INTRODUCTION TO OMEGA-CATEGORICAL GROUPS

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Description of the course

Those are the notes of a course given by the author at the American University in Cairo, for the *Cairo Logic Summer School*. We introduce the basics in the study of ω-categorical groups, leading to the fundamental question concerning ω-categorical groups: the Wilson Conjecture.

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Date: June 1, 2026.
The author is supported by the Ramon y Cajal grant RYC2023-042677-I funded by MICIU/AEI/10.13039/501100011033 and by ESF+.

1. Introductory Considerations

Let us start with the very definition of ω-categoricity, which is a property of a first-order theory T, in a given countable language $\mathscr{L}$. The theory T is ω-categorical if for all countable models M,N of T, we have $M \cong N$. In other words, T *admits a unique countable model up to isomorphism.* A given structure M is called ω-categorical if $\mathrm{Th}(M)$ is ω-categorical. This is equivalent to saying that for all other structure M in the same language

$$M \equiv N \iff M \cong N.$$

As usual in model theory we will assume that every structure we deal with (group, algebras) are infinite.

The "logical" definition of ω-categorical has an equivalent formulation, often more convenient to work with which is given by the so-called **Ryll-Nardzewski Theorem**. It states that the following conditions are equivalent:

(1) M is ω-categorical.
(2) for all $n \in \mathbb{N}$ there are only finitely many 0-definable subsets of the cartesian power $M^n = \underbrace{M \times \ldots \times M}_{n \text{ times}}$.
(3) for all $n \in \mathbb{N}$ there are only finitely many orbits in the action of $\mathrm{Aut}(M)$ on the cartesian power M^n.

Recall. Let us just recall basics on orbits and actions of the automorphism group $\mathrm{Aut}(M)$ on M. Each automorphism $\sigma \in \mathrm{Aut}(M)$ acts on M via $a \mapsto a^\sigma = \sigma(a)$. This action extends to $\vec{a} = (a_1, \ldots, a_n) \in M^n$ via $(\vec{a})^\sigma = (a_1^\sigma, \ldots, a_n^\sigma)$. The more relevant notion for us is that of an *orbit*. If $\vec{a} \in M^n$, the *orbit of* $\vec{a}$ is the set

$$\mathrm{Orb}(\vec{a}) = \{\vec{b} \in M^n \mid \vec{b} = \vec{a}^\sigma, \text{ for some } \sigma \in \mathrm{Aut}(M)\}$$

An *orbit* is an set of the form $\mathrm{Orb}(\vec{a})$ for some $\vec{a}$. The properties that we will use here are:

- For any two orbits O_1, O_2 of M^n, we have either $O_1 = O_2$ or $O_1 \cap O_2 = \emptyset$. In particular, M^n can be partitionned into orbits.
- Every subset S of M^n which is closed under the action of $\mathrm{Aut}(M)$ (i.e. $S^\sigma = S$ for all $\sigma \in \mathrm{Aut}(M)$) is a union of orbits.

Let us readily make use of those equivalent definitions by proving a first and fundamental property of ω-categorical structures.

Theorem 1.1. *Let M be an ω-categorical structure in a countable language. Then M is uniformly locally finite, by which we mean that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n$, the structure $A = \langle a_1, \ldots, a_n \rangle$ generated by $a_1, \ldots, a_n$ is finite, with $|A| \le f(n)$.*

We will give two proofs of this result, using the two equivalent definitions of ω-categoricity.

Proof Number 1. Fix a number n and let $\mathrm{Ter} = \mathrm{Ter}(x_1, \ldots, x_n)$ be the family of all $\mathscr{L}$-terms in variables $x_1, \ldots, x_n$ of the language $\mathscr{L}$. For each $t \in \mathrm{Ter}$, set $D_t \subseteq M^{n+1}$ to be the set defined by the formula "$t(x_1, \ldots, x_n) = y$". In other words, $(a_1, \ldots, a_n, b) \in D_t$ if and only if $t(a_1, \ldots, a_n) = b$. Bye the statement (2) above, there are only finitely many distinct elements in the family $\{D_t \mid t \in \mathrm{Ter}\}$. In other words, there exists $t_1, \ldots, t_s \in \mathrm{Ter}$ such that

$$\bigcup_{t \in \mathrm{Ter}} D_t = D_{t_1} \cup \ldots \cup D_{t_s}$$

Then, for any $a_1, \ldots, a_n \in M$ and $b \in \langle a_1, \ldots, a_n \rangle$, there is $1 \le i \le s$ such that $b = t_i(\vec{a})$, hence $\langle \vec{a} \rangle = \{t_1(\vec{a}), \ldots, t_s(\vec{a})\}$. Setting $f(n) = s$, we get the result. □

