GENERIC IMAGINARY SORT, NSOP$_1$ AND FRANK OLAF WAGNER

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1. Generic imaginary sort

Let $\mathcal{L}$ be a language and let T be any model-complete $\mathcal{L}$-theory which eliminates $\exists^\infty$. Let $\mathcal{L}' = \mathcal{L} \cup \{P, f\}$ and T' be the $\mathcal{L}'$-theory whose models are two sorted $(M, P, f : M \to P)$, consisting of a model M of T in one sort, a second sort P without structure, and a unary function $f : M \to P$. In the sort M (the *home* sort) tuples of variables will be denoted $x, y, z, ...$ and tuples of elements $a, b, c, ...$ In the sort P (the *imaginary* sort) tuples of variables will be denoted $X, Y, Z, ...$ and tuples of elements $\alpha, \beta, ...$ For tuples a, x, α, X we denote the coordinates a_i, x_i, α_i, X_i.

Fact 1.1 (Lemma 2.3 [CP998] or Lemma 6.3.2 [Wag00])**.** *As T eliminates $\exists^\infty$, given any $\mathcal{L}$-formula $\varphi(x, y)$ there exists an $\mathcal{L}$-formula $\delta(y)$ such that in an ω-saturated model $M \models T$ we have $M \models \theta(b)$ if and only if there exists $a \in \varphi(M, b)$ such that $a \cap \mathrm{acl}_T(b) = \emptyset$ and $\bigwedge_{i\neq j} a_i \neq a_j$. We denote this formula $\exists^{na} x\varphi(x, y)$.*

Theorem 1.2. *T' has a model-companion T^o, axiomatized by the following axiom-scheme. For each $\mathcal{L}$-formula $\varphi(x, y)$:*

$$\forall y[(\exists^{na} x\varphi(x, y)) \to (\forall X_1, \dots, X_n \exists x\varphi(x, y) \wedge \bigwedge_i f(x_i) = X_i)].$$

Proof. Any existential formula in y, Y is of the following form:

$$\exists x\exists X\varphi(x, y) \wedge \bigwedge_{i,j} t_i(x, y, X, Y) = t_j(x, y, X, Y) \wedge \bigwedge_{k,l} t_k(x, y, X, Y) \neq t_l(x, y, X, Y)$$

where $\varphi(x, y)$ is an $\mathcal{L}$ quantifier-free formula, and t_i, t_j, t_k, t_l are $\mathcal{L}'$-terms. Observe that $\mathcal{L}'$-terms are either $\mathcal{L}$-terms, P-variables $X_i, Y_j, ..,$ or of the form $f(t(x))$, for t an $\mathcal{L}$-terms. In particular there are no new terms in the home sort. By adding conditions of the form $\exists z\ z = t(x, y)$, it is enough to consider formulas of the form

$$\exists x\exists X\varphi(x, y)\wedge \bigwedge_{i,j,k} f(x_i) = t_i(X, Y)\wedge f(x_j) = f(x_k)\wedge \bigwedge_{l,m,n} f(x_l) \neq t_l(X, Y)\wedge f(x_m) \neq f(x_n)$$

for some existential $\mathcal{L}$-formula $\varphi(x, y)$, and $t_i(X, Y)$ some coordinate function (i.e. $t_i(X, Y) = X_k$ or Y_l). Conditions of the form $f(x_j) = f(x_k)$ are equivalent to $\exists X X = f(x_j) \wedge X = f(x_k)$, hence we may only consider formulas of the form

$$\exists x\exists X\varphi(x, y) \wedge \bigwedge_i f(x_i) = t_i(X, Y) \wedge \bigwedge_j f(x_j) \neq t_j(X, Y)$$

for $t_i(X, Y)$ some coordinate function.

