbbk \setminus \{0\}$$

Notice that we can generalise the above results by

$$\mathcal{F}_\Bbbk = \Bbbk_{\leq 0}(X) = \left\{ \frac{p(X)}{q(X)} \mid (p(X), q(X)) \in \Bbbk[X] \times \Bbbk[X] \setminus \{0\},\ deg(p) \leq deg(q) \right\}$$

$$\mathcal{I}_\Bbbk = \Bbbk_{< 0}(X) = \left\{ \frac{p(X)}{q(X)} \mid (p(X), q(X)) \in \Bbbk[X] \times \Bbbk[X] \setminus \{0\},\ deg(p) < deg(q) \right\}$$

So in fact, by the theorem 1.4.6 we get that $\mathcal{F}_\Bbbk / \mathcal{I}_\Bbbk$ *is an archimedean ordered field.*

4. Recall that the degree of a rational fraction is defined as the degree of the numerator minus the degree of the denominator.

2 Various constructions of the real number system

2.1 The real number system

To construct something, we need a deep understanding of what we want to construct. The idea of this section is to list all the properties we expect the real number system to have. First of all, the notion of ordered fields match our ideas of how we can add, multiply and compare the reals. Further, we expect the set of reals to be like "a line", as we represent it most of the time when we are younger, and this natural notion is how we expect the real ordered field to be complete. So we want to construct a complete ordered field and, from theorem 1.3.1, we know that it will represent the unique complete ordered field, the real number system.
A little about constructivism : In the first chapter, we explored the *theoretical* study of what we expect the real number system to be. So in fact we worked with the philosophy "if the real number system exists, it has these properties, it is unique". In this chapter, we shall explicitely construct the complete ordered field and, from these constructions, we can consider the real number system as *existing*. However to produce our constructions, we will accept the *existence* of other structures (such as the rational numbers), which in fact can themselves be constructed from others structures, and eventually : the natural number system. All the objects whose existence we assume can be explicitely constructed from the natural numbers, but where does the natural number system come from ? The existence of the natural numbers is most of the time based on the acceptance of the axioms of *Peano arithmetic* [1]. Meaning there is always a point in which we have to accept the existence of mathematical objects, but a mathematician's philosophy is to find a way to accept as little as possible.

2.2 Construction via $\mathbb{Q}$-Cauchy sequences

2.2.1 The ordered field $\mathbb{Q}$

In this section, we will describe the rational number system, as the construction of $\mathbb{R}$ via Cauchy sequences is based on it. We consider $\mathbb{Q}$ as the set of rational numbers, with the two natural binary operations $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$. We also have the predicate $<_{\mathbb{Q}}$. We specify the notation since we will later deal with different binary operations and predicate. The construction of $\mathbb{R}$ will need some assumptions about $\mathbb{Q}$, which can be proved from the construction of $\mathbb{Q}$ from the integers, and similarly, these properties of $\mathbb{Z}$ can be proved by the construction of $\mathbb{N}$. So in fact, the assumptions we are going to make about $\mathbb{Q}$ are provable from the construction of $\mathbb{N}$, but this is not the purpose of this report. We have the following assumption :

$$(\mathbb{Q}, <_{\mathbb{Q}}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}) \textit{ is an ordered field}$$

1. One can also construct the natural numbers from set theory, by what is called the *Von Neumann ordinal numbers.*

Note that $a \cdot_{\mathbb{Q}} b$ will be denoted ab for $a, b \in \mathbb{Q}$ when it is unambiguous. Another important assumption about $\mathbb{Q}$ must be stated, it is that $\mathbb{Q}$ has the archimedean property. All that we assume and will not be proved in this report about $\mathbb{Q}$ is summarized in the following.

Assumption 2.2.1. *We have :*

- $(\mathbb{Q}, <_{\mathbb{Q}}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}})$ *is an ordered field*
- $(\mathbb{Q}, <_{\mathbb{Q}}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}})$ *is archimedean, i.e* $\forall a, b \in \mathbb{Q}^+ \; \exists n \in \mathbb{N}, \; b < an$ *(see definition 1.2.4)*

Now from the first chapter (definition 1.1.4), we have a function $|\cdot| : \mathbb{Q} \to \mathbb{Q}^+$ (recall that $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid 0 <_{\mathbb{Q}} x\}$) which is a norm, so it define a distance and then we have a metric topology on $\mathbb{Q}$. This allows us to talk about the notion of convergent sequences and Cauchy sequences. Let $\mathbb{Q}^{\mathbb{N}}$ denote the set of all sequences of $\mathbb{Q}$, that is

$$\mathbb{Q}^{\mathbb{N}} = \{x : \mathbb{N} \to \mathbb{Q}\}$$

An element $x : \mathbb{N} \to \mathbb{Q}$ of $\mathbb{Q}^{\mathbb{N}}$ will be denoted (x_n) and $x(n)$ will be denoted x_n.

Definition 2.2.2 (Limit, convergence and $\mathbb{Q}$-Cauchy sequence). *Let* $(x_n) \in \mathbb{Q}^{\mathbb{N}}$*,* $l \in \mathbb{Q}$*, we say that* l is the limit of the sequence (x_n) *if and only if*

$$\forall \epsilon \in \mathbb{Q}^+ \; \exists N_\epsilon \in \mathbb{N} (n \geq N_\epsilon \to |x_n - l| < \epsilon)$$

We say that $(x_n) \in \mathbb{Q}^{\mathbb{N}}$ *is convergent if it has a limit* $l \in \mathbb{Q}$*.*
We say that (x_n) *is a* $\mathbb{Q}$-Cauchy sequence *(or Cauchy sequence) if it satisfies the following property*

$$\forall \epsilon \in \mathbb{Q}^+ \; \exists N_\epsilon (n, m \geq N_\epsilon \to |x_n - x_m| < \epsilon)$$

Remark 9. Every Cauchy sequence is bounded. Given any Cauchy sequence (x_n) and $\epsilon \in \mathbb{Q}^+$ we have a $N \in \mathbb{N}$ such that for any $n \geq N$, $|x_N - x_n| < \epsilon$ and this is equivalent to $|x_n| < \epsilon + |x_N|$, so (x_n) is bounded after N and before we can take the maximum value of (x_n). Therefore,

$$X = max\,\{|x_0|, |x_1|, \ldots, |x_{N-1}|, |x_N| + \epsilon\}$$

is a bound of the sequence (x_n).

We know that every convergent sequence is a Cauchy sequence [2] but we can find a $\mathbb{Q}$-Cauchy sequence that has no limit in $\mathbb{Q}$, for example define $x_0 = 1, \; x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$, which is clearly an element of $\mathbb{Q}^{\mathbb{N}}$, with the limit being $\sqrt{2}$ which is not in $\mathbb{Q}$. But the notion of Cauchy sequence is interesting since we expect that a Cauchy sequence has a limit somewhere, and the fact that in $\mathbb{Q}$ there are non convergent Cauchy sequences is a weakness of $\mathbb{Q}$. We expect that the complete ordered field $\mathbb{R}$ has this property but in fact we have the equivalence of complete ordered field and ordered fields in which every Cauchy sequence is convergent [3]. The construction of $\mathbb{R}$ via Cauchy sequences is based on this problem, we want $\mathbb{R}$ to be an ordered field containing $\mathbb{Q}$ but in which every Cauchy sequence is convergent. So the idea is to take $\mathbb{Q}$ and add all the "limit of Cauchy sequences" in it, to get $\mathbb{R}$. To this end, we remark first that the number $\sqrt{2}$ can be seen as "the limit of the sequence $(x_n) \in \mathbb{Q}^{\mathbb{N}}$" defined above, and that every $\lambda \in \mathbb{Q}$ can be seen as "the limit of the sequence $(\lambda_n) \in \mathbb{Q}^{\mathbb{N}}$" with $\lambda_n = \lambda \forall n$. With this idea, we can imagine that every

2. Take a convergent sequence (x_n), so for $\epsilon > 0$ there is N such that $|x_n - l| < \frac{\epsilon}{2}$ for all $n \geq N$, therefore, for $n, m \geq N$ $|x_n - x_m| \leq |x_n - l| + |x_m - l| \leq \epsilon$ so (x_n) is Cauchy.

3. Indeed, in a metric space, the notion of completeness (sometimes called Cauchy completeness) says that every Cauchy sequence is convergent. This notion of convergences coincide in archimedean field with the one we have seen before (sometimes called Dedekind completeness) and the proof is very standard, involving the famous Bolzano-Weierstrass theorem.

real number can be identified by a Cauchy sequence of rational numbers. To convince yourself of that, one can think every real number as a list of digit before and after a dot, for example :

$$2.5 \quad 3.333333\ldots \quad 2121.2121\ldots \quad 3.141592\ldots$$

Now one can easily consider the sequence :

$$\begin{aligned} x_0 &= 3 \\ x_1 &= 3.1 \\ x_2 &= 3.14 \\ &\vdots \end{aligned}$$

and define a Cauchy sequence of rational numbers that converge to π (such a sequence is clearly Cauchy, since we add at each n^{th} term a number lower than 10^{n-1}). So we have a way to think about real numbers inside the set of $\mathbb{Q}$-Cauchy sequences. We denote with $\mathcal{C}_\mathbb{Q}$ the set of all $\mathbb{Q}$-Cauchy sequences and to define $\mathbb{R}$ we need some properties of $\mathcal{C}_\mathbb{Q}$.

2.2.2 The space $\mathcal{C}_\mathbb{Q}$

We define over $\mathcal{C}_\mathbb{Q}$ two binary operations which will be used to construct the operations on the reals.

Proposition 2.2.3 (Addition and Multiplication on $\mathcal{C}_\mathbb{Q}$). *The operation*

$$\begin{aligned} \mathcal{C}_\mathbb{Q} \times \mathcal{C}_\mathbb{Q} &\longrightarrow \mathcal{C}_\mathbb{Q} \\ (x_n), (y_n) &\longmapsto (x_n) \dotplus (y_n) = (z_n) \text{ with } z_n = x_n +_\mathbb{Q} y_n \forall n \in \mathbb{N} \end{aligned}$$

is a well defined binary operation over $\mathcal{C}_\mathbb{Q}$, we call it addition on $\mathcal{C}_\mathbb{Q}$.
The operation

$$\begin{aligned} \mathcal{C}_\mathbb{Q} \times \mathcal{C}_\mathbb{Q} &\longrightarrow \mathcal{C}_\mathbb{Q} \\ (x_n), (y_n) &\longmapsto (x_n) * (y_n) = (z_n) \text{ with } z_n = x_n \cdot_\mathbb{Q} y_n \ \forall n \in \mathbb{N} \end{aligned}$$

is a well defined binary operation over $\mathcal{C}_\mathbb{Q}$, we call it multiplication on $\mathcal{C}_\mathbb{Q}$.
Further, these two operations are associative, commutative, and $*$ is distributive over $\dotplus$.

Proof : We want to show that if (x_n) and (y_n) are Cauchy sequences, then $(x_n) \dotplus (y_n)$ and $(x_n) * (y_n)$ are also Cauchy sequences. Let $\epsilon \in \mathbb{Q}^+$, then, by hypothesis, there are $N_{\epsilon,x}$ and $N_{\epsilon,y}$ such that $n, m \geq N_{\epsilon,u} \rightarrow |u_n - u_m| < \frac{\epsilon}{2}$ for $u = x$ or $u = y$. Now let $N_\epsilon = max\{N_{\epsilon,u}\}_{u=x,y}$ and assume $n, m \geq N_\epsilon$, then

$$\begin{aligned} |(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \text{by associativity of the field } \mathbb{Q} \\ &\leq |x_n - x_m| + |y_n - y_m| \text{as } |\cdot| \text{ is a norm} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

So $(x_n) \dotplus (y_n) \in \mathcal{C}_\mathbb{Q}$. Similarly we know by remark 9 that there are $X, Y \in \mathbb{N}$ such that $|x_n| \leq X$ and $|y_n| \leq Y \ \forall n \in \mathbb{N}$, so we can find for each $\epsilon \in \mathbb{Q}^+$ a $N_\epsilon \in \mathbb{N}$ such that $|x_n - x_m| < \frac{\epsilon}{2Y}$ and $|y_n - y_m| < \frac{\epsilon}{2X}$ and we have :

$$\begin{aligned} |(x_n y_n) - (x_m y_m)| &= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\ &= |x_n(y_n - y_m) + (x_n - x_m) y_m| \\ &\leq |x_n(y_n - y_m)| + |(x_n - x_m) y_m| \text{as } |\cdot| \text{ is a norm} \\ &\leq |x_n||(y_n - y_m)| + |(x_n - x_m)||y_m| \\ &< X\frac{\epsilon}{2X} + \frac{\epsilon}{2Y} Y = \epsilon \end{aligned}$$

So $(x_n) * (y_n)$ is a Cauchy sequence. The associativity, commutativity and distributivity come immediately from the associativity, commutativity and distributivity of $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$. □

Remark 10. With the observations at the end of the last subsection, we see that there is a natural injection of $\mathbb{Q}$ in $\mathcal{C}_{\mathbb{Q}}$ given by

$$\begin{aligned} \iota : \mathbb{Q} &\hookrightarrow \mathcal{C}_{\mathbb{Q}} \\ \lambda &\mapsto (\lambda_n) \triangleq (\lambda, \dots, \lambda, \dots) \end{aligned}$$

We easily understand that $\dot{+}$ and $*$ match the addition and multiplication inside $\mathbb{Q}$ in the sense that $\iota(\lambda + \mu) = \iota(\lambda) \dot{+} \iota(\mu)$ and $\iota(\lambda \cdot \mu) = \iota(\lambda) * \iota(\mu)$. So we see that a copy $\iota(\mathbb{Q})$ of the set $\mathbb{Q}$ lies inside $\mathcal{C}_{\mathbb{Q}}$, and we can add and mulptiply in the same way that we do in $\mathbb{Q}$.

Nota bene 2. Basic set theory tells us that the cardinality of $\mathbb{R}$ is strictly more than that of $\mathbb{Q}$ and in fact, it is known that $|\mathbb{R}| = |\mathbb{Q}^{\mathbb{N}}|$ so this gives us a good feeling about the choice of constructing the reals from the set of sequences.

Definition 2.2.4 (Zero sequence, one sequence, opposite and inverse)**.** *We say that $(x_n) \in \mathcal{C}_{\mathbb{Q}}$ is a* zero sequence *if it converges to* 0*. We denote by $\mathcal{Z}$ the set of all zero-sequences.*
We say that $(x_n) \in \mathcal{C}_{\mathbb{Q}}$ is a one sequence *if it converges to* 1*. We denote by $\mathcal{E}$ the set of all one-sequences.*
Given $(x_n) \in \mathcal{C}_{\mathbb{Q}}$ we define $\dot{-}(x_n) = (-x_n)$, i.e $\dot{-}(x_n)$ is the sequence whose n^{th} term is given by $-x_n$. We also define :

$$(x_n)^{\dot{-}1} = \begin{cases} \frac{1}{x_n} & \text{if } x_n \neq 0 \\ 0 & \text{if } x_n = 0 \end{cases}$$

We will denote $(x_n) \dot{+} \dot{-}(y_n)$ by $(x_n) \dot{-} (y_n)$.

Lemma 2.2.5. *The operation $\dot{-}(x_n)$ is well defined on $\mathcal{C}_{\mathbb{Q}}$ and $(x_n)^{\dot{-}1}$ is well defined on $\mathcal{C}_{\mathbb{Q}} \setminus \mathcal{Z}$. Furthermore, we have $\forall (x_n) \in \mathcal{C}_{\mathbb{Q}}\ (x_n) \dot{-} (x_n) = (0_n)$ and $\forall (x_n) \in \mathcal{C}_{\mathbb{Q}} \setminus \mathcal{Z}\ (x_n) * (x_n)^{\dot{-}1} \in \mathcal{E}$*

Proof : If (x_n) is a Cauchy sequence, then so is $\dot{-}(x_n)$ as

$$\forall \epsilon \in \mathbb{Q}^+\ \exists N_\epsilon (n, m \geq N_\epsilon \to |x_n - x_m| < \epsilon)$$

is equivalent to

$$\forall \epsilon \in \mathbb{Q}^+\ \exists N_\epsilon (n, m \geq N_\epsilon \to |-x_n - -x_m| < \epsilon)$$

as $|-\lambda| = |\lambda| \forall \lambda \in \mathbb{Q}$. Now if $(x_n) \in \mathcal{C}_{\mathbb{Q}} \setminus \mathcal{Z}$ then (x_n) does not converge to 0 and so by contrapositive, there is a $C \in \mathbb{Q}^+$ and $K \in \mathbb{N}$ such that for each $n \geq K$ $|x_n| \geq C$. However (x_n) is Cauchy so given an $\epsilon \in \mathbb{Q}^+$, we have a N_ϵ such that $|x_n - x_m| < \epsilon C^2 \ \ \forall n, m \geq N_\epsilon$. Now consider $(y_n) = (x_n)^{\dot{-}1}$ and for $n, m \geq max(K, N_\epsilon)$

$$\begin{aligned} |y_n - y_m| &= |\frac{1}{x_n} - \frac{1}{x_m}| \text{ as } |x_n| > 0 \text{ for } n \geq K \\ &= |\frac{x_m - x_n}{x_n x_m}| \\ &\leq \frac{|x_m - x_n|}{C^2} \\ &\leq \frac{\epsilon C^2}{C^2} = \epsilon \end{aligned}$$

Hence $(x_n)^{\dot{-}1}$ is Cauchy. Clearly $(x_n)\dot{-}(x_n)$ is the sequence of general term $x_n - x_n = 0$ so $(x_n)\dot{-}(x_n) = (0_n)$.
Consider $(x_n) \notin \mathcal{Z}$. By definition this implies that for each $N \in \mathbb{N}$ there is a $n \geq N$ with $|x_n|$ more than a certain positive constant, so in particular, for each N there is a $x_n \neq 0$ with $n \geq N$. Now consider $(a_n) = (x_n) * (x_n)^{\dot{-}1}$, by definition, $\forall n\ a_n = 0$ *or* 1, and by the previous observation, we can find a 1 as far as we can go, ie $\forall N$ there is an $n \geq N$ with $a_n = 1$. However as it is Cauchy (by previously) and has only values 0 or 1, it is convergent either to 0 or 1. Assume it converges to 0, this will contradict the fact that we can always find a 1 for a certain $n \geq N$ for every $N \in \mathbb{N}$, so (a_n) converges to 1, so $(x_n) * (x_n)^{\dot{-}1} \in \mathcal{E}$. □

Remark 11. Notice that $\mathcal{Z} \subseteq \mathcal{C}_\mathbb{Q}$ since every convergente sequence is Cauchy by footnote 2. We will see that the element $\dot{-}(x_n)$ and $(x_n)^{\dot{-}1}$ will respectively play the role of the opposite and the inverse of the element (x_n). The two operations $\dot{-}$ and $^{\dot{-}1}$ have a good behaviour on $\iota(\mathbb{Q})$ since it is easy to verify that $\forall \lambda \in \mathbb{Q}\ \iota(-\lambda) = \dot{-}\iota(\lambda)$ and $\lambda \neq 0 \implies \iota(\frac{1}{\lambda}) = \iota(\lambda)^{\dot{-}1}$. So for now, we have a copy $\iota(\mathbb{Q})$ of the *field* $\mathbb{Q}$ inside $\mathcal{C}_\mathbb{Q}$. This means that our former idea, "add the limit of Cauchy sequences to $\mathbb{Q}$" seems to be followed, we have a set containing the field $\mathbb{Q}$ and in which the $\mathbb{Q}$-Cauchy sequences are represented. For example the $\mathbb{Q}$-Cauchy sequence that converges to π and the one that converges to $\sqrt{2}$ are elements of $\mathcal{C}_\mathbb{Q}$, so they are represented in our set.