Proof Number 2. The beginning is the same, fix a number n and let $\mathrm{Ter} = \mathrm{Ter}(x_1, \ldots, x_n)$ be the family of all $\mathscr{L}$-terms in variables $x_1, \ldots, x_n$ of the language $\mathscr{L}$. For each $t \in \mathrm{Ter}$, set $D_t \subseteq M^{n+1}$ to be the set defined by the formula "$t(x_1, \ldots, x_n) = y$". Observe now that D_t is closed under the action of $\mathrm{Aut}(M)$: for all $\sigma \in \mathrm{Aut}(M)$, if $(a_1, \ldots, a_n, b) \in D_t$, i.e. $t(a_1, \ldots, a_n) = b$, then $t(\sigma(a_1), \ldots, \sigma(a_n)) = \sigma(b)$ so $(a_1, \ldots, a_n, b)^\sigma \in D_t$. It follows that D_t is a union of orbits of M^{n+1} under the action of $\mathrm{Aut}(M)$, for all $t \in \mathrm{Ter}$. Using (3), let $O_1, \ldots, O_m$ be a list of all the orbits of M^{n+1}:

$$O_1 \sqcup \ldots \sqcup O_m = M^{n+1}.$$

Then, there are at most 2^m distinct possible sets D_t, and hence we may conclude as in the previous proof. □

Of course, the two arguments are dual to one another. A possible advantage of the statement (3) is that it seems as it does not talk about 1st-order formulas, (although, in reality, it is). At any rate, we will always consider that either of the three statements above can serve as a working definition of ω-categoricity, refering to them as "by ω-categoricity" rather than as "by the Ryll-Nardzewski Theorem".

2. The algebraic Structure of ω-categorical groups

2.1. Basics on ω-categorical groups.

Let us start with a list of notations, conventions and prerequisites in group theory.

(1) A *group* $(G,\cdot,1)$ is a structure where the composition law $\cdot$ is associative ($x(yz)=(xy)z$), 1 is a neutral element ($1x=x$) and every element x has an inverse, denoted x^{-1} ($xx^{-1}=1$). In model theory, those are generally and implicitly studied in the language $\{\cdot,1,^{-1}\}$, where the function $x\mapsto x^{-1}$ is there to ensure that substructures are subgroups. $(G,\cdot)$ will always be a group.

(2) If $g\in G$ is such that $g^n=1$, then the smallest such n is called the *order of* g. A group G is of *of bounded exponent* if there exists a global bound n on the order of each element of G. The smallest such bound is called the *exponent* of G.

(3) For $g,h\in G$, the *commutator of* g *and* h is denoted $[g,h]:=g^{-1}h^{-1}gh$. Iterated commutators are defined by induction as follows: $[g_1,\dots,g_n]=[[g_1,\dots,g_{n-1}],g_n]$. The group G is abelian if $gh=hg$ for all $g,h\in G$, equivalently, $[g,h]=1$ for all $g,h\in G$. The number c is called the *nilpotency of class of* G if $c\in\mathbb{N}$ is the smallest such that $[g_1,\dots,g_{c+1}]=1$ for all $g_1,\dots,g_{c+1}\in G$. We will also say that a group is n-nilpotent if its nilpotency class is $\leq n$.

(4) Define the series $(\gamma_i)_{i\in\mathbb{N}}$ of subgroups of G as follows: $\gamma_1=\gamma_1(G)=G$, and $\gamma_{n+1}(G)=[\gamma_n(G),G]$, where the notation $[A,B]$ denotes the group generated by the commutators $[a,b]$ with $a\in A,b\in B$. Then G is nilpotent of class c if and only if c is the smallest such that $\gamma_{c+1}(G)=1$.

(5) Given a prime number p, a p-group is a group in which every element has a finite, p-divisible, order. Every finite p-group is nilpotent (see Exercise 2.1).

(6) Given $n\in\mathbb{N}$, a group G is called n-Engel if $[g,h^{(n)}]:=[g,\underbrace{h,\dots,h}_{n\text{ times}}]=1$ for all $g,h\in G$. A classical theorem of Zorn states that every finite n-Engel group is nilpotent (see [4, Theorem 12.3.4]).