We show that $(M, P, f) \models T^o$ if and only if (M, P) is an existentially closed model of T'. Let $(M, P, f) \models T^o$ and (N, Q, g) be an expansion of (M, P, f). Assume that $\psi(b, \beta)$ is an existential formula satisfied in (N, Q, g), for $b\beta$ a tuple from MP. Then there exists $a\alpha \in N$ such that

$$\varphi(a, b) \wedge \bigwedge_{i\in I} f(a_i) = t_i(\alpha, \beta) \wedge \bigwedge_{j\in J} f(a_j) \neq t_j(\alpha, \beta).$$

We may assume that $\{a_i \mid i \in I\} \supseteq \{a_j \mid j \in J\}$, simply by adding one equations $f(a_j) = \gamma_j$ (for each value of a_j, $j \in J$) in the formula and increasing the tuple $\alpha\beta$ by $(\gamma_j)_{j\in J}$. We may assume that $a\alpha \cap M = \emptyset$. We may also assume that for all $i, k \in I$ we have $a_i \neq a_k$. It follows that for all $i \in I$, if $j \in J$ is such that $a_i = a_j$, then $t_i(\alpha, \beta) \neq t_j(\alpha, \beta)$, so $\forall x(f(x) = t_i(\alpha, \beta) \to f(x) \neq t_j(\alpha, \beta))$.

First, there exists $\alpha' \subseteq P$ such that for all $i, j \in I \cup J$, we have $t_j(\alpha', \beta) = t_i(\alpha', \beta)$ if and only if $t_j(\alpha, \beta) = t_i(\alpha, \beta)$.

By the axioms, there exists a' such that $\varphi(a', b)$ and $\bigwedge_{i\in I} f(a'_i) = t_i(\alpha', \beta)$. For all $j \in J$, we have $a'_i = a'_j$ for some $i \in I$, hence as $t_j(\alpha', \beta) \neq t_i(\alpha', \beta)$ and $f(a'_i) = t_i(\alpha', \beta)$, we have $f(a_j) \neq t_j(\alpha', \beta)$. So $\psi(b, \beta)$ holds in (M, P, f).

Conversely, assume that (M, P, f) is existentially closed, and let $\varphi(x, y)$ be any $\mathcal{L}$-formula. Assume that for some $b \in M$ we have $\theta_\varphi(b)$. Hence there exists an elementary extension $N \succ M$ and $a \in N \setminus M$ such that $\varphi(a, b)$ and $a_i \neq a_j$ for all $i \neq j$. Let $\alpha_1, \ldots, \alpha_n \in P$, for some $n \leq |a|$. Let c be some fixed element of M, define

$$f' = f \cup \{(a_i, \alpha_i) \mid i = 1, \ldots, n\} \cup \{(d, c) \mid d \in N \setminus M \cup \{a_1, \ldots, a_n\}\}.$$

Then (N, P, f') is an extension of (M, P, f), so by hypothesis, $(M, P, f) \models \exists x \varphi(x, b) \wedge \bigwedge_i f(x_i) = \alpha_i$. □

Remark 1.3. Suppose T is the theory of torsion-free divisible abelian groups. Let $\varphi(x; y, z)$ be the $\mathcal{L}'$-formula $f(x+y) = z$. We show that $\varphi(x; y, z)$ has TP_2 in T^o. It suffices to fix $n \geq 1$ and build an $n \times n$ TP_2 array for φ with 2-inconsistency along rows. Let M be a model of T. Fix pairwise distinct elements $b_1, \ldots, b_n \in M$. We can then find elements $(a_\sigma)_{\sigma \in [n]^{[n]}}$ from M such that $a_\sigma + b_i \neq a_\tau + b_j$ whenever $\sigma(i) \neq \tau(j)$. (Enumerate $[n]^{[n]}$ and select each a_σ inductively.) We expand M to an $\mathcal{L}'$-structure M' as follows. Let $P(M')$ consist of points $c_1, \ldots, c_n$. Let f be defined so that $f(a_\sigma + b_i) = c_{\sigma(i)}$ for all $i \in [n]$ and $\sigma[n] \to [n]$ (and f does whatever on the rest of the elements of M). Since M' is a model of T', we may assume that M' is substructure of the monster model $(\mathcal{U}, P, f)$ of T^o. For any $\sigma\colon [n] \to [n]$, the set $\{\varphi(x; b_i, c_{i,\sigma(i)}) : i \in [n]\}$ is consistent since it is realized by a_σ. On the other hand, for any fixed $i \in [n]$, the set $\{\varphi(x; b_i, c_j) : j \in [n]\}$ is 2-inconsistent since $c_j \neq c_k$ for all $j \neq k$. Combinatorially, the argument is similar to the proof of TP_2 for the model completion of the empty $\mathcal{L}$-theory whenever $\mathcal{L}$ is a language with at least one binary function symbol ([KR18, Proposition 3.14.]). Note that the proof would work whenever T has a definable binary function h that is sufficiently non-degenerate (e.g., it sufficient for T to prove that there are arbitrarily large finite sets A and B such that h is injective on $A \times B$).