Now that we have two operations that will behave like addition and multiplication of reals, we need to define a binary predicate which will play the role of the comparison relation. We will do that through a set of positive elements which will be construct from the set of *ultimately positive sequences.*

Definition 2.2.6 (Ultimately positive Cauchy sequence)**.** *Let $(x_n) \in \mathcal{C}_\mathbb{Q}$, we say that (x_n) is an* ultimately positive sequence *if $(x_n) \notin \mathcal{Z}$ and there exists a $N \in \mathbb{N}$ such that*

$$n \geq N \implies x_n > 0$$

We call $\mathcal{C}_\mathbb{Q}^+$ the set of all ultimately positive sequences.

Lemma 2.2.7.
Let $(x_n) \in \mathcal{C}_\mathbb{Q} \setminus \mathcal{Z}$ then one of the following holds

$$(x_n) \in \mathcal{C}_\mathbb{Q}^+ \text{ or } \dot{-}(x_n) \in \mathcal{C}_\mathbb{Q}^+$$

Besides, $\mathcal{C}_\mathbb{Q}^+$ is closed under $\dot{+}$ and $*$.

Proof : As (x_n) doesn't converges to 0, we have that $\exists v \in \mathbb{Q}^+$ such that $\forall N \in \mathbb{N}$ there is an $n \geq N$ such that $|x_n| \geq v$. Now, since (x_n) is Cauchy, we have that for this same v there is an $M \in \mathbb{N}$ such that $|x_n - x_m| < v$ for $n, m \geq M$. Now for $n, m \geq max(N, M)$ we have that x_n and x_m have the same sign since otherwise, we should have $|x_n - x_m| \geq 2v$ contradicting $|x_n - x_m| < v$. So in fact either $x_n \geq v > 0$ or $-x_n \geq v > 0$, so either (x_n) or $\dot{-}(x_n)$ is ultimately

positive.
If (x_n) and (y_n) are in $\mathcal{C}_{\mathbb{Q}}^+$ then there exists $N, M \in \mathbb{N}$ such that $x_n > 0$ for $n \geq N$ and $y_n > 0$ for $n \geq M$, so for $n \geq K = max\{N, M\}$ we have that $x_n + y_n$ is a sum of two positive elements of $\mathbb{Q}$ which is a positive elements of $\mathbb{Q}$, so the sequence whose nth term is $x_n + y_n$ is ultimately positive, ie $(x_n) \dot{+} (y_n) \in \mathcal{C}_{\mathbb{Q}}^+$. Similarly, for the same K, $n > K$ we have $x_n \cdot y_n$ is a product of two positive rationals and as $\mathbb{Q}$ is an ordered field, we have that $x_n \cdot y_n$ is positive, and so $(x_n) * (y_n) \in \mathcal{C}_{\mathbb{Q}}^+$. □

Following the end of remark 11, we have a set that represent $\mathbb{Q}$ (by $\iota(\mathbb{Q})$) and all the elements that we can imagine in our idea of $\mathbb{R}$. But we see that the system of Cauchy sequences at this point has a huge weakness, being that if we take the two sequences $(1/n)$ and (0_n) then both of them behave like an absorbing element for $*$, but we want zero to be unique. Moreover if we want to see element of $\mathbb{R}$ as limit of $\mathbb{Q}$-Cauchy sequence, then *an element of* $\mathbb{R}$ *is represented in* $\mathcal{C}_{\mathbb{Q}}$ *by all the sequences that converges to it*, so an element of $\mathbb{R}$ has many representations in $\mathcal{C}_{\mathbb{Q}}$, meaning we need to gather in one element the sequences that stand for the same element, and to do so, we define an equivalence relation over $\mathcal{C}_{\mathbb{Q}}$.

Definition 2.2.8. *We define the relation* $\sim$ *on* $\mathcal{C}_{\mathbb{Q}}$

$$(x_n) \sim (y_n) \iff (x_n) \dot{-} (y_n) \in \mathcal{Z}$$

This means that $(x_n) \sim (y_n) \iff (x_n) \dot{-} (y_n) = (x_n - y_n)$ *converge to zero, ie*

$$(x_n) \sim (y_n) \iff \forall \epsilon \in \mathbb{Q}^+ \ \exists N_\epsilon \in \mathbb{N}(n \geq N_\epsilon \to |x_n - y_n| < \epsilon)$$

Remark 12. Notice that for a sequence converging to a $\lambda \in \mathbb{Q}$, *the equivalent class is the set of all sequences converging to this* λ, and so the constant sequence $\iota(\lambda) = (\lambda_n)$ is in the equivalent class. In particular we will represent the equivalent class of (0_n) by $[0]$ and the equivalent class of (1_n) by $[1]$. We remark that $[0] = \mathcal{Z}$ and $[1] = \mathcal{E}$. Notice also that *the equivalent class of a sequence converging to* t *is the set of all sequences converging to this* t is true considering the limit $t \in \mathbb{R}$ but we cannot define two Cauchy sequence as equivalent if and only if they have same limit since the limit may not be in $\mathbb{Q}$ and this definition should assume the existence of $\mathbb{R}$, which is what we are constructing. But this is the idea, and talking about Cauchy sequence allow us to not talk about limits.

Proposition 2.2.9. *The predicate* $\sim$ *is an* equivalence relation *on* $\mathcal{C}_{\mathbb{Q}}$.

Proof : Reflexivity : $(x_n) \dot{-} (x_n) = (0_n)$ so $(x_n) \sim (x_n)$.
Symmetry : $(x_n) \sim (y_n) \iff \forall \epsilon \in \mathbb{Q}^+ \ \exists N_\epsilon \in \mathbb{N}(n \geq N_\epsilon \to |x_n - y_n| < \epsilon) \iff \forall \epsilon \in \mathbb{Q}^+ \ \exists N_\epsilon \in \mathbb{N}(n \geq N_\epsilon \to |y_n - x_n| < \epsilon) \iff (y_n) \sim (x_n)$.
Transitivity : Assume $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$, then for $\epsilon \in \mathbb{Q}^+$ there are $N, M \in \mathbb{N}$ such that $|x_n - y_n| < \frac{\epsilon}{2}$ and $|y_m - z_m| < \frac{\epsilon}{2}$ for $n \geq N$, $m \geq M$. So for $n \geq K = max(N, M)$, we have

$$\begin{aligned} |x_n - z_n| &= |x_n - y_n + y_n - z_n| \\ &\leq |x_n - y_n| + |y_n - z_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

so $(x_n) \sim (z_n)$. □

We have now the tools needed to construct a complete ordered field.

Definition 2.2.10. *We denote by* $\mathscr{R}$ *the set consisting of all equivalence classes of* $\mathcal{C}_{\mathbb{Q}}$ *for* $\sim$. *In other words*

$$\mathscr{R} = \mathcal{C}_{\mathbb{Q}} / \sim$$

We denote the class of an element $(x_n) \in \mathcal{C}_\mathbb{Q}$ by $[(x_n)]$ or $[x_n]$, and we have

$$[x_n] = \{(x'_n) \in \mathcal{C}_\mathbb{Q} \mid (x_n) \sim (x'_n)\}$$

Remark 13. Now we have gathered the sequences that stood for the same element in one class and we will define $\mathbb{R}$ to be this set of equivalent classes. It is noteworthy that $(\lambda_n), (\mu_n) \in \iota(\mathbb{Q})$, we have $(\lambda_n) \sim (\mu_n) \iff \lambda = \mu$ this means that we can extend ι on $\mathscr{R}$ by

$$\begin{array}{rcl} \widetilde{\iota} : \mathbb{Q} & \hookrightarrow & \mathscr{R} \\ \lambda & \mapsto & [\lambda] \triangleq [\iota(\lambda)] \end{array}$$

so we still have a copy of the set $\mathbb{Q}$ inside $\mathscr{R}$ and we will denote it $[\mathbb{Q}] \triangleq \widetilde{\iota}(\mathbb{Q})$. We have by the previous remark that every sequence (x_n) converging to a rational number λ is equivalent to $\iota(\lambda)$ so $[x_n] = [\lambda]$ and we will prefer the right hand side representative.

2.2.3 The complete ordered field $\mathscr{R}$

Proposition 2.2.11 (Addition and Multiplication on $\mathscr{R}$). *The operation*

$$\begin{array}{rcl} \mathscr{R} \times \mathscr{R} & \longrightarrow & \mathscr{R} \\ [x_n], [y_n] & \longmapsto & [x_n] + [y_n] = [(x_n) \dotplus (y_n)] \end{array}$$

is a well defined binary operation over $\mathscr{R}$, we call it addition on $\mathscr{R}$.
The operation

$$\begin{array}{rcl} \mathscr{R} \times \mathscr{R} & \longrightarrow & \mathscr{R} \\ [x_n], [y_n] & \longmapsto & [x_n] \cdot [y_n] = [(x_n) * (y_n)] \end{array}$$

is a well defined binary operation over $\mathscr{R}$, we call it multiplication on $\mathscr{R}$.
Furthermore, $+$ and $\cdot$ inherite the associativity, commutativity and distributivity properties from $\dotplus$ and $*$.

Proof : We show that $+$ and $\cdot$ are well defined. This means that if $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$ then $(x_n) \dotplus (y_n) \sim (x'_n) \dotplus (y'_n)$ and $(x_n) * (y_n) \sim (x'_n) * (y'_n)$. Let $\epsilon \in \mathbb{Q}^+$, we know that there exits a $N_{\epsilon,x}$ such that $|x_n - x'_n| < \epsilon/2$ for $n \geq N_{\epsilon,x}$, and there exists a $N_{\epsilon,y}$ such that $|y_n - y'_n| < \epsilon/2$ for $n \geq N_{\epsilon,y}$. Set $N_\epsilon = max(N_{\epsilon,x}, N_{\epsilon,y})$. Now we have for $n \geq N_\epsilon$:

$$\begin{array}{rcl} |(x_n + y_n) - (x'_n + y'_n)| & = & |(x_n - x'_n) + (y_n - y'_n)| \text{ by associativity of } +_\mathbb{Q} \\ & \leq & |x_n - x'_n| + |y_n - y'_n| \text{ by triangle inequality} \\ & < & \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon \end{array}$$

So $((x_n) \dotplus (y_n)) \dot{-} ((x'_n) \dotplus (y'_n))$ converge to zero, ie $(x_n) \dotplus (y_n) \sim (x'_n) \dotplus (y'_n)$.
By remark 9 there exists $X, Y \in \mathbb{Q}$ such that $|x_n| \leq X$ and $|y_n| \leq Y$ for every n. Let $\epsilon \in \mathbb{Q}^+$ and choose N_ϵ such that $|x_n - x'_n| < \frac{\epsilon}{2Y}$ and $|y_n - y'_n| \leq \frac{\epsilon}{2X}$ for all $n \geq N_\epsilon$. Now for such n we have

$$\begin{array}{rcl} |(x_n y_n) - (x'_n y'_n)| & = & |x_n y_n - x_n y'_n + x_n y'_n - x'_n y'_n)| \\ & = & |x_n(y_n - y'_n) + (x_n - x'_n) y'_n)| \\ & \leq & |x_n||y_n - y'_n| + |x_n - x'_n||y_n| \text{ by triangle inequality} \\ & < & Y\dfrac{\epsilon}{2Y} + \dfrac{\epsilon}{2X} X = \epsilon \end{array}$$

So we see that $(x_n) * (y_n) \dot{-} (x'_n) * (y'_n)$ converges to zero, so $(x_n) * (y_n) \sim (x'_n) * (y'_n)$.

Associativity of + and ·

$$\begin{aligned} [x_n] + ([y_n] + [z_n]) &= [x_n] + [(y_n) \dot{+} (z_n)] \\ &= [(x_n) \dot{+} ((y_n) \dot{+} (z_n))] \\ &= [((x_n) \dot{+} (y_n)) \dot{+} (z_n)] \text{ by associativity of } \dot{+} \\ &= [(x_n) \dot{+} (y_n)] + [z_n] \\ &= ([x_n] + [y_n]) + [z_n] \end{aligned}$$

$$\begin{aligned} [x_n] \cdot ([y_n] \cdot [z_n]) &= [x_n] \cdot [(y_n) * (z_n)] \\ &= [(x_n) * ((y_n) * (z_n))] \\ &= [((x_n) * (y_n)) * (z_n)] \text{ by associativity of } * \\ &= [(x_n) * (y_n)] \cdot [z_n] \\ &= ([x_n] \cdot [y_n]) \cdot [z_n] \end{aligned}$$

Commutativity of + and ·

$$\begin{aligned} [x_n] + [y_n] &= [(x_n) \dot{+} (y_n)] \\ &= [(y_n) \dot{+} (x_n)] \text{ by commutativity of } \dot{+} \\ &= [y_n] + [x_n] \end{aligned}$$

$$\begin{aligned} [x_n] \cdot [y_n] &= [(x_n) * (y_n)] \\ &= [(y_n) * (x_n)] \text{ by commutativity of } * \\ &= [y_n] \cdot [x_n] \end{aligned}$$

Distributivity of · over +

$$\begin{aligned} [x_n] \cdot ([y_n] + [z_n]) &= [x_n] \cdot [(y_n) \dot{+} (z_n)] \\ &= [(x_n) * ((y_n) \dot{+} (z_n))] \\ &= [(x_n) * (y_n) \dot{+} (x_n) * (z_n)] \text{ by distributivity of } * \text{ over } \dot{+} \\ &= [(x_n) * (y_n)] + [(x_n) * (z_n)] \\ &= [x_n] \cdot [y_n] + [x_n] \cdot [z_n] \end{aligned}$$

□

Now we can easily represent element of $\mathscr{R}$ by a representative of the class. For example, we have seen that $(\frac{1}{n})$ and (0_n) behave both like an absorbing element, and we need it to be unique, but from now on we have $[\frac{1}{n}] = [0]$.

Theorem 2.2.12.

$(\mathscr{R}, +, \cdot)$ is a field

$[0]$ and $[1]$ behave respectively like additive neutral elements and multiplicative neutral element. The opposite of $[x_n] \in \mathscr{R}$ noted $-[x_n]$ is given by $[\dot{-}(x_n)]$ and the inverse of $[x_n] \in \mathscr{R} \setminus [0]$ noted $[x_n]^{-1}$ is given by $[(x_n)^{\dot{-1}}]$. As usual, $[x_n] + (-[y_n])$ will be denoted $[x_n] - [y_n]$.

PROOF : Let $[x_n] \in \mathscr{R}$.
We have $[x_n] \cdot [0] = [(x_n) * (0)]$ which is the class of the sequence whose nth term is $x_n \cdot 0 = 0$ for each n, so $[x_n] \cdot [0] = [0]$, so $[0]$ is the absorbing element.
$[x_n] + [0] = [(x_n) \dot{+} (0)] = [(x_n + 0)] = [x_n]$ so $[0]$ is the neutral element for addition.
$[x_n] \cdot [1] = [(x_n) * (1)] = [(x_n \cdot 1)] = [x_n]$ so $[1]$ is the neutral element for multiplication.
Clearly, for each $[x_n] \in \mathscr{R}$, $[x_n] + -[x_n] = [x_n] + [\dot{-}x_n] = [(x_n)\dot{-}(x_n)] = [0]$ (see remark 11).
Now if $[x_n] \neq [0]$, then

$$\begin{aligned} [x_n] \cdot [x_n]^{-1} &= [x_n] \cdot [(x_n)^{\dot{-1}}] \\ &= [(x_n) * (x_n)^{\dot{-1}}] \\ &= [1] \end{aligned}$$

The last equality is justified by lemma 2.2.5, since we know that $(x_n) * (x_n)^{\dot{-1}} \in \mathcal{E}$ and $\mathcal{E}$ is clearly the class of all sequences converging to 1, and a representant of this class is the constant sequence (1_n). □

Remark 14. Now that we know that $(\mathscr{R}, +, \cdot)$ is a field, and that $[\mathbb{Q}]$ is in $\mathscr{R}$, we have to check that the operation coincides with the natural operations on $\mathbb{Q}$. This means that we have to check that $\widetilde{\iota}(\lambda +_{\mathbb{Q}} \mu) = \widetilde{\iota}(\lambda) + \widetilde{\iota}(\mu)$, $\widetilde{\iota}(\lambda \cdot_{\mathbb{Q}} \mu) = \widetilde{\iota}(\lambda) \cdot \widetilde{\iota}(\mu)$, $\widetilde{\iota}(-\lambda) = -\widetilde{\iota}(\lambda)$ and $\widetilde{\iota}(\frac{1}{\lambda}) = \widetilde{\iota}(\lambda)^{-1}$. But all that comes from remark 10, remark 11 and remark 13 :

$$\begin{aligned} \widetilde{\iota}(\lambda +_{\mathbb{Q}} \mu) &= [\iota(\lambda +_{\mathbb{Q}} \mu)] \text{by definition (remark 13)} \\ &= [\iota(\lambda) \dot{+} \iota(\mu)] \text{by remark 10} \\ &= [\iota(\lambda)] + [\iota(\mu)] \text{by proposition 2.2.11} \\ &= \widetilde{\iota}(\lambda) + \widetilde{\iota}(\mu) \end{aligned}$$

(similar proof for $\widetilde{\iota}(\lambda \cdot_{\mathbb{Q}} \mu) = \widetilde{\iota}(\lambda) \cdot \widetilde{\iota}(\mu)$)

$$\begin{aligned} \widetilde{\iota}(-\lambda) &= [\iota(-\lambda)] \text{ by definition} \\ &= [\dot{-}\iota(\lambda)] \text{by remark 11} \\ &= -[\iota(\lambda)] \text{by theorem 2.2.12} \\ &= -\widetilde{\iota}(\lambda) \end{aligned}$$

(similar proof for $\widetilde{\iota}(\frac{1}{\lambda}) = \widetilde{\iota}(\lambda)^{-1}$). So we have that $([\mathbb{Q}], +, \cdot)$ is isomorphic as a field to $(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}})$.