(7) Given a property P (typically P could mean being nilpotent, or being finite), we say that G is *locally* P if every finitely generated subgroup has property P. For instance, G is finitely nilpotent if every finitely generated subgroup is nilpotent.

Exercise 2.1. In this exercise, we prove that finite p-groups are nilpotent. Let G be a finite p-group.

(1) The class equation gives that the action of G on itself has non-central representant $x_1,\dots,x_s$ and that

$$|G|=|Z(G)|+\sum_i|G/C_G(x_i)|.$$

Prove that $Z(G)\neq 1$.

(2) Prove that if $G/Z(G)$ is nilpotent then G is nilpotent.

(3) Prove by induction on $|G|$ that a finite p-group is nilpotent.

Let us now gather some simple but very useful consequences of Theorem 1.1.

Corollary 2.2. *Let G be an ω-categorical group, then G has bounded exponent.*

Proof. Theorem 1.1 tells us that there is a number $k=f(1)$ such that $|\langle a\rangle|\leq k$ for all $a\in G$. Hence, there exists $l\leq k$ such that $a^{k+1}=a^l$ hence $a^{k+1-l}=1$ i.e. a has order bounded by k. □

Corollary 2.3. *Let G be an ω-categorical p-group. Then G is locally nilpotent.*

Proof. By Theorem 1.1, any finitely generated subgroup is a finite p-group, and therefore nilpotent. □

Lemma 2.4. *Let G be an ω-categorical locally nilpotent group, then G is uniformly locally nilpotent: there exists a function $g:\mathbb{N}\to\mathbb{N}$ such that for all $a_1,\dots,a_n$, the group $\langle a_1,\dots,a_n\rangle$ is nilpotent of class bounded by $f(n)$.*

Proof. Fix $n\in\mathbb{N}$ and let $W(x_1,\dots,x_n)$ be the set of words in variables $(x_1,\dots,x_n)$. For each $k\in\mathbb{N}$, define $D_k\subseteq G^n$ to be the set

$$\{(a_1,\dots,a_n)\mid [w_1(\vec{a}),\dots,w_k(\vec{a})]=1,\ \text{for all } w_1,\dots,w_k\in W\}$$

A tuple $(a_1,\dots,a_n)$ belongs to D_k if and only if $\langle\vec{a}\rangle$ is nilpotent of class bounded by $k+1$. By local nilpotency of G, for any $\vec{a}=(a_1,\dots,a_n)$, there is c such that $\vec{a}\in D_{c+1}$. In other words, we have:

$$G^n=\bigcup_{k\in\mathbb{N}}D_k$$

Each D_k is closed under $\mathrm{Aut}(G)$ and therefore, is a union of orbits. As such there are only finitely many distinct D_k's, hence for some $k_1,\dots,k_s$, we have

$$G^n=D_{k_1}\cup\ldots\cup D_{k_s}.$$

Taking $g(n)$ to be the maximal of $k_1,\dots,k_s$, we have that any tuple $\vec{a}=(a_1,\dots,a_n)$ generates a group of nilpotency class bounded by $g(n)$. □

The sets D_k are in fact not only type-definable but indeed definable, by Theorem 1.1. We do not use that fact in the proof. We may now conclude this section with:

Corollary 2.5. *Let G be an ω-categorical group. The following statements are equivalent.*

(1) G is locally nilpotent.
(2) G is uniformly locally nilpotent.
(3) G is n-Engel, for some $n \in \mathbb{N}$.

Proof. (1) implies (2) follows from Lemma 2.4. (2) implies (3) is trivial. (3) implies (1): a finitely generated subgroup H of an ω-categorical n-Engel group is finite by Theorem 1.1, and clearly again n-Engel. Therefore, it is nilpotent by Zorn's Theorem. □

2.2. **Examples.** We now get into the study of particular example of ω-categorical groups.

Example 2.6 (Elementary abelian groups)**.** Let G be any infinite elementary abelian group, that is a group of exponent, for some prime p. Then G is ω-categorical. The easiest way to check this is to observe that an elementary abelian group is exactly the same notion as an $\mathbb{F}_p$-vector space. Then any two countable infinite $\mathbb{F}_p$-vector space have a countable basis, therefore, any bijection between the bases extends to an isomorphism. Another way of proving that G is ω-categorical is by using Fraïssé theory.

In fact, more is true.