2. An unnoticed early contribution of Wagner to NSOP_1 theories

NSOP_1 theories where introduced by Dzamonja and Shelah in 2004. A first Kim-Pillay style theorem for NSOP_1 theories appeared in 2016 [CR16] (followed later by [KR20, DK21], and other results around Kim-independence). In [CR16], Chernikov and Ramsey used their result to prove that earlier studied theories with a well-behaved independence relation were NSOP_1. The main examples for this are the work of Chatzidakis on ω-free PAC and the work of Granger on generic bilinear forms. However, it was not noticed that in 2000, Wagner had studied what should be seen as a weak form of interpolative fusion [KTW21] (the case where the languages intersects in equality), and proved that essentially, the fusion of two simple theories is NSOP_1. We use this result to prove that T^o is NSOP_1, although we could have used the machinery of interpolative fusion to do so.

Wagner proves the following [Wag00, Theorem 6.3.4, Theorem 6.3.9, Theorem 6.3.10].

{fact_wagner_nsop} **Fact 2.1.** *(Wagner) Let T_1 and T_2 be two model-complete theories in relational languages $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, with $\mathcal{L}_1 \cap \mathcal{L}_2 = \{=\}$. Suppose that both theories eliminates $\exists^\infty$. Then*

(1) $T_1 \cup T_2$ has a model-companion;
(2) $T_1 \cup T_2$ has elimination of $\exists^\infty$ and the algebraic closure $\mathrm{acl}(.)$ is the transitive closure of the algebraic closures in the sense of T_1 and of T_2.

Assume that T_1 and T_2 are simple, then the model companion of $T_1 \cup T_2$ is NSOP$_1$. More precisely, let $\mathcal{U} \models T_1 \cup T_2$ be a saturated existentially closed model, and $A, B, C \subseteq \mathcal{U}$. Let $\mathop{\smile\hskip-0.9em^{|}}$ be the ternary relation defined by $A \mathop{\smile\hskip-0.9em^{|}}_C B$ if $\mathrm{acl}(AC)$ and $\mathrm{acl}(BC)$ are forking independent over $\mathrm{acl}(C)$ in the sense of both T_1 and T_2. Then $\mathop{\smile\hskip-0.9em^{|}}$ is forking independence over models.

Proof. [Wag00, Theorem 6.3.9, Theorem 6.3.10] yields that $\mathop{\smile\hskip-0.9em^{|}}$ satisfies the following properties: invariance, extension, symmetry, finite character, local character, transitivity (Wagner calls it partial transitivity: if $A \subseteq B \subseteq C$ and $a \mathop{\smile\hskip-0.9em^{|}}_A B$ and $a \mathop{\smile\hskip-0.9em^{|}}_B C$ then $a \mathop{\smile\hskip-0.9em^{|}}_A C$) and the independence theorem over models. We will use the following recent version of [CR16, KR20], proved in [DK21] in the broader context of positive logic: {fact:KPnsop1}

Fact 2.2 (Chernikov-Ramsey; Kaplan-Ramsey; Dobrowolski-Kamsma)**.** *A complete theory T is* NSOP$_1$ *if and only if there is an invariant ternary relation $\mathop{\smile\hskip-0.9em^{|}}$ on small subsets of $\mathcal{U}$, which satisfies symmetry over models, existence over models, finite character over models, monotonicity over models, transitivity over models, extension over models, independence theorem over models, and*
Chain local character: *let a be a finite tuple and $\kappa > |T|$ a regular cardinal. For every continuous chain $(M_i)_{i<\kappa}$ of models with $|M_i| < \kappa$ for all $i < \kappa$ and $M = \bigcup_{i<\kappa} M_i$, there is $j < \kappa$ such that $a \mathop{\smile\hskip-0.9em^{|}}_{M_j} M$.*
Moreover, in this case $\mathop{\smile\hskip-0.9em^{|}}$ is Kim-independence over models.