Now we want to provide $\mathscr{R}$ with an order which will turn it into an ordered field. We do that through the definition of an ultimately positive sequence (see definition 2.2.6).

Definition 2.2.13. *We define $\mathscr{R}^+$ to be the set of the class of all ultimately positive sequences. In other words*

$$\mathscr{R}^+ \triangleq \left\{[x_n] \in \mathscr{R} \mid (x_n) \in \mathcal{C}_{\mathbb{Q}}^+\right\}$$

We say that $[x_n]$ is positive if $[x_n] \in \mathscr{R}^+$. We say $[x_n] < [y_n] \iff [y_n] - [x_n] \in \mathscr{R}^+$.

We need the following lemma to prove that this definition is well defined.

Lemma 2.2.14. *If $(x_n) \sim (y_n)$ and $(x_n) \in \mathcal{C}_{\mathbb{Q}}^+$ then $(y_n) \in \mathcal{C}_{\mathbb{Q}}^+$.*

PROOF :
As $(x_n) \notin \mathcal{Z}$ (by definition) there is a $N_1 \in \mathbb{N}$ and $e \in \mathbb{Q}^+$ such that $\forall n \geq N_1$ $|x_n| \geq e$. Also by definition, there is $N_2 \in \mathbb{N}$ such that $x_n > 0$ for $n \geq N_2$, so for $n \geq N = max(N_1, N_2)$ we have $x_n \geq e$. Since $(x_n) \sim (y_n)$, for $\epsilon = \frac{e}{2}$ there is a $M \in \mathbb{N}$ such that $|x_n - y_n| < \frac{e}{2}$, for $n \geq M$. So

for $n \geq K = max(N, M)$, $x_n - y_n < \frac{e}{2}$ for $x_n \geq e > \frac{e}{2}$. Hence, $y_n > x_n - \frac{e}{2} > \frac{e}{2} > 0$ so (y_n) is ultimately positive.

□

Theorem 2.2.15.

$(\mathscr{R}, <, +, \cdot)$ *is an ordered field*

PROOF :
We need to check the tree axioms of order. (Ω1) follows from lemma 2.2.7. Indeed, if $[x_n] \neq 0$ then $(x_n) \notin \mathcal{Z}$ and so either $(x_n) \in \mathcal{C}^+_{\mathbb{Q}}$ and so $[x_n] \in \mathscr{R}^+$ or $\dot{-}(x_n) \in \mathcal{C}^+_{\mathbb{Q}}$ and so $[\dot{-}(x_n)] = -[x_n] \in \mathscr{R}^+$ (but not both). Moreover $[x_n] = [x_n] - [0]$ so $[x_n] \in \mathscr{R}^+$ is equivalent to $[x_n] - [0] \in \mathscr{R}^+$ which means $[x_n] > 0$. So we have the trichotomy law.
For (Ω2), it suffice to observe that if $[x_n] < [y_n]$ and $[y_n] < [z_n]$ then $[y_n] - [x_n]$ and $[z_n] - [y_n]$ are two elements of $\mathscr{R}^+$ and that

$$[z_n] - [y_n] \quad = \quad ([z_n] - [y_n]) + ([y_n] - [x_n]$$

So we only need to show that the sum of two elements of $\mathscr{R}^+$ is in $\mathscr{R}^+$ and that is equivalent to say that the sum of two element of $\mathcal{C}^+_{\mathbb{Q}}$ is in $\mathcal{C}^+_{\mathbb{Q}}$, which is what lemma 2.2.7 says. For (Ω3) observe that

$$([z_n] + [y_n]) - ([z_n] + [x_n]) = [y_n] - [x_n]$$

so $[x_n] < [y_n] \implies [y_n] - [x_n] \in \mathscr{R}^+ \implies ([z_n] + [y_n]) - ([z_n] + [x_n]) \in \mathscr{R}^+ \implies [z_n] + [x_n] < [z_n] + [y_n]$. Now for the product, if we have $[x_n] < [y_n]$ and $[z_n] \in \mathscr{R}^+$, then $[z_n]([y_n] - [x_n]) = [z_n * ((y_n) \dot{-} (x_n))]$ and $(y_n) \dot{-} (x_n) \in \mathcal{C}^+_{\mathbb{Q}}$ so by lemma 2.2.7, $(z_n) * ((y_n) \dot{-} (x_n)) \in \mathcal{C}^+_{\mathbb{Q}}$ and so $[z_n]([y_n] - [x_n]) \in \mathscr{R}^+$. But this means that $[z_n][y_n] - [z_n][x_n] \in \mathscr{R}^+$ so $[z_n][x_n] < [z_n][y_n]$ and (Ω3) is proven. □

Remark 15. From the first part, we know that a copy of $\mathbb{Q}$ lies in any ordered field. We can easily show that on $[\mathbb{Q}]$ this order is the same as the natural one of $\mathbb{Q}$. Indeed, assume that $\lambda < \mu \in \mathbb{Q}$ then $\mu - \lambda \in \mathbb{Q}^+$ and the constant sequence $(\mu - \lambda)$ is in $\mathcal{C}^+_{\mathbb{Q}}$ since each term is in $\mathbb{Q}^+$, so $\widetilde{\iota}(\mu - \lambda) \in \mathscr{R}^+$, ie $[\lambda] < [\mu]$. This shows that ι is an *ordered field embedding*, and proves that the copy of $(\mathbb{Q}, <_{\mathbb{Q}}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}})$ that lies inside $(\mathscr{R}, <, +, \cdot)$ is precisely $([\mathbb{Q}], <, +, \cdot)$. Notice we have a copy of the natural numbers $[\mathbb{N}]$ and the integers $[\mathbb{Z}]$ in $\mathscr{R}$, whose elements are clearly the class of $\mathbb{Q}$-Cauchy sequences that converges to a natural number or an integer.

Theorem 2.2.16.

$(\mathscr{R}, <, +, \cdot)$ *is archimedean*

PROOF :
We have to show that for $[x_n], [y_n] \in \mathscr{R}^+$ there is a $[m] \in [\mathbb{N}]$ such that $[x_n] < [m] \cdot [y_n]$. To this end, observe that (x_n) is bounded above from remark 9 so we have $u \in \mathbb{Q}^+$ such that $x_n < u \forall n \geq N$. Besides, as (y_n) doesn't converge to zero, there is a $v \in \mathbb{Q}^+$ such that $v < y_n \forall n$. Now as $\mathbb{Q}$ is archimedean, by assumption 2.2.1, there is a $m \in \mathbb{N}$ such that (notice that we apply the archimedean property to the rationals $u + 1$ and v) $u + 1 < mv$. Now by the previous observations, we have for all $n \geq N$:

$$x_n + 1 < my_n$$

so

$$my_n - x_n > 1 \forall n \geq N$$

this means that $(my_n - x_n) = (m) * (y_n) \dot{-} (x_n)$ is in $\mathcal{C}^+_{\mathbb{Q}}$ and so $[(m) * (y_n) \dot{-} (x_n)] \in \mathscr{R}^+$. Further $[(m) * (y_n) \dot{-} (x_n)] = [m] \cdot [y_n] - [x_n] \in \mathscr{R}^+$ so $[x_n] < [m][y_n]$.

□

The following follows from the first chapter, theorem 1.2.6.

Corollary 2.2.17.

$$([\mathbb{Q}], <, +, \cdot) \text{ is dense in } (\mathscr{R}, <, +, \cdot)$$

Now we state and prove the main result of this construction, as so far all we know about $\mathscr{R}$ is that it is an ordered field which has at least the same properties as the rational numbers, but in fact, unlike $\mathbb{Q}$, $\mathscr{R}$ is complete.

Theorem 2.2.18.

$$(\mathscr{R}, <, +, \cdot) \text{ is complete}$$

PROOF :

We prove that $\mathscr{R}$ has the least upper bound property. To do that, consider any set $A \neq \emptyset \subseteq \mathscr{R}$ bounded above by $X = [X_n]$. The idea is to construct two equivalent sequences, one increasing and the other decreasing such that the least upper bound lies between them. The least upper bound of A will actually be the class of these sequences.

First, we need two elements $a < b \in \mathbb{Q}$, such that the least upper bound lies between them. As X may not be an element of $[\mathbb{Q}]$, we can find a $M \in [\mathbb{Q}]$ such that $X < M$ using the archimedean property of $\mathscr{R}$ (with X and $[1]$), so we can set $b = M$. To find the a, it suffices to take an element in $[\mathbb{Q}]$ lower than some element in A, and to be sure that there is such an element, use the archimedean property of $\mathscr{R}$. Now for each $n \in \mathbb{N}$ we apply the archimedean property of $\mathbb{Q}$ to $b - a \in \mathbb{Q}^+$ and $\frac{1}{n}$ to have a $r \in \mathbb{N}$ such that $(b-a) < \frac{r}{n}$ and so $b < a + \frac{r}{n}$. So the set $\left\{r \in \mathbb{N} \mid [a + \frac{r}{n}] \text{ is an upper bound of } A\right\} \subseteq \mathbb{N}$ is non empty and as $\mathbb{N}$ is well-ordered, there is a minimal element, we denote it r_n. Now we define

$$x_n = a + \frac{r_{n+1}}{n+1} \quad y_n = x_n - \frac{1}{n+1} = a + \frac{r_{n+1} - 1}{n+1} \forall n \in \mathbb{N}$$

By construction, for each n we have that the constant sequence $[\lambda = x_n]$ is an upper bound of A and $[\beta = y_n]$ is not, otherwise, the minimality of r_{n+1} would be contradicted. So in fact, $\forall n, m \in \mathbb{N}$ we have

$$y_n < x_m$$

Now we show that (x_n) and (y_n) are Cauchy. Observe that $x_n - x_m < x_n - y_n = \frac{1}{n+1}$ and $x_m - x_n < x_m - y_m = \frac{1}{m+1}$ so for $n+1, m+1 > k$, $|x_n - x_m| < \frac{1}{k}$ so (x_n) is Cauchy. Similarly, (y_n) is Cauchy.

We have $(x_n) \sim (y_n)$ since for $n + 1 > k$, we have $|x_n - y_n| = \frac{1}{n+1} < \frac{1}{k}$.

Now we have $[x_n] = [y_n]$ and we will prove that this element of $\mathscr{R}$ is the least upper bound of A.

First we prove that $[x_n]$ is an upper bound of A. By way of contradiction, assume that there is $[z_n] \in A$ such that $[x_n] < [z_n]$, then $(z_n) \dot{-} (x_n)$ is ultimately positive, and we have a $N \in \mathbb{N}$ and $e \in \mathbb{Q}^+$ such that $z_n - x_n \geq e$ ie $z_n \geq x_n + e$, for $n \geq N$. Further, as (x_n) is Cauchy, there is a $K \in \mathbb{N}$ such that $|x_m - x_n| < \frac{e}{2}$ for $n, m \geq K$. Now let $L = max(N, K)$, the last result implies $x_n > x_L - \frac{e}{2}$ for $n \geq L$, and combine it with the former to get

$$z_n > x_L + \frac{e}{2}$$

Consequently, we have that $[z_n] > [x_L]$ where $[x_L]$ is the class of the constant sequence $(\lambda = x_L)$. But $[x_L]$ is an upper bound of A by construction, so $[z_n]$ can't be in A, this is a contradiction. Now we prove by a similar argument that $[x_n] = [y_n]$ is the least upper bound. Again, by way of contradiction, assume that there is $[t_n] \in \mathscr{R}$ such that $[t_n] < [y_n]$ and $[t_n]$ is an upper bound of A. Then $(y_n) \dot{-} (t_n)$ is ultimately positive and there is a $U \in \mathbb{N}$ and a $d \in \mathbb{Q}^+$ such that $y_n - t_n \geq d$ for $n \geq U$. Since (y_n) is Cauchy, we have a $V \in \mathbb{N}$ such that $|y_n - y_m| < \frac{d}{2}$ for $n, m \geq V$ and for $n \geq W = max(U, V)$, $y_W > y_n - \frac{d}{2}$. We combine again the two results for $n \geq W$ to get

$$y_W > t_n + \frac{d}{2}$$

However, this means that $[\lambda = y_W] > [t_n]$ and we know that $[y_W]$ is not an upper bound of A by construction. So $[t_n]$ can't be an upper bound of A. So $[x_n]$ is the least upper bound of A. So $(\mathscr{R}, <, +, \cdot)$ has the least upper bound property, and so $(\mathscr{R}, <, +, \cdot)$ is complete. □

Consequently, $(\mathscr{R}, <, +, \cdot)$ is a complete ordered field and we know that this corresponds to our natural idea of $\mathbb{R}$, and since the complete ordered field is unique (up to isomorphism), we can say that we have constructed the real number system.

2.3 Construction via Dedekind cuts

In this section, the construction of the real number system based on Dedekind cuts is presented. This construction is more straightforward than the previous one, as the object (cuts) that we define in the first part shall in fact be the real numbers. Since the first construction has been completed in full detail, in the next constructions we shall leave more details to the reader, to present the general idea and prove the main theorems. Cuts are particular subset of $\mathbb{Q}$ so this construction needs a basic set theoretic knowledge of $\mathbb{Q}$.

2.3.1 Cuts

As in 2.2.1 we have the archimedean ordered field $(\mathbb{Q}, <_\mathbb{Q}, +_\mathbb{Q}, \cdot_\mathbb{Q})$, we will use it to define the notion of cut. When it is unambiguous, we will denote $<, +, \cdot$ the relation and operations in $\mathbb{Q}$ to avoid cumbersome notation (especially in the proofs).

Definition 2.3.1 (Cuts)**.** *A* cut *α is a set $\alpha \subsetneq \mathbb{Q}$ such that :*
- $\alpha \neq \emptyset$
- $\forall p \in \mathbb{Q}\ \forall q \in \alpha\ p <_\mathbb{Q} q \implies p \in \alpha$
- $\forall p \in \alpha\ \exists q \in \alpha\ p <_\mathbb{Q} q$

Remark 16. The second point tells us that every element not in a cut is in fact greater than every element in the cut. The third point means that every cut has no maximal element.

Informally, if we see the rational numbers as a discontinuous line printed over a continuous line (which represents the real numbers), we can see cuts as subset of $\mathbb{Q}$ with a characteristic imaginary point on the continuous line such that every element on the left of the point is in the set and every element on the right of the point is not. As we have no upper bound on this subset, this point on the line may not be in $\mathbb{Q}$ and if not, this imaginary point belongs to what we want to be the reals. So in fact this imaginary point is represented by the cut, and that is how we want to construct the real numbers in this section. Let us take some example of what a cut is. Let $X \triangleq \{q \in \mathbb{Q} \mid q \cdot_\mathbb{Q} q < 2\} \cup \mathbb{Q}^-$, we see that every element less than the square root of two is in this set, and if an element is more that square root of two, it is not in this set, so X represents the number $\sqrt{2}$. Now if cuts are meant to represent the reals, they also must represent the rational numbers, and it is done by the following : for every rational number $q \in \mathbb{Q}$, the set $\{p \in \mathbb{Q} \mid p < q\}$ is a cut, and represents the number q.

Remark 17. As in the last section, we can define a map ($\mathcal{P}\mathbb{Q}$ being the powerset of $\mathbb{Q}$, ie the set of subset) :

$$\begin{array}{rcl} \iota : \mathbb{Q} & \longrightarrow & \mathcal{P}\mathbb{Q} \\ q & \mapsto & \bar{q} \triangleq \{p \in \mathbb{Q} \mid p < q\} \end{array}$$

This map will gives us the embedding of the rationals in the reals. Further, one should note that the set $\mathbb{Q} \setminus \bar{q}$ is then the set of elements greater than or equal to q, so it has a greatest lower bound. We will denote by $\overline{\mathbb{Q}}$ the image of $\mathbb{Q}$ by ι (ie $\overline{\mathbb{Q}} = \iota(\mathbb{Q})$).

Nota bene 3. Basic set theory tells us that the cardinality of $\mathbb{R}$ is the same as the one of $\mathcal{P}\mathbb{N} = \mathcal{P}\mathbb{Q}$, so this can gives us a good feeling, at least as far as the cardinality is concerned, that this construction is well founded.

We will in this section prove all necessary lemmas to define addition, multiplication and order of cuts.

Definition 2.3.2. *Given two cuts α and β we define the* addition *of α and β by*

$$\alpha + \beta \triangleq \{p +_{\mathbb{Q}} q \mid p \in \alpha \ q \in \beta\}$$

Remark 18. It doesn't take long to check that $\alpha + \beta$ is again a cut.