Example 2.7 (Abelian groups of bounded exponent)**.** Any abelian group of bounded exponent is ω-categorical. In order to prove this, one needs something called the *Smielew Invariants*, which we will not cover here in details. The idea of the proof is as follows. For each $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$ and prime p, there is a set of sentences $\Theta^k_{p,n}$ which expresses the fact that, if A is any abelian group, $A \vDash \Theta^k_{p,n}$ if and only if $\mathbb{Z}/p^n\mathbb{Z}$ occurs exactly k times in the p-torsion factor of A. Now for a given abelian group A of bounded exponent, it is a torsion group and each torsion factor will be entirely determined by $\Theta^k_{p,n}$, which are subsets of $\mathrm{Th}(A)$. For any other abelian group B which satisfies $\mathrm{Th}(A)$ will have the same number of factors of the form $\mathbb{Z}/p^n\mathbb{Z}$, hence the p-torsion of A and B will be isomorphic. Since the p-torsion factors are in direct sum, we will indeed have A and B isomorphic.

Example 2.8 (Generic c-nilpotent groups)**.** Fix $c \in \mathbb{N}$. A *Lazard group G* is a group with a distinguished *Lazard series of length c*, that is a sequence of subgroups

$$G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_{c+1} = 1$$

satisfying $[G_i, G_j] \subseteq G_{max(i+j,c+1)}$, for all $1 \le i, j \le c+1$. Let $\mathscr{L}_c$ be the language of groups expanded by predicates $P_1, \ldots, P_{c+1}$. Then, for all prime number p and $c \in \mathbb{N}$, we define the class $C_{c,p}$ of all finite Lazard groups of exponent p in the language $\mathscr{L}_c$. Then, it is a nontrivial theorem that, for $c < p$, the class $C_{c,p}$ is a Fraïssé class and therefore, the limit $\mathbb{G}_{c,p}$ is an ω-categorical structure as a Lazard group in the language $\mathscr{L}_c$. Then, the Lazard predicates are in fact definable in the group structure (and coincide with the lower central series $(\gamma_i)_{1 \le i \le c+1}$.

Example 2.9 (Boolean Power of Finite Groups)**.** Let F be any finite group, and $B = 2^\omega$ be the Cantor space, i.e. the set of sequences of 0's and 1's, indexed by ω. We recall some basic properties of the topological space B:

- B has a basis of basic clopen sets (=balls) given by "having the same prefix": given a fixed sequence $x_0, \ldots, x_s$ of 0's and 1's, such a basic clopen is the set of all elements of 2^ω which starts with $x_0^\frown \cdots ^\frown x_s$.
- Compactness of B also gives that every clopen is a finite union of balls.
- For any two partitions $(X_1, \ldots, X_s)$ and $(Y_1, \ldots, Y_s)$ of B into clopens, there is a homeomorphism of B sending X_i to Y_i.

We denote by $B[F]$ the set of all continuous functions $B \to F$, where F is given the trivial topology. Given $f \in B[F]$ and $a \in F$, since $\{a\}$ is a clopen of F, $f^{-1}(a)$ is also a clopen of B hence it is a finite union of balls. Therefore,

$$\bigcup_{a \in F} f^{-1}(a) = B$$

can be refined into a partition of B into at most $|F|$ many clopen sets, where each is a finite union of balls. We equip the set $B[F]$ with the following product: for $f, g \in B[F]$, we define $fg \in B[F]$ to be the map defined by $fg(a) = f(a)g(a)$ for all $a \in F$. It is easy to check that this define a group law on $B[F]$. We call $B[F]$ the *boolean power of F*. Then:

• $B[F]$ *is countable.* Every element of $B[F]$ corresponds to a partition of B in at most $|F|$ clopen sets. Therefore, it is enough to prove that the number of clopens is countable, which is clear, are those are finite unions of balls, which are in bijections with possible prefixes, which are in countable numbers.