Four things have to be checked in order to apply Fact 2.2:

(1) Monotonicity ($A \mathop{\smile\hskip-0.9em^{|}}_C BD \implies A \mathop{\smile\hskip-0.9em^{|}}_C B$) is not proved in [Wag00];
(2) Existence ($A \mathop{\smile\hskip-0.9em^{|}}_C C$ for all A, C) is not proved in [Wag00];
(3) Chain local character is not proved in [Wag00];
(4) The extension property in [Wag00] (for all A, B, C there exists $A' \equiv_C A$ such that $A' \mathop{\smile\hskip-0.9em^{|}}_C B$) is not the same extension as the one in Fact 2.2 (if $A \mathop{\smile\hskip-0.9em^{|}}_C B$ and D is given, then there exists $A' \equiv_{BC} A$ such that $A' \mathop{\smile\hskip-0.9em^{|}}_C BD$).

Concerning (2) and (4), the version of extension proved by Wagner implies both existence and extension in the sense of Fact 2.2 (this uses invariance and (partial) transitivity). For (1), assume that $A \mathop{\smile\hskip-0.9em^{|}}_C BD$, which means that $\mathrm{acl}(AC)$ and $\mathrm{acl}(BCD)$ are forking independent over $\mathrm{acl}(C)$ in the sense of both T_1 and T_2. From monotonicity for forking independence in T_1 and T_2, as $\mathrm{acl}(BC) \subseteq \mathrm{acl}(BCD)$, we get $A \mathop{\smile\hskip-0.9em^{|}}_C B$ trivially. Concerning (3), Chain local character easily follows from local character. □

Remark 2.3. A small erratum concerning section *6.3.1 Amalgamating Simple Theories* in [Wag00].

- Proposition 6.3.13 and Proposition 6.3.15 should read $T_1 \cup T_2$ and not $T_1 \cap T_2$.

Remark 2.4. Proposition 6.3.9 and 6.3.10 [Wag00] does not include a proof of the property monotonicity, which is trivial. This is because as it is enounced, transitivity implies it (together with what is now called base monotonicity):

$$a \mathop{\smile\hskip-0.9em^{|}}_A BC \iff a \mathop{\smile\hskip-0.9em^{|}}_A B \text{ and } a \mathop{\smile\hskip-0.9em^{|}}_{AB} C.$$

Let $\mathcal{L}_0 = \{E\}$ where E is a binary relation and let T_0 be the model-companion of the theory of equivalence relations: T_0 is the theory of an equivalence relation with infinitely many infinite classes, T_0 has elimination of $\exists^\infty$.

Corollary 2.5. *Let T be a model-complete theory in a language $\mathcal{L}$ which eliminates $\exists^\infty$. Assume $E \notin \mathcal{L}$. Then $T \cup T_0$ has a model-companion, we call it T_E. If T is simple, then T_E is NSOP$_1$.*

Proof. Apply Fact 2.1. □

Proposition 2.6. *Let T be a model-complete theory in a language $\mathcal{L}$ which eliminates $\exists^\infty$. Then T^o and T_E are bi-interpretable.*

Proof. Define $E(x,y)$ by $f(x) = f(y)$ in one direction. For the other direction consider the quotient by E and define f to be the projection. Bi-interpretability is easy to check. □

Corollary 2.7. *Let T be a model-complete theory in a language $\mathcal{L}$ which eliminates $\exists^\infty$. Then T^o is NSOP$_1$.*

Remark 2.8. It is probably true that if T weakly eliminates imaginaries then so does T^o. Probably assuming T stable would help proving such statement.

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