Lemma 2.3.3. *$+$ is associative, commutative, has a neutral element $\zeta \triangleq \mathbb{Q}^-$ and every cut has an opposite for $+$. For a cut α the opposite is given by $\{-q \mid \exists p \in \mathbb{Q} \setminus \alpha \ q > p\}$.*

Proof : Associativity and commutativity are straightforward from the associativity and comutativity of $+_{\mathbb{Q}}$. Let α be a cut. Then $\alpha + \zeta = \{q + u \mid q \in \alpha \ \ u < 0\}$. Now take $q + u \in \alpha + \zeta$. From $u < 0$ follows $q + u < q$ so as $q \in \alpha$, $q + u \in \alpha$ so $\alpha + \zeta \subseteq \alpha$. Now take $q \in \alpha$. By definition 2.3.1 there is a $p \in \alpha$ such that $q < p$, ie $q - p < 0$ and by density of $\mathbb{Q}$, there is $u \in \mathbb{Q}$ such that $q - p < u < 0$ so $u \in \zeta$, $p \in \alpha$ and $q < p + u$ so $q \in \alpha + \zeta$ and $\alpha \subseteq \alpha + \zeta$. So we have proved that $\alpha + \zeta = \alpha$ so ζ is a neutral element for $+$.
Now we prove that there exists β such that $\alpha + \beta = \zeta$. We naturally define $\beta = \{-q \in \mathbb{Q} \mid \exists p \in \mathbb{Q} \setminus \alpha \ q > p\}$ (this definition comes naturally by representing geometrically what a cut is). Further, β is a cut, and this comes from the fact that α is a cut. Then we claim that $\alpha + \beta = \zeta$. Take $p - q \in \alpha + \beta$ ($p \in \alpha$ and $q \notin \alpha$). Assume $p - q \geq 0$, then $p \geq q$ but this contradict remark 16 as we must have $p < q$. So we have $\alpha + \beta \subseteq \zeta$. Now take $u \in \zeta$, then $u < 0$, so $v = -\frac{1}{2}u > 0$ and then we have a $n \in \mathbb{Q}$ such that $q = vn \in \alpha$ and $v(n+1) \notin \alpha$ (To see this, apply the archimedean property of $\mathbb{Q}$). Now set $p = -(n+2)v$ and $p \in \beta$ as $-p > (n+1)v \notin \alpha$. Now we get $p + q = nv - nv + 2v = u$ so $u \in \alpha + \beta$ and so we have $\alpha + \beta = \zeta$. □

Remark 19. We have clearly that $\overline{p+q} = \bar{p} + \bar{q}$ (one can consider the complement in $\mathbb{Q}$ of these sets, and it becomes very easy to see). So the map ι preserves the addition. In particular $\overline{\mathbb{Q}}$ is closed under addition. Besides every element in $\overline{\mathbb{Q}}$ has an opposite in $\overline{\mathbb{Q}}$, namely $-\bar{p} = \overline{-p}$ (as $\bar{p} + \overline{-p} = \overline{p-p} = \zeta$).

We will denote the opposite of α by $-\alpha$ and abbreviate $\alpha + -\beta$ by $\alpha - \beta$. Before defining the multiplication, we will need the notion of positive and negative cuts, so we define the comparison relation, and this is done thanks to the inclusion relation.

Lemma 2.3.4. *For α, β, γ three cuts, we have the following :*

- *$\alpha \subsetneq \beta$ and $\beta \subsetneq \gamma \implies \alpha \subsetneq \gamma$*
- *One and only one of the following hold :*

$$\alpha \subsetneq \beta \quad \alpha = \beta \quad \beta \subsetneq \alpha$$

PROOF : The first point holds because $\subsetneq$ is a transitive relation. For the second point, it is clear that at most one holds. Now take α, β two cuts. If $\alpha \neq \beta$, then there is a $p \in \mathbb{Q}$ that belongs to one and not the other, say $p \in \alpha$ and $p \notin \beta$. Then by remark 16 for all $q \in \beta$ $q < p$ so $q \in \alpha$, and so $\beta \subsetneq \alpha$. If $p \in \beta$ and $p \notin \alpha$ then we get $\alpha \subsetneq \beta$. □

With this lemma the following definition makes sense.

Definition 2.3.5. *Let α, β be two cuts. We say that α is lower than β (denoted $\alpha < \beta$) if and only if $\alpha \subsetneq \beta$. Similarly, α is greater than β (denoted $\alpha > \beta$) if $\beta \subsetneq \alpha$. We say that α is positive if $\alpha > \zeta$, and negative if $\alpha < \zeta$.*

Remark 20. It is easy to see that if a cut is negative, then there are no positive rationals in it. Conversely, if all the elements in a cut are negative, then the cut is included in ζ so it is negative. Similarly, a cut is positive if and only if there is a positive rational number in it. Now considere $\alpha > \zeta$, then $-\alpha$ contains only the opposite of elements not in α, and the element not in α are all positive, so $-\alpha$ is negative. Similarly, if α is negative, then $-\alpha$ is positive. Observe that $- - \alpha$ is the set of opposite elements of opposite element of elements that are not in $\mathbb{Q} \setminus \alpha$, so it is $- - \alpha = \alpha$. Finally, the lemma 2.3.4 implies that $<$ satisfies (Ω_1) and (Ω_2).

Remark 21. Consider $p, q \in \mathbb{Q}$, and $p < q$. Now clearly the elements lower than p are lower than q, so we get $\bar{p} \subsetneq \bar{q}$ (the equality case quickly leads to a contradiction). So in fact the map $\iota : \mathbb{Q} \to \overline{\mathbb{Q}}$ is order preserving. Moreover, this implies that ι is injective and it is clearly surjective by definition of $\overline{\mathbb{Q}}$.

Now we shall define the multiplication, however, we begin by doing it for positive cuts.

Definition 2.3.6. *Let α, β be two positive cut. We define the* multiplication of α and β

$$\alpha \cdot \beta \triangleq \{r \in \mathbb{Q} \mid \exists p > 0 \ q > 0 \ p \in \alpha \ q \in \beta \ r \leq p \cdot_{\mathbb{Q}} q\}$$

We denote $\upsilon = \bar{1} = \{p \in \mathbb{Q} \mid p < 1\}$.

Remark 22. Notice that $\alpha \cdot \beta$ is a positive cut.

Lemma 2.3.7. *The operation $\cdot$ defined over positive cuts is associative, commutative, has neutral element υ and every positive cut α has an inverse element, given by*

$$\alpha^{-1} \triangleq \zeta \cup \{0\} \cup \left\{p \in \mathbb{Q}^+ \mid \exists r \in \mathbb{Q} \setminus \alpha \ p^{-1} > r\right\}$$

PROOF : Again, associativity and commutativity of $\cdot$ come from associativity and commutativity of $\cdot_{\mathbb{Q}}$. Let α be a positive cut. We claim that $\alpha \cdot \upsilon = \alpha$. First pick $x \in \alpha \cdot \upsilon$, then $x \leq pu$ for $p \in \alpha$, $p > 0$ and $0 < u < 1$, so $pu < p$ and so $x < p$, so $x \in \alpha$, consequently, $\alpha \cdot \upsilon \subseteq \alpha$. Now pick $x \in \alpha$. If $x \leq 0$, then, as α is positive, by remark 16 there is $y \in \alpha$ such that $y > 0$. Then, as $\frac{1}{2} \in \upsilon$ we have $\frac{y}{2} > 0$ so $x < \frac{y}{2}$ so $x \in \alpha \cdot \upsilon$. If $x > 0$, by definition of a cut, there is a $y \in \alpha$ such that $0 < x < y$. Further as $y^{-1} > 0$ we have $xy^{-1} < 1$ so $u = xy^{-1} \in \upsilon$, and then $x = uy$ so $x \in \alpha \cdot \upsilon$, so $\alpha = \alpha \cdot \upsilon$.
Let α be a positive cut, and $\beta = \zeta \cup \{0\} \cup \left\{p \in \mathbb{Q}^+ \mid \exists r \in \mathbb{Q} \setminus \alpha \ p^{-1} > r\right\}$. We show that $\alpha \cdot \beta = \upsilon$. Let $x = pq$ be an element of $\alpha \cdot \beta$. Then by definition, $p \in \alpha$ and $q^{-1} \notin \alpha$, so by remark 16 $p < q^{-1}$ ie $x = pq < 1$ so $x \in \upsilon$. Now take $x \in \upsilon$. Choose $0 < x < 1$, the case $x \leq 0$ being trivial as there is obviousely a positive element in $\alpha \cdot \beta$. We have $x^{-1} > 1$, and we claim that there is a $p \in \alpha$ such that $px^{-1} \notin \alpha$ (as in the proof of lemma 2.3.3 this comes from the archimedean property of $\mathbb{Q}$). Now if px^{-1} is the greatest lower bound of $\mathbb{Q} \setminus \alpha$, (this is possible in only one case : $\alpha = \bar{q}$ for some $q \in \mathbb{Q}$ and so $px^{-1} = q$) then as α is a cut, there is a $p < p' \in \alpha$ and then $p'x^{-1}$ is no longer an infimum. So we assume that px^{-1} is not an infimum, then there is a $r \notin \alpha$ such that $r < px^{-1}$. Now by density, we can choose $q \in \mathbb{Q}$ such that $r < q < px^{-1}$. As $r < q$, $q' = q^{-1} \in \beta$ and we have $x < pq'$ so in fact $x \in \alpha \cdot \beta$. □

Now we can extend the definition of $\cdot$ to all the cuts.

Proposition 2.3.8. *Let α β be two cuts, we define the multiplication as the following :*

$$\alpha\cdot\beta=\begin{cases}-(-\alpha\cdot\beta) & \text{if } \alpha<\zeta \text{ and } \beta>\zeta\\ -(\alpha\cdot-\beta) & \text{if } \alpha>\zeta \text{ and } \beta<\zeta\\ -\alpha\cdot-\beta & \text{if } \alpha<\zeta \text{ and } \beta<\zeta\\ \zeta & \text{if } \alpha=\zeta \text{ or } \beta=\zeta\end{cases}$$

Remark 23. Each time, on the right hand side, the multiplication is defined as the elements are positive.

Remark 24. We show that $\overline{pq}=\bar{p}\cdot\bar{q}$ (first for $\bar{p},\bar{q}>\zeta=\bar{0}$ ie $p,q>0$ by remark 17). Take $x\in\overline{pq}$, then $x<pq$ and as $p^{-1}>0$ we have $xp^{-1}<q$ so $xp^{-1}\in\bar{q}$. Now $x=xp^{-1}p$ so by definition $x\in\bar{p}\cdot\bar{q}$. Similarly if $x\in\bar{p}\cdot\bar{q}$ then $x\leq uv$ $u\in\bar{p}$ and $v\in\bar{q}$ so $uv<pq$ (u,v are positive) so $x\in\overline{pq}$. So in fact we have $\overline{pq}=\bar{p}\cdot\bar{q}$ for every $p,q\in\mathbb{Q}$ (we extend the definition to all rational as in proposition 2.3.8), which means that ι preserve the multiplication. In particular, $\overline{\mathbb{Q}}$ is closed under multiplication, and every element (*non* ζ) has an inverse (as $\upsilon=\bar{1}=\overline{pp^{-1}}=\bar{p}\overline{p^{-1}}$). ζ has no inverse since $\zeta\cdot\alpha=\zeta\neq\upsilon$.

Now we have to check that the operations are compatible with $<$.

Lemma 2.3.9. *Let α, β and γ be three cuts. We have the following :*
- $\alpha<\beta\implies\alpha+\gamma<\beta+\gamma$
- $\alpha<\beta$ *and* $\gamma>\zeta\implies\alpha\cdot\gamma<\beta\cdot\gamma$

PROOF : Let $x\in\alpha+\gamma$. Then $x=p+q$ with $p\in\alpha$ and $q\in\gamma$. Now by definition there is a $r\in\beta$ such that $p<r$ so $p+q<r+q$, so $x\in\beta+\gamma$. So $\alpha+\gamma\leq\beta+\gamma$. Now assume they are equal, then add $-\gamma$ on each side of the equation and we get the contradiction $\alpha=\beta$. So the first point is shown.

Sublemma. The first point gives us that if $\alpha<\beta$ then $-\alpha>-\beta$, by adding $-\alpha-\beta$ on each side.

Now assume α,β,γ three positive cuts and $\alpha<\gamma$. If $x\in\alpha\cdot\gamma$ then $x\leq pq$ with $p\in\alpha$ and $q\in\gamma$ $p,q>0$. Now as $\alpha<\beta$, we get $p\in\beta$ so $x\leq pq$ with $p\in\beta q\in\alpha$ which means $x\in\beta\cdot\gamma$. Again if we assume the equality, we can multiply by γ^{-1} to lead to a contradiction, so finally, $\alpha\cdot\gamma<\beta\cdot\gamma$. If $\alpha<\zeta$ and $\beta<\zeta$ and $\alpha<\beta$ then by the sublemma $-\beta<-\alpha$ and $-\alpha,-\beta$ are positive. Consequently, from above we get then $-\beta\cdot\gamma<-\alpha\cdot\gamma$ so again by sublemma $\alpha\cdot\gamma=-(-\alpha\cdot\gamma)<-(-\beta\cdot\gamma)=\beta\cdot\gamma$. The last case, if $\alpha<\beta$, $\alpha<\zeta$ and $\beta>\zeta$. Then $-\alpha\cdot\gamma>\zeta\cdot\gamma=\zeta$ so $\alpha\cdot\gamma=-(-\alpha\cdot\gamma)<\zeta$ by the sublemma. Now as β is positive, so is $\beta\cdot\gamma$, so $\alpha\cdot\gamma<\beta\cdot\gamma$. □

Lemma 2.3.10. *Let α, β and γ be three cuts,*

$$\alpha\cdot(\beta+\gamma)=\alpha\cdot\beta+\alpha\cdot\gamma$$

PROOF : First we do the proof for α β, γ positive cuts. Let $x\in\alpha\cdot(\beta+\gamma)$. Then $x\leq pq$ with $p\in\alpha$ $p>0$ and $0<q=r+s$ with $r\in\beta$ and $s\in\gamma$ (we can choose r,s positive) . So by distributivity in $\mathbb{Q}$, $x\leq pr+ps$, and as p,r,s are positive, we have that $pr\in\alpha\cdot\beta$, $ps\in\alpha\cdot\gamma$ so by definition $pr+ps\in\alpha\cdot\beta+\alpha\cdot\gamma$ so $x\in\alpha\cdot\beta+\alpha\cdot\gamma$ so we have the first inclusion. Now take $x=pr+qs\in\alpha\cdot\beta+\alpha\cdot\gamma$, $0<p,q\in\alpha$, $0<r\in\beta$, $0<s\in\gamma$. Then p and q are comparable, say $p\geq q$, so as we deal with positive elements, we get $pr+ps=p(r+s)\geq pr+qs=x$. Clearly $(r+s)\in\beta+\gamma$ so we get $p(r+s)\in\alpha\cdot(\beta+\gamma)$ and we have $x\in\alpha\cdot(\beta+\gamma)$ as $\alpha\cdot(\beta+\gamma)$ is a cut. Therefore, the distributive law holds for positive cuts.
Now we break into cases. First observe that if β or γ is ζ, then the equation becomes trivial ($\alpha\cdot\beta=\alpha\cdot\beta$ or $\alpha\cdot\gamma=\alpha\cdot\gamma$). If $\alpha=\zeta$, the equation is again trivial : $\zeta=\zeta$. So we can assume

α, β, γ non ζ.
Assume $\alpha > \zeta$ $\beta < \zeta$ $\gamma > \zeta$ and $\beta + \gamma > \zeta$. Then we have $\gamma = (\beta + \gamma) + -\beta$ and observe that every term is positive, so we can apply the distributive law : $\alpha \cdot \gamma = \alpha \cdot (\beta + \gamma) - \alpha \cdot \beta$ which gives $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
Assume $\alpha > \zeta$, $\beta < \zeta$ $\gamma > \zeta$ and $\beta + \gamma < \zeta$, then apply the same trick with $-\beta = \gamma + -(\beta + \gamma)$.
Now if $\alpha < \zeta$, we have two more cases, but we can again do the same trick by multiplying by $(-\alpha)$, and we will get the distributive law multiplied on each side by $-\zeta$, which is the same.
Remark that β and γ play symmetric roles, so we don't forget any cases. □

Remark 25. The result of lemma 2.3.10 is true in particular for $\overline{\mathbb{Q}}$ so from remarks 19 and 24 we have that $(\overline{\mathbb{Q}}, +, \cdot)$ is a field. Furthermore, from remark 20 and lemma 2.3.9, $(\overline{\mathbb{Q}}, <, +, \cdot)$ is an ordered field.

Finally, the completeness result, which is in this case a property easy to prove.

Proposition 2.3.11. *Let $\mathscr{C} \subset \mathcal{P}\mathbb{Q}$ be a set of cuts and γ a cut such that $\forall \alpha \in \mathscr{C}$ $\alpha \leq \gamma$, then there is a cut γ_0 such that for all γ such that $\forall \alpha \in \mathscr{C}$ $\alpha \leq \gamma$, we have $\gamma_0 \leq \gamma$ and $\gamma \geq \alpha$ $\forall \alpha \in \mathscr{C}$.*

Proof : Set

$$\gamma_0 = \bigcup_{\alpha \in \mathscr{C}} \alpha$$

We first show that γ_0 is a cut. It is clearly non-empty as each cut in $\mathscr{C}$ is non-empty. Besides if it were $\mathbb{Q}$ there should not be such γ. Now assume $p \in \gamma_0$ and $q \in \mathbb{Q}$ $q < p$. Observe that by definition, there is a cut $\alpha \in \mathscr{C}$ such that $p \in \alpha$, and as $q < p$, $q \in \alpha$. Now as $\alpha \subseteq \gamma_0$, $q \in \gamma_0$. It remains to show the third point. Take $p \in \gamma_0$, then again there is a $\alpha \in \mathscr{C}$ such that $p \in \alpha$ and as α is a cut, there is a $q \in \alpha$ such that $p < q$, and as $\alpha \subseteq \gamma_0$ we get $q \in \gamma_0$. Therefore, γ_0 is a cut. Now we show that it is included in every γ such that $\forall \alpha \in \mathscr{C}$ $\alpha \leq \gamma$. Take such a γ. Then take $p \in \gamma_0$, there exists $\alpha \in \mathscr{C}$ such that $p \in \alpha$, and by hypothesis, $\alpha < \gamma$ so $\alpha \subsetneq \gamma$ so $p \in \gamma$, therefore $\gamma_0 \subseteq \gamma$. Finally by definition $\forall \alpha$, $\alpha \subseteq \gamma_0$. □

2.3.2 The Dedekind Field

In this subsection, we summarise the construction of $\mathbb{R}$ by Dedekind cuts by gathering the informations in the previous subsection.