• $B[F]$ *is ω-categorical.* We will prove that there are only finitely many orbits in the action of $\mathrm{Aut}(B[F])$ on $B[F]$, but the argument is similar in the general case. Let $m = |F|$ and let $X_1, \ldots, X_m$ be a partition of B in to m

clopens. Let H be the subgroup of $B[F]$ of elements $f \in B[F]$ such that $f \restriction X_i$ is constant, for all $i = 1, \ldots, m$. Then H is isomorphic to F^m, via the map $f \mapsto (f(X_1), \ldots, f(X_m))$. Let $g \in B[F]$ and set $Y_a = g^{-1}(a)$, for each $a \in F$. Now refine the familly $(Y_a)_{a \in F}$ into a partition $Y_1 \sqcup \ldots \sqcup Y_m$ of length m of $\bigcup_{a \in F} Y_a = B$. Let σ be an homeomorphism of B which maps X_i to Y_i. Define $\tilde{\sigma} : B[F] \to B[F]$ to map $h \in B[F]$ to the map $(u \mapsto h(\sigma^{-1}(u)))$. Then, $\tilde{\sigma}$ is an automorphism of $B[F]$, and $\tilde{\sigma}(g) \in H$ by definition. So every element of $B[F]$ is in the orbit of an element of H, so the number of orbits is at most $|H| = m^m$.

Another variant of the construction is given by $B^-[F]$, which consists in elements of $B[F]$ which fix a given element $x_0 \in B$. It is easy to check that $B^-[F]$ is a subgroup of $B[F]$, and it is again ω-categorical.

2.3. The Apps-Wilson analysis and the Wilson Conjecture.

Definition 2.10. • A subgroup H of a group G is called *characteristic* if H is closed under the action of $\mathrm{Aut}(G)$, i.e. $H^\sigma = H$ for all $\sigma \in \mathrm{Aut}(G)$.

• A group G is called *characteristically simple* if it admits no nontrivial characteristic subgroups.

Example 2.11. The center $Z(G)$ of a group, or any member of the lower central series γ_k is always characteristic.

An elementary abelian group is always characteristic simple. (Assume that $H \subseteq G$ is a nontrivial characteristic subgroup of an elementary abelian group G, then there is $a \in H$ nontrivial and $b \in G \setminus H$. By extending a to a basis of G and b tp a basis of G, we see that the map $b \mapsto b$ can always be extended to an automorphism of G. But as H is assumed to be characteristic, we have $a^\sigma = b \in H$, contradicting our hypotheses.)

Lemma 2.12. *Let G be an ω-categorical group, and H a characteristic subgroup. Then H and G/H are also ω-categorical.*

Proof. Any automorphism of G induces an automorphism of H by restriction, hence any orbit $O \subseteq H^n$ under $\mathrm{Aut}(H)$ is closed under the action of $\mathrm{Aut}(G)$, hence O is a union of $\mathrm{Aut}(G)$-orbits, which are in finite numbers, hence the number of orbits under $\mathrm{Aut}(H)$ is finite. Let us now turn to G/H. A first observation is that if H is characteristic in G, then it is normal in G, so that the quotient G/H is meaningful. Consider now the component-wise canonical projection $\pi : G^n \to (G/H)^n$. Let $O \subseteq (G/H)^n$ be an orbit under $\mathrm{Aut}(G/H)$.

Claim 1. $\pi^{-1}(O)$ is closed under the action of $\mathrm{Aut}(G)$.

Proof of the claim. Any automorphism σ of G induces an automorphism of G/H via $\tilde{\sigma}(gH) = g^\sigma H^\sigma = g^\sigma H$, because $H^\sigma = H$. This extends component-wise via $(g_1H, \ldots, g_nH)^\sigma = (g_1^\sigma H, \ldots, g_n^\sigma H)$. In particular, if $(a_1H, \ldots, a_nH)$ is in an orbit O of $(G/H)^n$, then for any $\sigma \in \mathrm{Aut}(G)$ and $(a_1, \ldots, a_n) \in \pi^{-1}(O)$, we have $(a_1H, \ldots, a_nH) \in O$ hence $(a_1^\sigma H, \ldots, a_n^\sigma H) \in O$ so $(a_1^\sigma, \ldots, a_n^\sigma) \in \pi^{-1}(O)$. □

By ω-categoricity and Claim 1 ,there is only a finite number of such O, so G/H is ω-categorical. □

Theorem 2.13. **??** *Let G be an ω-categorical group. Then there exists subgroups $G_1, \ldots, G_{s+1}$ of G with*

$$G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_{s+1} = 1$$

such that

(1) *each G_i is characteristic in G, in particular ω-categorical;*
(2) *G_i/G_{i+1} is characteristically simple and ω-categorical.*

Proof. First, observe that every characteristic subgroup of G is a union of orbits under the action of $\mathrm{Aut}(G)$, therefore, G admits only finitely many characteristic subgroup. This imply that there exists a chain $G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_{s+1} = 1$ of maximal length (i.e. such that s is maximal such). By Lemma 2.12, we only need to prove that G_i/G_{i+1} is characteristically simple. Let K be a subgroup G_i/G_{i+1} which is characteristic in G_i/G_{i+1}. Then for $\pi : G_i \to G_i/G_{i+1}$, the subgroup $H = \pi^{-1}(K)$ of G_i is such that $K = H/G_{i+1}$. By Claim 1, H because K is closed under the action of $\mathrm{Aut}(G_i/G_{i+1})$, H is closed under the action of $\mathrm{Aut}(G_i)$, and in particular it is also closed under the action of $\mathrm{Aut}(G)$, so H is characteristic in G. By maximality of the chain $(G_i)_i$, we have either $H = G_i$ or $H = G_{i+1}$ and we conclude that G_i/G_{i+1} is characteristically simple. □

The following was proved by Apps and Wilson, it describes the possibilities for the quotients G_i/G_{i+1}.

Theorem 2.14 (Apps-Wilson analysis, 1981)**.** *Let G be a countably infinite characteristically simple ω-categorical group. Then, one of the following holds:*

(1) *G is elementary abelian*
(2) *$G \cong B[S]$ or $G \cong B^-[S]$ for some finite simple group S.*
(3) *G is a non-abelian p-group.*

This theorem is rather non-trivial, we will not include the proof. The biggest mystery of the study of ω-categorical groups is the third case, which is conjectured to be vacuitous.

Conjecture 2.15 (Wilson, 1981, v1)**.** *Every ω-categorical characteristically simple p-group is abelian.*

An (a priori) more general statement of the conjecture is as follows.

Conjecture 2.16 (Wilson, 1981, v2)**.** *Every ω-categorical n-Engel group is nilpotent.*

Proof that $v2$ implies $v1$. Start with an ω-categorical characteristically simple p-group G. By Corollary 2.3, G is locally nilpotent and by Corollary 2.5, G is n-Engel for some n. Using $v2$, G is nilpotent, so there is c such that $\gamma_{c+1} = 1$. Because G is characteristically simple, this implies that $G = 1$. □

We will in fact see that $v1$ and $v2$ are equivalent statements. There are very few positive results in the direction of the this conjecture of Wilson. The conjecture has been checked with extra model-theoretic assumption (e.g. NSOP [3]) but in general it is open. Note that it is believed by a few specialists that the conjecture might be false.

3. Omega-categorical Lie and associative algebras

We will now try to expand a little but the scope of the conjecture of Wilson.

Definition 3.1. An *algebra* $(V,+,0,*)$ *over a field* $\mathbb{F}$ is an $\mathbb{F}$-vector space $(V,+)$ equipped with a bilinear map $(x,y) \mapsto x*y$, that is, which satisfies

$$(\lambda x+\mu y)*z = \lambda x*z+\mu y*z \qquad z*(\lambda x+\mu y) = \lambda z*x+\mu z*y$$

• An algebra $(A,+,\cdot)$ is *associative* if for all $a,b,c \in A$ we have $(ab)c = a(bc)$. A is called *of bounded nilexponent* if there exists $n \in \mathbb{N}$ such that $a^n = 0$ for all $a \in A$. It is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $a_1 \cdots a_n = 0$ for all $a_1,\dots,a_n \in A$.

• A *Lie algebra* is an algebra $(L,+,0,[.,.])$ in which the bilinear map $(x,y) \mapsto [x,y]$ satisfies $[x,x] = 0$ as well as the Jacobi identity

$$[x,y,z]+[y,z,x]+[z,x,y] = 0.$$

Note that as in groups, we use the left-normed conventions for the Lie bracket: $[a_1,\dots,a_n] = [[a_1,\dots,a_{n-1}],a_n]$ and $[a,b^{(n)}] := [a,b,\dots,b]$ (n times). As for groups, a Lie algebra L is called *n-Engel* if $[a,b^{(n)}] = 0$ for all $a,b \in L$. It is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $[a_1 \cdots a_n] = 0$ for all $a_1,\dots,a_n \in L$.

The statement $V2$ of the Wilson conjecture readily translates as questions in the categories of associative and Lie algebras:

Question 1. Is every ω-categorical n-Engel Lie algebra nilpotent?
Question 2. Is every ω-categorical associative algebra of bounded nilexponent nilpotent?
While the answer to Question 1 is still unknown, the answer to Question 2 is an old theorem of Cherlin.