Definition 2.3.12. *Let $\mathscr{R}$ be the set of all cuts.*

Proposition 2.3.13. *With $+$, $\cdot$ being the operations defined in the previous subsection*

$$(\mathscr{R}, +, \cdot) \text{ is a field}$$

Proof : (Φ_1) is given by remarks 18 and 22. (Φ_2), (Φ_3), (Φ_4), (Φ_5) come from lemma 2.3.3 and lemma 2.3.7 (combined with Proposition 2.3.8) observe that with proposition 2.3.8, associativity, commutativity, the existence of neutral element and inverse element are extended to negative slopes. Finally (Φ_6) follows from lemma 2.3.10. □

Proposition 2.3.14. *With $<$ being the relation defined in the previous subsection*

$$(\mathscr{R}, <, +, \cdot) \text{ is an ordered field}$$

Proof : (Ω_1) and (Ω_2) have been discussed in remark 20. (Ω_3) comes from lemma 2.3.9. □

Theorem 2.3.15.

$$(\mathscr{R},<,+,\cdot) \textit{ is complete}$$

PROOF : This is precisely what proposition 2.3.11 shows. □

Remark 26. It follows from remark 25 that $(\overline{\mathbb{Q}},<,+,\cdot)$ is an ordered subfield of $(\mathscr{R},<,+,\cdot)$. In fact, from the first part, we know that in every complete ordered field, there is a copy of $\mathbb{Q}$. This copy can be constructed directly from the ordered field as has been done in the first part by considering $\mathcal{N}_{\mathscr{R}}$, $\mathcal{Z}_{\mathscr{R}}$ and $\mathcal{Q}_{\mathscr{R}}$, and by isomorphism, $\mathcal{Q}_{\mathscr{R}} = \overline{\mathbb{Q}}$.

So by theorem 1.3.1, we get that $(\mathscr{R},<,+,\cdot)$ is isomorphic to our idea of real numbers. Therefore, we can say that we have constructed the real numbers system.

2.4 Construction via slopes

The construction we are going to present in this section is a remarkable construction for it is constructed directly from the group of integers $(\mathbb{Z},+)$. Instead of assuming the existence of the ordered field $(\mathbb{Q},<,+,\cdot)$ like in the two previous constructions, we only assume that we are given the additive group of integers. Even though $(\mathbb{Q},<,+,\cdot)$ can be obtained easily from $(\mathbb{Z},+)$, it seems a rather more elegant assumption.

2.4.1 A study of the slopes

This construction is based on a certain kind of function $\mathbb{Z}\to\mathbb{Z}$ called *slopes*. Slopes are sometimes called *almost homomorphism* or *quasihomomorphism*, we will see why later.

Definition 2.4.1. *A* slope *is a function $\lambda:\mathbb{Z}\to\mathbb{Z}$ such that the set*

$$\Lambda(\lambda) \triangleq \{\lambda(m+n)-\lambda(m)-\lambda(n) \mid m,n\in\mathbb{Z}\}$$

is finite. We denote by $\mathscr{S}_{\mathbb{Z}}$ the set of all slopes.

We remark that linear functions are slopes, and in fact, we will see that linear functions will represents the integer numbers. We shall deal with sets of the form $\{f(n)\mid n\in\mathbb{Z}\}$ and when it is unambiguous, we will denote such a set $\{f(n)\}_{n\in\mathbb{Z}}$ or even sometimes $\{f(n)\}$. We shall also use the notation $-X$ for $\{-n\mid n\in X\}$.

Remark 27. Let $\lambda\in\mathscr{S}_{\mathbb{Z}}$, then by definition $\Lambda(\lambda)$ is finite. This means that there exists $k\in\mathbb{N}$ such that

$$\Lambda(\lambda)=\{\lambda_1,\dots,\lambda_k\}\subsetneq\mathbb{Z}$$

So in fact, for each $n,m\in\mathbb{Z}$ there exists $j\in\{1,\dots,k\}$, $j=j(m,n)$ such that

$$\lambda(m+n)=\lambda(n)+\lambda(n)+\lambda_j$$

We can understand better why slopes are sometimes called almost homomorphism. We shall keep this notation, ie for each $\lambda\in\mathscr{S}_{\mathbb{Z}}$ is associated a $k\in\mathbb{N}$ and a finite set $\{\lambda_1,\dots,\lambda_k\}$ of integers such that for every $m,n\in\mathbb{Z}$ there is a $1\le j\le k$ such that $\lambda(m+n)=\lambda(m)+\lambda(n)+\lambda_j$. It follows that $\lambda(0)=\lambda(0+0)=\lambda(0)+\lambda(0)+\lambda_j$ so we have always $-\lambda(0)\subseteq\Lambda(\lambda)$.

Definition 2.4.2. *Let $\lambda,\mu\in\mathscr{S}_{\mathbb{Z}}$. We say that λ and μ are equivalent denoted $\lambda\sim\mu$ if the set $\{\lambda(n)-\mu(n)\mid n\in\mathbb{Z}\}$ is finite.*

Remark 28. Assume $\lambda,\mu\in\mathscr{S}_{\mathbb{Z}}$ and $\lambda\sim\mu$, then $\{\lambda(n)-\mu(n)\}$ is finite and so there is a finite number of $\epsilon_1,\dots,\epsilon_n$ such that for each $n\in\mathbb{Z}$ there is a $1\le i\le n$ such that $\lambda(n)=\mu(n)+\epsilon_i$.

Proposition 2.4.3. *$\sim$ is an equivalence relation on $\mathscr{S}_{\mathbb{Z}}$.*

PROOF : Reflexivity : $\{\lambda(n)-\lambda(n)\}_{n\in\mathbb{Z}} = \{0\}$ for every $\lambda \in \mathscr{S}_{\mathbb{Z}}$ so $\lambda \sim \lambda$.
Symmetry : If $\lambda \sim \mu$ holds then $\{\lambda(n)-\mu(n)\}_{n\in\mathbb{Z}}$ is finite. Now this means that $\{\mu(n)-\lambda(n) \mid n \in \mathbb{Z}\} = -\{\lambda(n)-\mu(n) \mid n \in \mathbb{Z}\}$ is also finite, so $\mu \sim \lambda$.
Transitivity : Assume $\lambda \sim \mu$ and $\mu \sim \nu$, then

$$\{\lambda(n)-\nu(n)\}_{n\in\mathbb{Z}} = \{(\lambda(n)-\mu(n))+(\mu(n)-\nu(n))\}_{n\in\mathbb{Z}} = \{x_n+y_n\}_{n\in\mathbb{Z}}$$

where (x_n) and (y_n) take only a finite number of values, so it is finite and $\lambda \sim \nu$. □

In the second subsection, we will define the real numbers as equivalent class of slopes. Recall the notation for two functions : $\alpha+\beta(n) = \alpha(n)+\beta(n)$.

Lemma 2.4.4. *Let $\lambda, \mu, \nu \in \mathscr{S}_{\mathbb{Z}}$*
- $\lambda+\mu \in \mathscr{S}_{\mathbb{Z}}$
- *If $\lambda \sim \mu$, then $\lambda+\nu \sim \mu+\nu$.*

PROOF : For the first item, observe that $\Lambda(\lambda+\mu) = \{\lambda+\mu(n+m)-\lambda+\mu(n)-\lambda+\mu(m)\} = \{[\lambda(m+n)-\lambda(m)-\lambda(n)]+[\mu(m+n)-\mu(m)-\mu(n)]\} = \{x_n+y_n\}$ when (x_n) and (y_n) takes only a finite number of values. The second point is straightforward as

$$\{\lambda(n)-\mu(n)\} = \{\lambda(n)+\nu(n)-\mu(n)-\nu(n)\} = \{\lambda+\nu(n)-\mu+\nu(n)\}$$

. □

Remark 29. So we have that $+$ is compatible with $\sim$. Further, observe that the slope $\sigma_0 : n \mapsto 0$ behave as a neutral element for $+$ for $\lambda+\sigma_0(n) = \lambda(n)+0 = \lambda(n)$. Furthermore, $\nu \sim \sigma$ is equivalent to $\{\nu(n)-\sigma_0(n)\} = \nu(\mathbb{Z})$ is finite, so the zero elements of $\mathscr{S}_{\mathbb{Z}}$ are the slope that take a finite number of values. We set $\mathfrak{z} = \{\lambda \in \mathscr{S}_{\mathbb{Z}} \mid \lambda \sim \sigma_0\} = \{\beta : \mathbb{Z} \to \mathbb{Z} \mid \beta(\mathbb{Z}) \text{ is finite}\}$. Now as $+$ in $\mathbb{Z}$ is associative and commutative, we get that $+$ in $\mathscr{S}_{\mathbb{Z}}$ is associative and commutative.

Remark 30. Let λ be a slope, and define $(-\lambda)$ as the slope such that $(-\lambda)(n) = -\lambda(n)$. We clearly have $\lambda+(-\lambda) = \sigma_0$. Besides if $\lambda+\mu \sim \sigma_0$ then by lemma 2.4.4 we have $\mu \sim -\lambda$ (note that this needs associativity and commutativity of $+$, discussed in remark 29). Conversely, if $\lambda-\mu \sim \sigma_0$ again by adding μ on each side by lemma 2.4.4 commutativity and associativity (remark 29) we get $\lambda \sim \mu$. So we have that $\lambda \sim \mu \iff \lambda-\mu \sim \sigma_0$. Further by remark 29 this means that $\lambda-\mu(\mathbb{Z})$ is finite, so there is a constant $C > 0$ such that $|\lambda(n)-\mu(n)| < C$.

Now for two functions α and β, recall the notation $\alpha \circ \beta$ defined by $\alpha \circ \beta(x) = \alpha(\beta(x))$.

Lemma 2.4.5. *Let $\lambda, \mu, \nu \in \mathscr{S}_{\mathbb{Z}}$, we have the following :*
- $\lambda \circ \mu \in \mathscr{S}_{\mathbb{Z}}$
- *if $\lambda \sim \mu$ then $\nu \circ \lambda \sim \nu \circ \mu$.*

PROOF : Let $\lambda, \mu \in \mathscr{S}_{\mathbb{Z}}$, we show that $\Lambda(\lambda \circ \mu)$ is finite. To do so, take $p \in \Lambda(\lambda \circ \mu)$. There are $n, m \in \mathbb{Z}$ such that $p = \lambda(\mu(m+n))-\lambda(\mu(m))-\lambda(\mu(n))$. From the remark 27 there is $i \in \mathbb{Z}$ such that $\mu(m+n) = \mu(n)+\mu(n)+\mu_i$ and similarly

$$\begin{aligned}\lambda(\mu(m+n)) &= \lambda(\mu(m)+\mu(n)+\mu_i)\\ &= \lambda(\mu(m))+\lambda(\mu(n)+\mu_i)+\lambda_j\\ &= \lambda(\mu(m))+\lambda(\mu(n))+\lambda(\mu_i)+\lambda_j+\lambda_k\end{aligned}$$

Now $p = \lambda(\mu_i)+\lambda_j+\lambda_k$. Finally as there is a finite number of μ_i, there is a finite number of $\lambda(\mu_i)$ and there is a finite number of λ_j so there is a finite number of element of the form $\lambda(\mu_i)+\lambda_j+\lambda_k$, so $\Lambda(\lambda \circ \mu)$ is finite and $\lambda \circ \mu$ is a slope.

For the second point we have to show that $\{\nu \circ \lambda(n) - \nu \circ \mu(n)\}_{n\in\mathbb{Z}}$ is finite. Pick $p \in \{\nu \circ \lambda(n) - \nu \circ \mu(n)\}_{n\in\mathbb{Z}}$, then there is a $n \in \mathbb{Z}$ such that $p = \nu \circ \lambda(n) - \nu \circ \mu(n)$. Following the remark 28, there is a ϵ_j such that $\lambda(n) = \mu(n) + \epsilon_j$ and we get

$$\begin{aligned} p &= \nu(\lambda(n)) - \nu(\mu(n)) \\ &= \nu(\mu(n) + \epsilon_j) - \nu(\mu(n)) \\ &= \nu(\mu(n)) + \nu(\epsilon_j) + \nu_k - \nu(\mu(n)) \text{ from remark 27} \\ &= \nu(\epsilon_j) + \nu_k \end{aligned}$$

Again, there is only a finite number of ϵ_j and μ_k so only a finite number of such p. So $\{\nu \circ \lambda(n) - \nu \circ \mu(n)\}_{n\in\mathbb{Z}}$ is finite and the second point is proven. □

Remark 31. So the composition is compatible with $\sim$. The associativity property comes from the associativity of $\circ$. We would like to prove other good properties of $\circ$, such as a commutativity and existence of inverse element and this will be done with a little more work. However we can already say that $\iota : \mathbb{Z} \to \mathbb{Z} : n \mapsto n$ acts like a neutral element, as does every element of the set $\mathfrak{u} \triangleq \{\lambda \in \mathscr{S}_{\mathbb{Z}} \mid \lambda \sim \iota\}$.

Definition 2.4.6. *For $\lambda \in \mathscr{S}_{\mathbb{Z}}$ we define $\mathfrak{b}_\lambda = max(\{|x| \mid x \in \Lambda(\lambda)\} \cup \{1\})$.*

Remark 32. This definition makes sense because if λ is a slope, $\Lambda(\lambda)$ is finite and now for every $\lambda_j \in \Lambda(\lambda)$ we have $\lambda_j \leq \mathfrak{b}_\lambda$. Notice that we don't allow $\mathfrak{b}_\lambda$ to be 0 so $\mathfrak{b}_\lambda$ is always positive.

Definition 2.4.7. *A slope is said* odd *if it satisfies the following property*

$$\forall x \in \mathbb{Z}\ \lambda(-n) = -\lambda(n)$$

Remark 33. Observe that for every slope, we have for $x \in \mathbb{Z}$ and $n \in \mathbb{N}$

$$\lambda(nx) = \lambda(\underbrace{x + \cdots + x}_{n\ times}) = \lambda(\underbrace{x + \cdots + x}_{n-1\ times}) + \lambda(x) + \lambda_{j_1}$$

for some $\lambda_{j_1} \in \Lambda(\lambda)$ and we can repeat this idea to get

$$\lambda(nx) = n\lambda(x) + \sum_{i=1}^{n-1} \lambda_{j_i}$$

Now if $n < 0$

$$\lambda(nx) = \lambda(\underbrace{-x + \cdots + -x}_{|n|=-n\ times}) = -n\lambda(-x) + \sum_{i=1}^{|n|-1} \lambda_{j_i}$$

so if λ is odd, $\lambda(-x) = -\lambda(x)$ and then $\lambda(nx) = n\lambda(x) + \sum_{i=1}^{|n|-1} \lambda_{j_i}$. So in fact, if λ is odd we have

$$\forall x, y \in \mathbb{Z}\ \lambda(xy) = x\lambda(y) + \sum_{i=1}^{|x|-1} \lambda_{j_i}$$

And finally observe that $|\sum_{i=1}^{|x|-1} \lambda_{j_i}| \leq (|x| - 1)\mathfrak{b}_\lambda$ as $\lambda_j < \mathfrak{b}_\lambda\ \forall j$.

Now we see that every slope is equivalent to an odd slope.

Lemma 2.4.8 (First concentration lemma)**.** *Every slope is equivalent to an odd slope.*

PROOF : Take any $\lambda \in \mathscr{S}_\mathbb{Z}$. Define $\delta : \mathbb{Z} \to \mathbb{Z}$ such that $\delta(0) = 0$, $\delta(n) = \lambda(n) \forall n > 0$ and $\delta(-n) = -\delta(-n) \forall n < 0$. Clearly δ is an odd slope, and we claim that $\delta \sim \lambda$. Take $p \in \{\lambda(n) - \delta(n)\}_{n\in\mathbb{Z}}$, then $p = \lambda(n) - \delta(n) = 0$ if $n > 0$. If $n = 0$ $p = \lambda(0)$. If $0 > n$,

$$p = \lambda(n) - \delta(n) = \lambda(n) + \delta(-n) = \lambda(n) + \lambda(-n) = \lambda(0) - \lambda_j$$

so as there is only a finite number of λ_j we have that $\lambda \sim \delta$. □

Remark 34. Notice that odd slopes are only determined by their value on $\mathbb{N}$. In particular, to show that an odd map is a slope, we have only to prove that $\{\lambda(n+m) - \lambda(n) - \lambda(m)\}_{n\in\mathbb{N}}$ is finite.

Lemma 2.4.9. *Let $\lambda, \mu \in \mathscr{S}_\mathbb{Z}$, then we have*

$$\lambda \circ \mu \sim \mu \circ \lambda$$

PROOF : By the first concentration lemma (2.4.8), we can choose λ, μ odd. Now from remark 33, for any $x, y, n \in \mathbb{Z}$

$$\lambda(nx) = n\lambda(x) + \sum_{i=1}^{|n|-1} \lambda_{j_i} \quad \mu(ny) = n\mu(y) + \sum_{i=1}^{|n|-1} \mu_{j_i}$$

And

$$\lambda(nx) = x\lambda(n) + \sum_{i=1}^{|x|-1} \lambda'_{j_i} \quad \mu(ny) = y\mu(n) + \sum_{i=1}^{|y|-1} \mu'_{j_i}$$

So

$$n\lambda(x) = x\lambda(n) + \sum_{i=1}^{|x|-1} \lambda'_{j_i} - \sum_{i=1}^{|n|-1} \lambda_{j_i} \quad n\mu(y) = y\mu(n) + \sum_{i=1}^{|y|-1} \mu'_{j_i} - \sum_{i=1}^{|n|-1} \mu_{j_i}$$

Now substract these expressions with $x = \mu(n)$ and $y = \lambda(n)$ to get

$$\begin{aligned}
|n\lambda(\mu(n)) - n\mu(\lambda(n))| &= |\sum_{i=1}^{|\mu(n)|-1} \lambda'_{j_i} - \sum_{i=1}^{|n|-1} \lambda_{j_i} - \sum_{i=1}^{|\lambda(n)|-1} \mu'_{j_i} + \sum_{i=1}^{|n|-1} \mu_{j_i}| && (2.1)\\
&\leq \sum_{i=1}^{|\mu(n)|-1} |\lambda'_{j_i}| + \sum_{i=1}^{|n|-1} |\lambda_{j_i}| + \sum_{i=1}^{|\lambda(n)|-1} |\mu'_{j_i}| + \sum_{i=1}^{|n|-1} |\mu_{j_i}| && (2.2)\\
&\leq |\mu(n)|\mathfrak{b}_\lambda + |n|\mathfrak{b}_\lambda + |\lambda(n)|\mathfrak{b}_\mu + |n|\mathfrak{b}_\mu && (2.3)\\
&\leq |n|(|\mu(1)| + \mathfrak{b}_\mu)\mathfrak{b}_\lambda + |n|\mathfrak{b}_\lambda + |n|(|\mu(1)| + \mathfrak{b}_\lambda)\mathfrak{b}_\mu + |n|\mathfrak{b}_\mu && (2.4)
\end{aligned}$$

(2.2) comes from the classical triangle inequality, (2.3) comes from the end of remark 33 (and adding a positive term) and (2.4) from

$$|\lambda(n)| = |n\lambda(1) + \sum_{i=1}^{|n|-1} \lambda_{j_i}| \leq (|n| - 1)(\lambda(1) + \mathfrak{b}_\lambda) \leq |n|(\lambda(1) + \mathfrak{b}_\lambda)$$

So we get in fact

$$|\lambda(\mu(n)) - \mu(\lambda(n))| \leq (|\mu(1)| + \mathfrak{b}_\mu)\mathfrak{b}_\lambda + \mathfrak{b}_\lambda + (|\mu(1)| + \mathfrak{b}_\lambda)\mathfrak{b}_\mu + \mathfrak{b}_\mu$$

which is a constant so $\lambda \circ \mu \sim \mu \circ \lambda$ by remark 30.

□

So, up to equivalence, the composition of slopes is commutative. Now comes an important lemma.

Lemma 2.4.10 (Finite differences lemma)**.** *For every slope λ non-equivalent to σ_0, there exists $b_\lambda, B_\lambda > 0$ such that we have the following inequalities*

$$|\lambda(n+k) - \lambda(n)| \leq kb_\lambda \quad \forall n \in \mathbb{Z}\ k > 0$$

$$|\lambda(n+B_\lambda k) - \lambda(n)| \geq k \quad \forall n \in \mathbb{Z}\ k > 0$$

PROOF : For the first inequality, we have that

$$\lambda(n+k) = \lambda(n) + \lambda(k) + \lambda_j = \lambda(n) + k\lambda(1) + \sum_{i=1}^{k-1} \lambda_{j_i} + \lambda_j$$

Now clearly $|\sum_{i=1}^{k-1} \lambda_{j_i} + \lambda_j| \leq k\mathfrak{b}_\lambda$ so

$$|\lambda(n+k) - \lambda(n)| \leq kb_\lambda$$

by setting $b_\lambda = |\lambda(1)| + \mathfrak{b}_\lambda$.
For the second inequality, since $\lambda \notin \mathfrak{z}$, there is a $B_\lambda \in \mathbb{N}$ such that $|\lambda(B_\lambda)| > \mathfrak{b}_\lambda$, so $|\lambda(B_\lambda)| \geq \mathfrak{b}_\lambda + 1$. Now as previously

$$\begin{aligned} |\lambda(n+kB_\lambda) - \lambda(n)| &= |\lambda(n) + k\lambda(B_\lambda) - \lambda(n) + \sum_{i=1}^{k} \lambda_{j_i}| \\ &\geq |k(\mathfrak{b}_\lambda + 1) - k\mathfrak{b}_\lambda| \\ &\geq k \end{aligned}$$

□

Remark 35. The interpretation of this lemma is very important, for it tells us a lot about the shape of any slope. The first inequality tells us that the "gradiant" or the "slope" of a slope is in fact bounded above linearly. The second inequality gives us a lower bound of the gradiant. Therefore, if we draw the graph of a slope in $\mathbb{Z} \times \mathbb{Z}$, the above lemma tells us that the graph lies between two lines (not horizontal as $b_\lambda, B_\lambda \neq 0$). The most important thing to conclude from this lemma is that a slope can only take each value a finite number of time and this come from the fact that $B_\lambda \neq 0$.

We shall show that we can find an inverse for the composition, as well as define an order over the set of slopes. However, this will be a little bit more subtle and to do so we need to improve our knowledge of slopes, by defining the well-adjusted slopes.

Definition 2.4.11. *We say that a slope λ is* well-adjusted *if and only if it is odd and $\Lambda(\lambda) \subseteq \{-1, 0, 1\}$.*

Notice that a well-adjusted slope is even closer to a linear map than a slope, since the λ_j are $-1, 0$ *or* 1, so we always have $\mathfrak{b}_\lambda = 1$. Moreover, the interesting result is that every slope is equivalent to a well-adjusted slope.

Lemma 2.4.12 (Second Concentration Lemma)**.** *Every slope is equivalent to a well-adjusted slope.*

PROOF :
The proof is a bit tedious and needs the *optimal euclidean division.*

Definition. Let $p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}$, the *optimal euclidean division* of p by q (denoted $p : q$) is the integer r such that

$$r = p : q \iff 2p - |q| \leq 2qr \leq 2p + |q|$$

Remark 36. The essential property is that

$$|\frac{p}{q} - p : q| < \frac{1}{2} \qquad (\star)$$

This comes from divising $2p - |q| \leq 2q(p : q) \leq 2p + |q|$ by $2|q|$. This property helps us to understand what the optimal euclidean division is used for. Indeed, the number $p : q$ is so close to the factor $\frac{p}{q}$ that we will use it to "*divide*", and use the fact that it is an integer. More precisely, we have if $x \in \Lambda(\mu)$ $|x| \leq \mathfrak{b}_\mu$, and in fact, for a well-adjusted slope, we would rather have $|x| \leq 1$, this will be done in some sense by "*dividing*" (with :) on both side by $\mathfrak{b}_\mu$.

Sublemma. Let $q > 0$ and $a, b, c \in \mathbb{Z}$ such that $|a - b - c| \leq q$ then $|a : 3q - b : 3q - c : 3q| \leq 1$.

Proof : By hypothesis, $|\frac{a}{3q} - \frac{b}{3q} - \frac{c}{3q}| \leq \frac{1}{3}$. For $x = a, b$ *or* c we have $|\frac{x}{3q} - x : 3q| \leq \frac{1}{2}$ by $(\star)$, so

$$\begin{aligned} |a : 3q - b : 3q - c : 3q - (\frac{a}{3q} - \frac{b}{3q} - \frac{c}{3q})| &\leq |a : 3q - \frac{a}{3q}| + |b : 3q - \frac{b}{3q}| + |c : 3q - \frac{c}{3q}| \\ &\leq \frac{3}{2} \end{aligned}$$

Now as $|\frac{a}{3q} - \frac{b}{3q} - \frac{c}{3q}| \leq \frac{1}{3}$ we get that $|a : 3q - b : 3q - c : 3q| \leq \frac{1}{3} + \frac{3}{2} = \frac{11}{6} < 2$, and as $|a : 3q - b : 3q - c : 3q|$ is an integer, we get the stated result.

□

Nota bene 4. We can wonder about the reason why we choose this operation instead of using the "standard" euclidean division : $a//b = q$ with $a = bq + r$ $0 \leq r < |b|$. The answer is because the property $|\frac{a}{b} - a : b| < \frac{1}{2}$ is essential for our proofs, and with the standard euclidean division, we only get $|\frac{a}{b} - a//b| = |q + \frac{r}{b} - q| = |\frac{r}{b}| < 1$. So in fact, $a : b$ gives us an integer closer to $\frac{a}{b}$ than $a//b$ does.

We can now prove the second concentration lemma. Take any $\mu \in \mathscr{S}_\mathbb{Z}$, we define $\bar{\mu} : \mathbb{Z} \to \mathbb{Z}$ by $\bar{\mu}(n) = \mu(3\mathfrak{b}_\mu n) : 3\mathfrak{b}_\mu$. We show that $\bar{\mu} \sim \mu$. We have first from remark 33 that for every $k \geq 1$

$$|\mu(kn) - k\mu(n)| \leq \mathfrak{b}_\mu(k-1) \qquad (\dagger)$$

Now we use $(\dagger)$ with $k = 3\mathfrak{b}_\mu$ to get

$$|\mu(3\mathfrak{b}_\mu n) - 3\mathfrak{b}_\mu \mu(n)| \leq \mathfrak{b}_\mu(3\mathfrak{b}_\mu - 1)$$

Now we have $|\frac{\mu(3\mathfrak{b}_\mu)}{3\mathfrak{b}_\mu} - \mu(n)| \leq \mathfrak{b}_\mu$ and by $(\star)$ $|\mu(3\mathfrak{b}_\mu n) : 3\mathfrak{b}_\mu - \frac{\mu(3\mathfrak{b}_\mu n)}{3\mathfrak{b}_\mu}| \leq \frac{1}{2}$. Therefore

$$\begin{aligned} |\mu(3\mathfrak{b}_\mu n) : 3\mathfrak{b}_\mu - \mu(n)| &= |\mu(3\mathfrak{b}_\mu n) : 3\mathfrak{b}_\mu - \frac{\mu(3\mathfrak{b}_\mu n)}{3\mathfrak{b}_\mu} + \frac{\mu(3\mathfrak{b}_\mu n)}{3\mathfrak{b}_\mu} - \mu(n)| \\ &\leq |\mu(3\mathfrak{b}_\mu n) : 3\mathfrak{b}_\mu - \frac{\mu(3\mathfrak{b}_\mu n)}{3\mathfrak{b}_\mu}| + |\frac{\mu(3\mathfrak{b}_\mu n)}{3\mathfrak{b}_\mu} - \mu(n)| \\ &\leq \mathfrak{b}_\mu + \frac{1}{2} \end{aligned}$$

So we have $\mu \sim \bar{\mu}$.
The fact that $\bar{\mu}$ is well-adjusted follows from the sublemma applied to the trivial (by definition of $\mathfrak{b}_\mu$)

$$|\mu(3\mathfrak{b}_\mu n + 3\mathfrak{b}_\mu m) - \mu(3\mathfrak{b}_\mu n) - \mu(3\mathfrak{b}_\mu m)| \leq \mathfrak{b}_\mu$$

which gives

$$|\mu(3\mathfrak{b}_\mu n + 3\mathfrak{b}_\mu m) : 3\mathfrak{b}_\mu - \mu(3\mathfrak{b}_\mu n) : 3\mathfrak{b}_\mu - \mu(3\mathfrak{b}_\mu m) : 3\mathfrak{b}_\mu| \leq 1$$

The left hand side stands for any element of $\Lambda(\bar{\mu})$ so $\bar{\mu}$ is a well-adjusted slope.

□

So every slope is, up to equivalence, a well-adjusted slope. We will henceforth deal with well-adjusted slopes as they have some nice properties.

Remark 37. For instance, if λ is a well adjusted slope, we have that

$$|\lambda(n)| \leq (|n| - 1)(|\lambda(1)| + 1)$$

This follows from remark 33 with $y = 1$ since $\mathfrak{b}_\lambda = 1$, we can also write this result :

$$\forall k, n; k > 0 \quad -(k-1) \leq \lambda(kn) - k\lambda(n) \leq (k-1)$$

We can give an interpretation of the finite differences lemma in term of well-adjusted slope. Clearly, from the definition, the two lines will bound the shape of a well-adjusted slope very closely, ie we can imagine the graph between two lines such that the distance betwenn them is less than two.

Lemma 2.4.13. *i* $\exists a > 0 \lambda(a) > \mathfrak{b}_\lambda$*, then for every* $k > 0$

$$\lambda(ka) > \lambda((k-1)a) > \cdots > \lambda(a) > \mathfrak{b}_\lambda$$

ii If λ *is well adjusted and* $\exists a > 0 \lambda(a) > 1$*, then for every* $k > 0$

$$\lambda(ka) > \lambda((k-1)a) > \cdots > \lambda(a) > 1$$

iii For λ *well adjusted, if for some* $a \in \mathbb{Z}$ *we have* $\lambda(a) > 1$ *then for any* $l \in \mathbb{Z}$ *there exists* $n_l \in \mathbb{Z}$ *such that* $|l - \lambda(n_l)| \leq |\lambda(1)| + 1$.

PROOF : (i) Observe that

$$\begin{aligned} \lambda(ka) &= \lambda((k-1)a) + \lambda(a) + \lambda_j \\ &\geq \lambda((k-1)a) + \lambda(a) - \mathfrak{b}_\lambda \\ &> \lambda((k-1)a) \text{ as } \lambda(a) > \mathfrak{b}_\lambda \end{aligned}$$

So finally by an immediate induction we have the result.
(ii) Particular case of i as if λ is well adjusted, $\mathfrak{b}_\lambda = 1$.
(iii) First observe that

$$|\lambda(x+1) - \lambda(x)| = |\lambda(x) + \lambda(1) + \lambda_j - \lambda(x)| \leq |\lambda(1)| + |\lambda_j| \leq |\lambda(1)| + 1$$

since $\lambda_j \leq \mathfrak{b}_\lambda = 1$. Now, if $l = 0$ take $n_l = 1$. If $l > 0$, from ii, we know that there is a $k \geq 0$ such that $l \leq \lambda(ka)$ so set $n_l = ka - 1$ and then

$$|l - \lambda(n_l)| \leq |\lambda(ka) - \lambda(ka-1)| \leq |\lambda(1)| + 1$$

The case $l < 0$ is done similarly since a well-adjusted slope is an odd slope. □

We can now define an "inverse" for the operation $\circ$.

Lemma 2.4.14. *Let* λ *be a well-adjusted slope non-equivalent to* σ_0*, then the there exists a slope* λ' *such that*

$$\lambda \circ \lambda' \sim \iota$$

PROOF : Define $\lambda' : \mathbb{Z} \to \mathbb{Z} : x \mapsto n_x$ where n_x as in lemma 2.4.13 iii. First, we show that λ' is a slope.

$$\begin{aligned} |\lambda(\lambda'(x+y) - \lambda'(x) - \lambda'(y))| &= |\lambda(n_{x+y} - n_x - n_y)| && (2.5)\\ &= |\lambda(n_{x+y}) - \lambda(n_x) - \lambda(n_y) + \lambda_{j_1} + \lambda_{j_2}| && (2.6)\\ &\leq |\lambda(n_{x+y}) - \lambda(n_x) - \lambda(n_y)| + |\lambda_{j_1}| + |\lambda_{j_1}| && (2.7)\end{aligned}$$

Now by the lemma, $|\lambda(n_z)| \leq |z| + |\lambda(1)| + 1$, so

$$\begin{aligned} |\lambda(\lambda'(x+y) - \lambda'(x) - \lambda'(y))| &\leq |x+y-x-y+3(|\lambda(1)|+1)| + 2 \text{ as } \lambda_j \leq 1 \ \forall j\\ &\leq 3|\lambda(1)| + 5\end{aligned}$$

But we know from remark 35 that λ takes each value only a finite amount of times, so λ' is a slope. Now again by the lemma

$$|\iota(x) - \lambda(\lambda'(x))| = |x - \lambda(n_x)| \leq |\lambda(1)| + 1$$

so $\lambda \circ \lambda' \sim \iota$. □

Remark 38. We will denote $\lambda^{\circ 1}$ for any slope such that $\lambda \circ \lambda^{\circ 1} \sim \iota$.

Now we will try to define a way to compare different slopes, namely positive and negative slopes.

Definition 2.4.15. *We say that $\lambda \in \mathscr{S}_\mathbb{Z}$ is* positive *if $\lambda \notin \mathfrak{z}$ and the set*

$$\{\lambda(n) \mid n \in \mathbb{N} \ \lambda(n) \leq 0\}$$

is finite. We say that λ is greater than μ (denoted $\lambda \succ \mu$ or $\mu \prec \lambda$) if $\lambda - \mu$ is positive. Naturally, a slope is positive if $\lambda \succ \sigma_0$. If $\lambda \prec \sigma_0$ we say that λ is negative.

Remark 39. Intuitively, if we follow the idea of the remark 35, it comes that a slope is positive if on the half plane $n \geq 0$ we have only finitely many negative values, ie infinitely many positive values. Drawing a graph, this means that a positive slope is increasing overall, ie the two lines that bound the graph are increasing. Conversely a negative slope will be decreasing overall.

The following lemma tells us more about this ordering.

Lemma 2.4.16. *For any slope λ, μ :*

i $\lambda \succ \sigma_0 \iff \exists a > 0 \ \lambda(a) > \mathfrak{b}_\lambda$

ii $\lambda \succ \sigma_0 \iff \exists a > 0 \ \lambda'(a) > 1$ *for λ' a well-adjusted slope equivalent to λ.*

iii $\lambda \succ \mu \iff \exists a > 0 \ \lambda(ka) > \mu(ka) + k \ \forall k > 0.$

PROOF : (i)Assume $\lambda \succ \sigma_0$, then the right hand side holds by definition, since there are infinitely many positive values, and $\mathfrak{b}_\lambda$ is always positive. Now assume that the right hand side holds, then by lemma 2.4.13 i, we have infinitely many positive values, so λ is positive.
(ii) Every slope is equivalent to a well adjusted slope by the second concentration lemma, and for well-adjusted slope, we have $\mathfrak{b}_\lambda = 1$, so we deduce the result from i
(iii) If $\lambda \succ \mu$ then $\lambda - \mu \succ \sigma_0$. Now by lemma 2.4.13 i, there is $a > 0$ such that $(\lambda - \mu)(ka) > k\mathfrak{b}_{\lambda-\mu} \geq k$ so we get the result. For the converse clearly if the right hand side holds, then $\lambda - \mu$ has infinitely many positive values, and so $\lambda \succ \mu$. □

Remark 40. Notice that using lemma 2.4.13 i and the previous lemma ii, we see that if $\lambda \succ \sigma_0$, then there is a $a > 0$ such that

$$\lambda(ka) > \lambda((k-1)a) > \cdots > \lambda(a) > \mathfrak{b}_\lambda > 0$$

So this confirms the idea that every positive slope is increasing overall.