3.1. Cherlin's Theorem.

In two beautiful papers [1, 2], Cherlin proved that the answer to Question 2 is positive.

Theorem 3.2 (Cherlin, 1980)**.** *Every ω-categorical associative algebra of bounded nilexponent is nilpotent.*

Th proof of this result is highly nontrivial. It is even more general, as Cherlin proves it for ω-categorical *nilrings*. We will only prove the commutative case, which is already quite a tour de force.

We start with a lemma.

Lemma 3.3. *Let A be an associative and commutative algebra generated by a set $S \subseteq A$ such that $a^2 = 0$ for all $a \in S$. If the action of $\mathrm{Aut}(A)$ on $A^3 = A \times A \times A$ has only finitely many orbits, then A is nilpotent.*

Proof. Consider the sets $C_k \subseteq A^3$ defined by

$$(a_1,a_2,a_3) \in C_k \iff \exists x_1 \dots x_k \in A \ (a_1x_1 = a_2a_3 \wedge \bigwedge_{i=1}^{k-1} x_{i+1}a_1 = x_ia_2)$$

and the set $D_k \subseteq A^3$:

$$(a_1,a_2,a_3) \in D_k \iff \exists y_1 \dots y_{k+1} \in A \ (a_2a_3y_1 \neq 0 \wedge a_1y_{k+1} = 0 \wedge \bigwedge_{i=1}^{k} a_1y_i = a_2y_{i+1})$$

Let $E_k = D_k \cap E_k$.

• *Step 1.* If $k \neq l$, then $E_k \cap E_l = \emptyset$. This follows from the fact that if $k < l$, then $\vec{a} = (a_1,a_2,a_3) \in D_k$ implies $\vec{a} \notin C_l$. Indeed, otherwise there exists $(a_1,a_2,a_3) \in D_k \cap C_l$ and we have witnesses $x_1,\dots,x_l \in A$ of C_l and $y_1 \dots, y_{k+1}$ of D_k. As $l > k$, there exists at least $x_1,\dots,x_{k+1}$. Using the definition of D_k, we have

$$0 \neq a_2a_3y_1$$

By the definition of C_k, $a_2a_3 = x_1a_1$ hence:

$$0 \neq (a_2a_3)y_1 = (x_1a_1)y_1$$

By associativity, we have $(x_1a_1)y_1 = x_1(a_1y_1)$ and by the definition of D_l, we have $a_1y_1 = a_2y_2$, so

$$0 \neq (a_2a_3)y_1 = (x_1a_1)y_1 = x_1(a_2y_2)$$

Again, $x_1(a_2y_2) = (x_1a_2)y_2$ hence by the definition of C_l we have $x_1a_2 = x_2a_1$ so that we get

$$0 \neq (a_2a_3)y_1 = (x_1a_1)y_1 = x_1(a_2y_2) = (x_2a_1)y_2$$

Keeping up the process, because we have witnesses $x_1,\dots,x_{k+1}$ and $y_1,\dots,y_{k+1}$, we can reach the point where we obtain

$$0 \neq (a_2a_3)y_1 = (x_1a_1)y_1 = x_1(a_2y_2) = (x_2a_1)y_2 = \dots (x_ka_2)y_{k+1} = (x_{k+1}a_1)y_{k+1}$$

Now $(x_{k+1}a_1)y_{k+1} = x_{k+1}(a_1y_{k+1}) = 0$ by the definition of D_k, so we reach a contradiction.

• *Step 2.* A is nilpotent. Otherwise, there exists arbitrarily long nonzero products in A, which implies that there are arbitrary long nonzero products of elements of S. In particular there is an arbitrarily long odd nonzero product

$$u_1 \cdots u_{2k+1} \neq 0$$

We prove that this implies that E_k is nonempty. Let

- $a_1 = \sum_{i=1}^{k} u_iu_{k+i} + u_1u_{k+1}u_{2k+1}$,
- $a_2 = \sum_{i=1}^{k-1} u_iu_{k+i+1} + u_ku_{k+1}$,
- $a_3 = u_1 \cdots u_{k-1}u_{2k+1}$.