Lemma 2.4.17. *For $\lambda, \mu, \nu \in \mathscr{S}_{\mathbb{Z}}$*

i If $\lambda \sim \mu$ and $\lambda \succ \sigma_0$ then $\mu \succ \sigma_0$.

ii One and only one of the following holds :

$$\lambda \prec \mu \quad \lambda \sim \mu \quad \lambda \succ \mu$$

iii if $\lambda \prec \mu$ and $\mu \prec \nu$ then $\lambda \prec \nu$.

Proof : (i) Assume μ is not positive, then there are infinitely many $\mu(n) \leq 0$. Now as $\lambda \sim \mu$, there is a $\epsilon \geq 0$ such that $|\lambda(n) - \mu(n)| \leq \epsilon \forall n \in \mathbb{Z}$ (just take $\epsilon = max(|\epsilon_i|)$ in remark 28). So $\mu(n) \geq \lambda(n) - \epsilon$ and so there are infinitely many negative $\lambda(n) - \epsilon$. But as there are only finitely many $\lambda(n) \leq 0$, there are infinitely many $0 \leq \lambda(n) \leq \epsilon$ which contradict the fact that a slope takes each value a finite amount of times (remark 35).

(ii) Let δ be a well-adjusted slope equivalent to $\lambda - \mu$. If $\delta(\mathbb{Z}) \subseteq \{-1, 0, 1\}$, then $\delta \sim \sigma_0$ by remark 29, and so $\lambda \sim \mu$. If not, then there is a $n > 0$ such that $\delta(n) > 1$ or $\delta(n) < -1$, and by lemma 2.4.16 this means that $\lambda \prec \mu$ or $\mu \prec \lambda$ respectively. Clearly, it can't be $\lambda \sim \mu$ and $\lambda \prec \mu$ or $\lambda \sim \mu$ and $\lambda \succ \mu$ in the same time by definition. Furthermore, $\lambda - \mu$ is positive and negative would mean that $\{\lambda - \mu(n)\}_{n \in \mathbb{N}}$ has only finitely many positive and negative elements, which implies that $\lambda \sim \mu$, but this is not possible as previously.

(iii) Let $\delta_1 = \mu - \lambda$ and $\delta_2 = \nu - \mu$ be well-adjusted, we have δ_1 and δ_2 positive, so by lemma 2.4.16 there are $n, m > 1$ such that $\delta_1(n) > 1$ and $\delta_2(m) > 1$. Now

$$\begin{aligned} (\nu - \lambda)(nm) &= (\nu - \mu + \mu - \lambda)(nm) \\ &= \delta_2(nm) + \delta_1(nm) \end{aligned}$$

and by lemma 2.4.13 $\delta_1(nm) \geq \delta_1(n) > 1$. Similarly $\delta_2(nm) > 1$, so $(\nu - \lambda)(nm) > 2$ and so by lemma 2.4.16 ii $\lambda \prec \nu$. □

We can now prove that the relation $\prec$ is compatible with the two operation $+$ and $\circ$.

Lemma 2.4.18. *Let $\lambda, \mu, \nu \in \mathscr{S}_{\mathbb{Z}}$. Then*

$$\lambda \succ \mu \implies \lambda + \nu \succ \mu + \nu$$

and

$$\lambda \succ \mu \text{ and } \nu \succ \sigma_0 \implies \nu \circ \lambda \succ \nu \circ \mu$$

Proof : If $\lambda \succ \mu$, from lemma 2.4.16 iii we get that there is $a > 0$ such that $\lambda(ka) > \mu(ka) + k$ so clearly, $\lambda + \nu(ka) = \lambda(ka) + \nu(ka) > \mu(ka) + \nu(ka) + k = \mu + \nu(ka) + k$ so $\lambda + \nu \succ \mu + \nu$ again by lemma 2.4.16 iii.
For the second implication, assuming the hypothesis, we have $a, a' > 0$ such that $\forall k > 0$ $\lambda(ka) > \mu(ka) + k$ and $\nu(a'k) > k$. Now using this we have also

$$\lambda(aa'k) > \mu(aa'k) + a'k$$

so

$$\begin{aligned} \nu(\lambda(aa'k)) &= \nu(\mu(aa'k) + a'k + \lambda(aa'k) - \mu(aa'k) - a'k) \\ &> \nu(\mu(aa'k) + a'k) + \nu(\lambda(aa'k) - \mu(aa'k) - a'k) - 1 \\ &> \nu(\mu(aa'k)) + \nu(a'k) - 1 + \nu(\lambda(aa'k) - \mu(aa'k) - a'k) - 1 \end{aligned}$$

Now observe that $\lambda(aa'k) - \mu(aa'k) - a'k \geq 1$ and so $\nu(\lambda(aa'k) - \mu(aa'k) - a'k) \geq 0$ as ν is positive well- adjusted slope. Besides $\nu(a'k) \geq k+1$ so in fact we get

$$\nu(\lambda(aa'k)) > \nu(\mu(aa'k)) + k - 1$$

which is not exactly the result in lemma 2.4.16 but yields the result anyway as it means that $\nu \circ \lambda - \nu \circ \mu$ takes infinitely many positive value on $\mathbb{N}$. □

We denote $\lambda \precsim \mu$ if $\lambda \prec \mu$ *or* $\lambda \sim \mu$, similarly for $\succsim$.

Lemma 2.4.19. *For λ, μ well-adjusted slopes, we have the following :*

$$\lambda \succ \mu \implies \forall n \geq 0 \ \lambda(n) + 2 > \mu(n)$$

$$\lambda \succ \mu \iff \exists n > 0 \ \lambda(n) > \mu(n) + 2$$

Proof : For the first result, assume that $\lambda \succ \mu$ and that there is a $n \geq 0$ such that $\mu(n) \geq \lambda(n)+2$. Then from lemma 2.4.16 iii there is $a > 0$ such that we have $\lambda(an) > \mu(an)+n$, and from remark 37, $-(a-1) \leq \nu(an) - a\nu(n) \leq (a-1)$ holds for every well-adjusted slope. So we have

$$\begin{aligned} \lambda(an) &> \mu(an) + n \\ &\geq a\mu(n) - a + 1 + n \\ &\geq a\lambda(n) + 2a - a + 1 + n \end{aligned}$$

So $\lambda(an) - a\lambda(n) > a + 1 + n$. However, $\lambda(an) - a\lambda(n) \leq a - 1$ so

$$\lambda(an) - a\lambda(n) > a + 1 + n > a - 1 \geq \lambda(an) - a\lambda(n)$$

which is a contradiction.
For the second result, $\implies$ follows from the fact that $\mathfrak{b}_{\lambda-\mu} \leq 2$ (as λ and μ are both well-adjusted) so as $\lambda - \mu$ is positive, there is a such that $\lambda - \mu(a) > 2$. Assume the right hand side, there is a n such that $\lambda - \mu(n) > 2 \geq \mathfrak{b}_{\lambda-\mu}$ so by lemma 2.4.16 ii $\lambda - \mu$ is positive (this result is a particular case of lemma 2.4.16 iii). □

Now we prove the theorem which will show the completeness of our system.

Theorem 2.4.20. *Let Δ be a set of well-adjusted slopes such that there is a μ such that $\forall \delta \in \Delta \ \delta \prec \mu$ then there is a μ_0 such that $\forall \delta \in \Delta \ \delta \precsim \mu_0$ and $\mu \prec \mu_0$ entails that there is a $\delta \in \Delta$ such that $\delta \succ \mu$.*

Proof : Assume the hypothesis. By lemma 2.4.19 we have that for all $n \geq 0$ and for all $\delta \in \Delta \ \delta(n) < \mu(n) + 2$. Now observe that the set $S_n \triangleq \{\delta(n) \mid \delta \in \Delta\}$ is bounded above by $\mu(n) + 2$. Therefore there is a $\delta_n \in \Delta$ such that $maxS_n = \delta_n(n)$. Now we define $\mu_0 : \mathbb{Z} \to \mathbb{Z}$ for each $n \geq 0$ by

$$\mu_0(n) = \delta_n(n)$$

and $\mu_0(-n) = -\mu_0(n)$ (n>0). We have $\mu_0(n) = maxS_n$.

Claim. μ_0 is a slope.

We will prove the claim later. Now for every slope $\delta \in \Delta$ and every $n > 0$ we have $\delta(n) \leq \delta_n(n) = \mu_0(n)$. Assume $\mu_0 \prec \delta$ for some $\delta \in \Delta$, then by lemma 2.4.16 iii , there is a $n > 0$ ($n = 2a$ for example) such that $\delta(n) > \mu_0(n) + 2 > \delta_n(n) > \delta(n)$, contradiction. So μ_0 is an upper bound for Δ. Now assume $\mu \prec \mu_0$ (μ well adjusted), then $\mu_0 - \mu \succ \sigma_0$ so $\mu_0 - \mu$

takes infinitely many positive values, so there is a $n \in \mathbb{N}$ such that $\mu_0(n) - \mu(n) > 2$. However, $\mu_0(n) = \delta_n(n)$ and so $\delta_n(n) > \mu(n)+2$ so $\delta_n \succ \mu$ by lemma 2.4.19, which proves the theorem.

PROOF OF THE CLAIM :

We will use the following result.

Sublemma. For all $n, m > 0$ and $c \in \mathbb{Z}$ we have

$$|c : nm - c : n(n+m) - c : m(m+n)| \leq 1$$

.

This comes from $\frac{c}{mn} - \frac{c}{n(n+m)} - \frac{c}{m(n+m)} = 0$. We have

$$\begin{aligned} &|c : nm - c : n(n+m) - c : m(m+n)| \\ &= |c : nm - c : n(n+m) - c : m(m+n) - (\tfrac{c}{mn} - \tfrac{c}{n(n+m)} - \tfrac{c}{m(n+m)})| \\ &\leq |c : nm - \tfrac{c}{mn}| + |c : n(n+m) - \tfrac{c}{n(n+m)}| + |c : m(n+m) - \tfrac{c}{m(n+m)}| \\ &\leq \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{3}{2} < 2 \end{aligned}$$

and we get the sublemma as $|c : nm - c : n(n+m) - c : m(m+n)|$ is an integer.

Now we prove the claim. First we have the following for $q, p, N > 0$ such that $q = pN$

$$|\delta_q(q) - N\delta_q(p)| \leq |\delta_q(q) - \delta_q(Np) - (N-1)| = |N-1|$$

using the result of remark 35. So $\delta_q(p) \leq \frac{N-1}{N} + \frac{\delta_q(q)}{N}$. Now we deduce

$$\begin{aligned} |\delta_q(q) : N - \delta_q(p)| &\leq |\delta_q(q) : N - \frac{\delta_q(q)}{N} - \frac{N-1}{N}| \\ &\leq |\delta_q(q) : N - \frac{\delta_q(q)}{N}| + |\frac{N-1}{N}| \\ &\leq \frac{1}{2} + 1 \text{ by definition of :} \end{aligned}$$

So $\delta_q(q) : N \leq 1 + \delta_q(p) \leq \delta_p(p) + 1$ by definition of δ_p and we conclude that for all $u, v > 0$ $|\delta_u(u) - \delta_{uv}(uv) : v| \leq 1$. So in particular,

$$|\delta_n(n) - \delta_{nm(n+m)}(nm(n+m)) : m(n+m)| \leq 1$$

$$|\delta_m(m) - \delta_{nm(n+m)}(nm(n+m)) : n(n+m)| \leq 1$$

$$|\delta_{n+m}(n+m) - \delta_{nm(n+m)}(nm(n+m)) : mn| \leq 1$$

However, $\sigma(k) = \delta_k(k)$ so put $c = \delta_{nm(n+m)}(nm(n+m))$

$$\begin{aligned} &|\sigma(n+m) - \sigma(n) - \sigma(m) - c : nm + c : n(n+m) + c : m(m+n)| \\ &\leq |\sigma(n+m) - c : nm| + |\sigma(n) - c : m(m+n)| + |\sigma(m) - c : n(n+m)| \\ &\leq 3 \end{aligned}$$

So using the sublemma :

$$|\sigma(n+m) - \sigma(n) - \sigma(m)| \leq 3 + |c : nm - c : n(n+m) - c : m(m+n)| \leq 4$$

and we conclude that σ is a slope. $\square$

2.4.2 Constructing the Eudoxus reals

Now we can formally define the real numbers, the tedious part is out of the way and everything continues smoothly.

Definition 2.4.21. *We define the set*

$$\mathscr{R} = \mathscr{S}_{\mathbb{Z}}/_{\sim}$$

The class of the slope $\lambda \in \mathscr{S}_{\mathbb{Z}}$ will be denoted $\tilde{\lambda}$.
Further, we define two binary operations[4] *$+$ and $\cdot$ over $\mathscr{R}$ by*

$$\tilde{\lambda} + \tilde{\mu} = \widetilde{\lambda + \mu}$$

$$\tilde{\lambda} \cdot \tilde{\mu} = \widetilde{\lambda \circ \mu}$$

where $+$ and $\circ$ are the operations defined in the first subsection. Furthermore, we define the binary predicate $<$ over $\mathscr{R}$ by

$$\tilde{\lambda} < \tilde{\mu} \iff \lambda \prec \mu$$

Notice that these definitions make sense by lemmata 2.4.4, 2.4.5 and 2.4.17. We shall now prove that $(\mathscr{R}, <, +, \cdot)$ is a complete ordered field but again, almost everything has been done in the first part, so we will allow ourself to pass quickly on trivial properties.

Proposition 2.4.22.

$(\mathscr{R}, +, \cdot)$ is a field

Proof : (Φ_1) is given by lemmata 2.4.4 and 2.4.5. Associativity of both $+$ and $\circ$ are discussed in remarks 29 and 31, and by compatibility with $\sim$ we have that $+$ and $\cdot$ are associative. Similarly, commutativity of $+$ has been discussed in remark 29 and commutativity of $\cdot$ comes from lemma 2.4.9. Indeed, take $\tilde{\lambda}, \tilde{\mu} \in \mathscr{R}$, then

$$\begin{aligned} \tilde{\lambda} \cdot \tilde{\mu} &= \widetilde{\lambda \circ \mu} \\ &= \widetilde{\mu \circ \lambda} \text{ by lemma 2.4.9} \\ &= \tilde{\mu} \cdot \tilde{\lambda} \end{aligned}$$

So $+$ and $\cdot$ are commutative and we have (Φ_2) and (Φ_3). (Φ_4) is a trivial consequence of remarks 29 and 31, since in fact $\tilde{\sigma_0} = \mathfrak{z}$ and $\tilde{\iota} = \mathfrak{u}$. (Φ_5) follows from remark 30 for $+$ and lemma 2.4.14 for $\cdot$. Finally we prove the distributivity law (Φ_3) : let $x = \tilde{\lambda}, y = \tilde{\mu}, z = \tilde{\nu} \in \mathscr{R}$. For every $n \in \mathbb{Z}$, we have $(\lambda + \mu) \circ \nu(n) = (\lambda + \mu)(\nu(n)) = \lambda(\nu(n)) + \mu(\nu(n))$ so $(\lambda + \mu) \circ \nu = \lambda \circ \nu + \mu \circ \nu$. But then

$$\begin{aligned} (x + y) \cdot z &= (\tilde{\lambda} + \tilde{\mu}) \cdot \tilde{\nu} \\ &= \widetilde{(\lambda + \mu) \circ \nu} \text{ by definition} \\ &= \widetilde{\lambda \circ \nu + \mu \circ \nu} \text{ by above} \\ &= \tilde{\lambda} \cdot \tilde{\nu} + \tilde{\mu} \cdot \tilde{\nu} \\ &= x \cdot z + y \cdot z \end{aligned}$$

and by commutativity of $\cdot$, we have (Φ_3). □

4. Remark that we denote by $+$ both the operation between slopes and equivalent classes of slope.

Remark 41. We denote $0 = \tilde{\sigma_0}$ and $1 = \tilde{\iota}$ the neutral elements of $\mathscr{R}$. The opposite of $x \in \mathscr{R}$ will be denoted as usual x^{-1} and in fact if $x = \tilde{\lambda}$, we have $x^{-1} = \widetilde{\lambda^{\circ 1}}$ since $x \cdot x^{-1} = \widetilde{\lambda \circ \lambda^{\circ 1}} = \tilde{\iota} = 1$. Similarly, we denote by $-x$ the opposite of x and it is of course given by $\widetilde{-\lambda}$.

Proposition 2.4.23.

$(\mathscr{R}, <, +, \cdot)$ *is an ordered field*

PROOF : First of all, notice that $<$ is well-defined on $\mathscr{R}$ thanks to lemma 2.4.17 i. Now we prove (Ω_1). Lemma 2.4.17 ii gives

$$\lambda \prec \mu \quad \lambda \sim \mu \quad \lambda \succ \mu$$

Which is the exact translation of

$$\tilde{\lambda} < \tilde{\mu} \quad \tilde{\lambda} = \tilde{\mu} \quad \tilde{\lambda} > \tilde{\mu}$$

notice that we have used the notation $>$ which is defined by $x < y \iff y > x$. Clearly lemma 2.4.17 iii gives exaclty (Ω_2), provided by the fact that $\prec$ is compatible with $\sim$ which is given by lemma 2.4.17 i. Finally (Ω_3) is exaclty what lemma 2.4.18 shows. □

Theorem 2.4.24.

$(\mathscr{R}, <, +, \cdot)$ *is complete*

PROOF : Let X be a subset of $\mathscr{R}$ which is bounded above. Then this corresponds to a set of well-adjusted slope Δ with an upper bound μ. Now by theorem 2.4.20 there is a μ_0 such that no other upper bound of Δ is below it. Then clearly $x = \tilde{\mu}_0$ is the least upper bound of the set X. □

So finally, $(\mathscr{R}, <, +, \cdot)$ is a complete ordered field so again, we have constructed the real number system. But if it is easy to see that cuts can represent numbers, it is much less obvious that slopes represent numbers too. We will now try to represent explicitly some real numbers by explicite slopes.
First, from definition 1.1.6, the natural numbers are obtained from any ordered field by adding the neutral element for multiplication. We have $1 = \tilde{\iota}$, so 1 is represented by the identity function $\iota : n \mapsto n$. So the number 2 will be represented by the function defined by $(\iota + \iota) : n \mapsto n + n$, we have $2 = \widetilde{(\iota + \iota)}$. We obtain every natural number by

$$n = \underbrace{\widetilde{\iota + \cdots + \iota}}_{n\ times} = \widetilde{k \mapsto nk}$$

Further, we obtain the integers by setting $-n = \widetilde{k \mapsto -nk}$. 0 is clearly represented by $k \mapsto 0$. Now for the rational numbers, we know that $x = \frac{p}{q} \iff px = q$. Define the odd slope $\lambda(n) = min\,\{k \geq 0 \mid qk \geq pn\}$. By definition, we have $|q\lambda(n) - pn| \leq q$ and it can be shown that $|\lambda(n+m) - \lambda(n) - \lambda(m)| \leq 3$, so it is a slope. Now, if $\bar{q} : k \mapsto qk$ and $\bar{p} : k \mapsto pk$, we have $\bar{q} \circ \lambda(n) \geq pn = \bar{p}(n)$, and in fact $\bar{q} \circ \lambda(n) \sim \bar{p}(n)$. So λ represents $\frac{p}{q}$.
Similarly, we can represent $\sqrt{2}$ by the slope $\rho(n) = min\left\{k \mid 2n^2 \leq k^2\right\}$ and π by $\mu(n) = \frac{|\left\{(p,q) \in \mathbb{Z}^2 | p^2 + q^2 \leq n^2\right\}|}{n}$.

RELATION WITH OTHER CONSTRUCTIONS

We can indeed relate this construction of the real numbers with the two others that we have done. As far as Cauchy sequences are concerned, the slope λ correspond to the sequence

$$x_n = \frac{\lambda(n+1)}{n+1}$$

We can see that it works well for the sequences that represent $\mathbb{Z}$.

For Dedekind cuts, we have to set

$$\alpha = \left\{ \frac{p}{q} \in \mathbb{Q} \mid \bar{p} \precsim \lambda \circ \bar{q} \right\}$$

We have that the Dedekind cut α represents the same real as λ.

2.5 Others constructions

In this section, we present some other constructions of the reals. We will just present the basic idea, without further details.

2.5.1 Construction from Hyperrational numbers

Given the archimedean ordered field $(\mathbb{Q}, <, +, \cdot)$, it is possible by a process called *ultraproduct* to construct a bigger ordered field $({}^*\mathbb{Q}, <, +, \cdot)$ usually called *hyperrational numbers* or *non-standard rational numbers*. This new ordered field contains the rational numbers but is not archimedean. So by the section 1.4 we get two sets $\mathcal{F}_{^*\mathbb{Q}}$ and $\mathcal{I}_{^*\mathbb{Q}}$ such that :

$$\mathcal{F}_{^*\mathbb{Q}} \big/ \mathcal{I}_{^*\mathbb{Q}} \textit{ is an archimedean ordered field}$$

Finally, it can be proved that $\mathcal{F}_{^*\mathbb{Q}} \big/ \mathcal{I}_{^*\mathbb{Q}}$ is *complete*, so it is isomorphic to $\mathbb{R}$ by the first chapter.

2.5.2 Construction from Surreal numbers

This construction is the same principle as the previous one, for it consists of constructing a bigger system and extracting the reals from it. The Surreal numbers form a proper class consisting of pairs of sets. It is defined by transfinite induction, starting from $0 = \{\emptyset \mid \emptyset\}$ and following the two rules :

Rule 1. Every number corresponds to two sets of previously created numbers, such that no member of the left set is greater than or equal to any member of the right set.

Rule 2. One number is less than or equal to another number if and only if no member of the first number's left set is greater than or equal to the second number, and no member of the second number's right set is less than or equal to the first number.

By construction, the class of Surreal numbers is ordered, and in fact this order is total. Furthermore an addition and a multiplication can be defined on it to turn it into a sort of big ordered field, supposedly containing every ordered fields. In particular, it contains $\mathbb{R}$ and $\mathbb{R}$ can be extracted from it, as the biggest archimedean subfield, leading to another construction of the complete ordered field.

2.5.3 Construction via continued fractions

This construction is motivated by the fact that every real number can be represented by a unique (simple) continued fraction, ie $\forall x \in \mathbb{R}$

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \quad \triangleq \langle a_0, a_1, a_2, \dots \rangle$$

for some $a_j \in \mathbb{Z}$. Recall some beautiful classical results :

$$\sqrt{2} = \langle 1, 2, 2, 2, \ldots \rangle \qquad \phi = \langle 1, 1, 1, 1, 1, \ldots \rangle \qquad e = \langle 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots \rangle$$

Clearly, such a result makes us think about a representation of every real number by an infinite sequence of integers.
This is the idea : introduce a symbol ω such that $\omega > n \ \forall n \in \mathbb{N}$ and let

$$\mathscr{R} = \{\langle a_0, a_1, \ldots \rangle \mid a_0 \in \mathbb{N} \ a_n \in \mathbb{N} \cup \{\omega\} \setminus \{0\} \ \ n \geq 1\}$$

such that the two following conventions are respected :

1. If $a_n = \omega$ then $\forall k > n \ a_k = \omega$
2. If n is maximal such that $a_n \neq \omega$ (if exists) then $a_n \neq 1$

For $a_n \neq \omega$, we identify $\langle a_0, \ldots, a_n, \omega, \omega, \ldots \rangle$ with $\langle a_0, \ldots, a_n \rangle$. Notice that the second point is motivated by the fact that $\langle a_0, \ldots, a_n, 1 \rangle = \langle a_0, \ldots, a_n + 1 \rangle$.
Now notice that $\mathscr{Q} \triangleq \{\langle a_0, \ldots \rangle \in \mathscr{R} \mid \exists j \ a_j = \omega\}$ is identified with $\mathbb{Q}$ and so has a structure of ordered field $(\mathscr{Q}, <, +, \cdot)$. The idea of this construction is to extend these operations over $\mathscr{R}$.
For example, we define the order $\dot{<}$ over $\mathscr{R}$ by

$$\langle a_0, a_1, \ldots \rangle \,\dot{<}\, \langle b_0, b_1, \ldots \rangle \quad \Longleftrightarrow \quad \begin{cases} a_{k(a,b)} < b_{k(a,b)} & \text{if } k(a,b) \text{ is even} \\ b_{k(a,b)} < a_{k(a,b)} & \text{if } k(a,b) \text{ is odd} \end{cases}$$

where $k(a,b)$ denotes the least index j such that $a_j \neq b_j$ and $\langle a_0, a_1, \ldots \rangle \neq \langle b_0, b_1, \ldots \rangle$.

A sketch of the proof of completeness :
Let S be a bounded above subset of $\mathscr{R}$. Then for every $j \in \mathbb{N}$, the set $S_j \triangleq \{a_j \mid \langle a_0, \ldots \rangle \in S\} \subseteq \mathbb{Z}$, is bounded above if j is even or bounded below if j is odd. Now set

$$s_j = \begin{cases} maxS_j & \text{if } j \text{ is even} \\ minS_j & \text{if } j \text{ is odd} \end{cases}$$

Then the continued fraction $\langle s_0, s_1, \ldots \rangle$ is the least upper bound of S.

2.5.4 Construction via alternating series

This construction is of the same spirit as the previous one. It is based on the following theorem :

Theorem (alternating-Sylvester series). Every real number $A \in \mathbb{R}$ has a unique representation of the form :

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} - \cdots = ((a_0, a_1, \ldots))$$

with $a_i \in \mathbb{Z}$, $a_1 \geq 1$ and $a_{i+1} \geq a_i(a_i + 1)$.

So the idea of this construction is to represent every real number as an infinite sequence $((a_0, a_1, \ldots))$ such that $a_i \in \mathbb{Z}$, $a_1 \geq 1$ and $a_{i+1} \geq a_i(a_i + 1)$. Addition, multiplication and order can be defined on the set

$$\mathscr{R} = \{((a_0, a_1, \ldots)) \ \textit{such that} \ a_i \in \mathbb{Z}, \ a_1 \geq 1 \ \textit{and} \ a_{i+1} \geq a_i(a_i + 1)\}$$

to get a complete ordered field.

3 Sources, comments and conclusion

In this last part, we shine a light on the sources and comments in each section. It is our choice to separate the mathematical content from this part so we can focus on each different aspect of the report.

On the first chapter

The purpose of the first section is to introduce the formal notion of ordered fields. It is mainly inspired by [8] and [21] although none of them present fully the isomorphism $\mathbb{Q} \cong \mathcal{Q}_{\mathbb{k}}$. Besides giving the first taste and tools for the isomorphism theorem, this result tells us that $\mathbb{Q}$ is indeed the smaller ordered field, and is contained in each ordered field. The second section consists of the introduction to the completeness axiom as well as the nice properties that it entails.
In the third section the proof of the uniqueness of the complete ordered field allow us to talk about *the* complete ordered field. The proof presented here differs in a subtle way from most of the proofs of uniqueness that can be found, for most of the time, the uniqueness is proved after the construction of the real number system. Here the idea was to prove formally the uniqueness first. However the proof has its roots in [21], and some technical details come from [1].
The fourth section has in fact three goals. First it was a way to introduce the notion of *infinitesimals* and *infinite* elements, that are fundamental tools for non-standard analysis, for which more can be found in [8]. Secondly, we saw an example of non-archimedean ordered field, something that can seems a bit unnatural, [10] and [2] were used for this. Finally, the results introduced were a nice preparation to get the idea of the construction via Hyperrational numbers (subsection 2.5.1).

On the second chapter

The first section presents in detail the construction of $\mathbb{R}$ by $\mathbb{Q}$-Cauchy sequences. It is mainly inspired by [4] and [17]. This construction is a classical way of getting the real numbers from the ordered field $\mathbb{Q}$. It is a bit tedious to do, and not very interesting mathematically speaking, but from my point of view it is the most intuitive way of constructing the real numbers.
In the second section, we present the other classical construction of the real numbers, by the Dedekind cuts. This construction follows [6] and [19] more than [21] and [9] even though in [6] the cuts were an unbounded subset of $\mathbb{Q}$ on the right instead of on the left as we have defined it. The main reason is because, the way we have done it, the comparison of two cuts is given by inclusion, whereas in [6] it is done by inclusion of complements. Now that we have the existence of $\mathbb{R}$ we can see that a cut is given by $]-\infty, x[\cap\mathbb{Q}$ and clearly such a cut represents the real x. In [21] and [9] a cut is a pair of sets, (A, B) and in fact if it represents $x \in \mathbb{R}$ then $(A, B) = (]-\infty, x] \cap \mathbb{Q}, [x, +\infty[\cap\mathbb{Q})$ so we see that it is entirely characterised by $]-\infty, x[\cap\mathbb{Q}$.
Historically the construction by Cauchy sequences and by Dedekind cuts have been published at the same time and they were the first rigorous constructions of $\mathbb{R}$. The construction by Dedekind cuts was in fact discovered by Richard Dedekind (1831-1916) in 1858 for his lecture on real analysis, but not published until 1872. In the same year Eduard Heine (1821-1881) published a

rigorous construction of $\mathbb{R}$ by equivalence classes of $\mathbb{Q}$-Cauchy sequences based on the work of Georg Cantor (1845-1918). However Karl Weierstrass (1815-1897) developped this idea in his lecture in 1863 but did not publish this idea (see Historical Remarks 1.8 in [6]).

After having presented the two most classical constructions, we have focused on a much more modern construction, the construction via slopes, or almost homomorphism. This construction is based on the 2003 paper of Norbert A'Campo (see [3]). This construction is not very well-known, and there is not a lot of literature about it. Rob Arthan and Ross Street (see [5] and [20]) have also done this construction by almost homomorphism. Only A'Campo [3] uses the terminology "slope" instead of almost homomorphism, but we adopt it since our presentation is mostly inspired by [3]. It appears that the proof of completeness in [5] is different from the one we present (inspired by [3]) for Arthan does not use well-adjusted slope, which he qualifies in [5] as "a very efficient normal form". Norbert A'Campo's construction has its roots in the definition of rotation number of an orientation preserving homeomorphism of the circle due to Henri Poincaré. The first to talk about real numbers constructed from functions of the additive group of integers was Stephen Shanuel and he is at the origin of the name Eudoxus reals, for this construction "seemed to reflect the relationship between the discrete and the continuous apparent in the ancient theory of proportion" (see section 5 in [5] for historical remarks and citations). The article [3] left a lot of the proofs to the reader, and we have tried to make an exposition as detailed as possible. At the end of this section, we have claimed some represention of real numbers (namely $\frac{p}{q}$, $\sqrt{2}$ and π) and the detail of these results can be found in [3].

The construction from hyperrational numbers is well presented in [8] as well as in [13] and [12]. For a good understanding of what Surreal numbers are, see [11] and [15]. In addition, [11] provides an isomorphism between a certain subset of the surreal numbers and $\mathbb{R}$. The construction by continued fractions is due to Johann Georg Rieger and is explained in [18]. Basic knowledge for continued fractions can be found in [7]. The construction by alternating series [14] was published in 1988 by Arnold and John Knopfmacher (father and son) and is inspired by [18]. They present in fact two constructions, both based on a certain kind of sequence.

We can observe that there are three different types of construction for the real. The first type is the construction directly from "existing" objects, as sets, or sequences. The constructions by Dedekind cuts, by continued fractions and by alternating series are of this type. In the first case, real numbers are particular subsets of $\mathbb{Q}$, and in the second and third case, they are particular infinite sequences of elements of $\mathbb{Z}$. The second type is by constructing something bigger, and then identifying $\mathbb{R}$ inside. The construction by Hyperrational and Surreal numbers are of this type. Finally the third type is by taking the equivalence classes of an "existing" set. The constructions by $\mathbb{Q}$-Cauchy sequences and by slopes are of this type. In the first case, the equivalence relation is made upon $\mathbb{Q}$-Cauchy sequences, while in the second case we look at the equivalent classes of a certain kind of function $\mathbb{Z} \to \mathbb{Z}$.

The interesting part in each construction is the proof of completeness. Notice that that for the proof of completeness in the construction by $\mathbb{Q}$-Cauchy sequences, we use the archimedean property, which is implied by the completeness by theorem 1.2.7. It is the only time we used this property to prove the completeness. The proof in this case is rather tedious. In the case of the construction by Dedekind cuts, the proof is immediate and that is why this construction is the most popular, even though the definition of multiplication is tedious. In the case of the construction by slopes, the proof is not very easy and needs some background work. The sketch of the proof for continued fractions appears to be in the same spirit as the one for slopes, since we construct the supremum by taking each component being in some sense the maximal of the component of the elements of the bounded above set.

Finally, notice that a trait of a construction lies in the amount of objects it consider as given, and while the construction based on $\mathbb{Q}$-Cauchy sequences, Dedekind cuts, Hyperrational numbers are based on the *ordered field* $(\mathbb{Q}, <, +, \cdot)$, the constructions by continued fractions and alternating series use the *ordered ring* $(\mathbb{Z}, <, +, \cdot)$. The construction by slopes takes as given nothing more than the *additive group* $(\mathbb{Z}, +)$. Finally the Surreals are constructed from basic set theory considering pairs of sets. Anyhow, $(\mathbb{Q}, <, +, \cdot)$, $(\mathbb{Z}, <, +, \cdot)$ and $(\mathbb{Z}, +)$ can be constructed from $(\mathbb{N}, s)$ (s being the succesor function) by the *axioms of Peano* (see an example among many [4]).

Bibliography

[1] Uniqueness of the real numbers. *http ://math.ucr.edu/ res/math205A/uniqreals.pdf.*

[2] *Mathématiques L3 Algèbre.* Pearson, 2009.

[3] Norbert A'Campo. A natural construction for the real numbers. *arXiv :math/0301015 [math.GN]*, 2003.

[4] A.G.HAMILTON. *Numbers Sets and Axioms : The Apparatus of Mathematics.* Cambridge University Press, 1992.

[5] R.D Arthan. The eudoxus real numbers. *arXiv :math/0405454v1 [math.HO]*, 2004.

[6] Ethan D. Bloch. *The real numbers and real analysis.* Springer Science, 2011.

[7] Darren C. Collins. Continued fractions. *MIT Undergraduate Journal of Mathematics.*

[8] Martin Davis. *Applied Nonstandard Analysis.* John Wiley and Sons,, 1977.

[9] Richard Dedekind. *Essays on the theory of numbers.* The open court publishing company, 1901.

[10] H.B. Enderton. Field of rational functions. *http ://www.math.ucla.edu/ hbe/resource/general/131a.3.06w/nonArch*

[11] Gretchen Grimm. An introduction to surreal numbers. *http ://www.whitman.edu/mathematics/SeniorProjectArchive/2012/Grimm.pdf*, 2012.

[12] James Forsythe Hall. Completeness of ordered fields. *arXiv :1101.5652v1 [math.LO]*, 2011.

[13] Todor D. Todorov James F. Hall. Another proof of the existence a dedekind complete totally ordered field. *arXiv :1101.3825v1 [math.LO]*, 2011.

[14] Arnold Knopfmacher and John Knopfmacher. Two concrete new constructions of the real numbers. *Rocky Mountain J. Math. Volume 18, Number 4*, 1988.

[15] D. E Knuth. *Surreal Numbers.* Addison-Wesley Publishing Company, 1974.

[16] Edmund Landau. *Fundation of Analysis.* AMS Chelsea, 1960.

[17] D.M. Ha M.A. Ackoglu, P.F.A. Bartha. *Analysis in Vector Spaces.* John Wiley and Sons, Inc., 2009.

[18] G.J Rieger. A new approach to the real numbers (motivated by continued fractions). *Abh. Braunschweig.Wiss. Ges. 33*, 1982.

[19] Walter Rudin. *Principle of Mathematical Analysis.* McGraw-Hill, 1976.

[20] Ross Street. Update on the efficient reals. 2003.

[21] Elias Zakon. *The Zakon series for Mathematical Analysis : Basic Concepts of Mathematics*, volume I. The Trillia Group, West Lafayette, Indiana, USA, 2001.