Claim 2. $(a_1,a_2,a_3) \in E_k = C_k \cap D_k$ for the following witnesses:

$$\begin{aligned} x_i &= u_1 \cdots \hat{u}_i \cdots u_k \quad (1 \leq i \leq k) \\ y_1 &= u_{k+2} \cdots u_{2k} \\ y_i &= u_{k+1} \cdots \widehat{u_{k+i}} \cdots u_{2k+1} \quad (2 \leq i \leq k) \\ y_{k+1} &= u_{k+1} \cdots u_{2k+1} \end{aligned}$$

Proof of the claim. It only consists of checking that the identities hold. For instance, to get $a_2a_3y_1 \neq 0$, we check that $a_2a_3y_1 = u_1 \dots u_{2k+1} \neq 0$. Using commutativity and the fact that $u_i^2 = 0$, we have:

$$a_2a_3 = \left(\sum_{i=1}^{k-1} u_iu_{k+i+1} + u_ku_{k+1} \right) u_1 \cdots u_{k-1}u_{2k+1} = u_1 \cdots u_{k-1}u_ku_{k+1}u_{2k+1}$$

hence $a_2a_3y_1 = u_1 \cdots u_{2k+1} \neq 0$. Let us also check, for instance, that $x_ia_2 = x_{i+1}a_1$. Because x_i is the product of every u_j for $1 \leq j \leq k$ except u_i, the product of x_i with $a_2 = \sum_{i=1}^{k-1} u_iu_{k+i+1} + u_ku_{k+1}$ will results in those monomials which selects monomials of a_2 which do not involve u_i, hence $x_ia_2 = u_1 \cdots \hat{u}_i \cdots u_ku_iu_{k+i+1} = u_1 \cdots u_ku_{k+i+1}$. On the other hand, the product of x_{i+1} and $a_1 = \sum_{i=1}^{k} u_iu_{k+i} + u_1u_{k+1}u_{2k+1}$ will yield $u_1 \cdots \widehat{u_{i+1}} \cdots u_ku_{i+1}u_{k+i+1} = u_1 \cdots u_ku_{k+i+1}$, which yields $x_ia_2 = x_{i+1}a_1$. The rest of the equations are proved similarly. □

Observe that E_k is closed by the action of $\mathrm{Aut}(A)$ on A^3, so by the claim, for each $k \in \mathbb{N}$, there exists a nonzero orbit $O_k \subseteq E_k \subseteq A_3$. By Step 1, those orbite are disjoint, which contradicts the hypotheses on A. □

Proof of the commutative case of Theorem 3.2. Let A be a commutative, ω-categorical associative algebra of bounded nilexponent. Let n be the exponent of A, that is, the smallest $n \in \mathbb{N}$ such that $a^n = 0$ for all $a \in A$. We prove by induction on n that A is nilpotent, so let us assume that every ω-categorical ring of nilexponent n is nilpotent. The initialisation starts with nilexponent 2, which is dealt with by Claim 2. For $n > 2$, consider the group I of A generated by the definable set $\{a \in A \mid a^2 = 0\}$. This set is clearly nonempty (if $a^n = 0$ then $(a^n/2+1)^2 = 0$), hence I is nonempty. Observe that I is an ideal of A. Further, I is closed under automorphisms of A, so that, reasoning as in Lemma 2.12 I and A/I are ω-categorical. By Claim 2, I is nilpotent. By induction hypothesis, A/I is also nilpotent. We conclude that A is also nilpotent.

□

3.2. Wilson's Theorem.

This result of Cherlin allows to deduce a fundamental theorem of Wilson on ω-categorical groups, which states as follows.

Corollary 3.4 (Wilson, 1981)**.** *Let G be an ω-categorical n-Engel group. Then G is solvable if and only if G is nilpotent.*

We will not see the proof, as it involves a notion we did not encounter yet: *interpretations*. But we may use it to deduce that $v2$ and $v1$ of the Wilson Conjecture are equivalent.

Proof that v1 implies v2. If G is any n-Engel ω-categorical characteristically simple group G. By Theorem **??**, there exists $G = G_1 \supseteq G_2 \supseteq \ldots \supsetneq G_{s+1} = 1$ such that each quotient G_i/G_{i+1} is characteristically simple. As G is n-Engel, each G_i/G_{i+1} is locally nilpotent, hence it cannot be of the form $B[S]$ not $B^-[S]$, hence each quotient is a characteristically simple p-group. Assuming $v1$, we conclude that each quotient G_i/G_{i+1} is nilpotent. By an immediate induction, we conclude that G is solvable, and hence also nilpotent by Corollary 3.4. □

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