NOTE ON A BOMB DROPPED BY MR CONANT AND MR KRUCKMAN, AND ITS CONSEQUENCES FOR THE THEORY ACFG

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Introduction

In [3] Conant and Kruckman dropped a bomb on the model theory community by exhibiting a counterexample (actually 3) to a folklore fact which can be stated as follows:

$$A \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{C} B \implies A \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{C} \mathrm{acl}(BC) \qquad (👻)$$

where $\mathop{\smile\!\!\!\!\!\!\mid}^{d}$ is dividing independence. In other words, they proved that it may happen that a formula divides over $\mathrm{acl}(B)$ but not over B. In the setting of [5], they prove that in general $\mathop{\smile\!\!\!\!\!\!\mid}^{d}$ fails right closure, see also Subsection 2.1 for axioms of independence relations. From this, statements such as $\mathop{\smile\!\!\!\!\!\!\mid}^{d} \to \mathop{\smile\!\!\!\!\!\!\mid}^{a\,M}$, where $A \mathop{\smile\!\!\!\!\!\!\mid}^{a\,M}_{C} B$ iff $\mathrm{acl}(AD) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$ for all $C \subseteq D \subseteq \mathrm{acl}(BC)$, are not longer true and this has great and unforeseen consequences on several other papers, see [3, Section 5]. Among the results that has to be revised with the failure of (👻) are several general description of $\mathop{\smile\!\!\!\!\!\!\mid}^{d}$ in strictly NSOP_1 theories. This is also the case for the authors paper [4] on the model-companion of the expansion of a field of fixed positive characteristic by an additive subgroup, otherwise known as ACFG. In particular, it was proved in [4] that $\mathop{\smile\!\!\!\!\!\!\mid}^{f} = \mathop{\smile\!\!\!\!\!\!\mid}^{d}$ and that those are obtained by 'forcing base monotonicity' on Kim-independence (see below for more details). In this note, we start in Section 1 by giving two counterexample to $\mathop{\smile\!\!\!\!\!\!\mid}^{f} = \mathop{\smile\!\!\!\!\!\!\mid}^{d}$ in ACFG. The first one (Subsection 1.2) is really happening in the home sort (using in particular that algebraically closed fields have build-in codes for finite sets). The second one (Subsection 1.3) is closer to the counterexample [3, Section 3.2] and uses that models (K, G) of ACFG interpret a generic (symmetric) binary map, namely $(x, y) \mapsto \pi(xy)$ where $\pi : K \to K/G$ is the canonical projection.

In Section 2, we expand on the subtleties around 'forcing base monotonicity' and how this can be done with or without forcing the closure axiom. Define the following extensions of a given independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ (essentially due to Adler [1, 2]):

$$A \mathop{\smile\!\!\!\!\!\!\mid}^{M}_{C} B :\iff A \mathop{\smile\!\!\!\!\!\!\mid}_{D} B \text{ for all } C \subseteq D \subseteq \mathrm{ac}(BC).$$
$$A \mathop{\smile\!\!\!\!\!\!\mid}^{m}_{C} B :\iff A \mathop{\smile\!\!\!\!\!\!\mid}_{D} B \text{ for all } C \subseteq D \subseteq BC.$$

Until now, the 'monotonisation' usually referred to $\mathop{\smile\!\!\!\!\!\!\mid}^{M}$ and was sometimes denoted $\mathop{\smile\!\!\!\!\!\!\mid}^{m}$ (sic), but we fix the notations above, following Adler and [3]. In this note, $\mathop{\smile\!\!\!\!\!\!\mid}^{m}$ will be referred to as the 'naive' monotonisation.

Caveat. In [4], there is a mistake in Definition 4.1, where the monotonisation $\mathop{\smile\!\!\!\!\!\!\mid}^{m}$ is defined as above but the intended definition and the definition used in the proofs of [4] really is the one of $\mathop{\smile\!\!\!\!\!\!\mid}^{M}$. This is a notationnaly fortunate mistake but makes the problems associated to the failure of (👻) even more confused and this note more necessary. The reader should read every mention of '$\mathop{\smile\!\!\!\!\!\!\mid}^{m}, \mathop{\smile\!\!\!\!\!\!\mid}^{wm}$' in [4] as '$\mathop{\smile\!\!\!\!\!\!\mid}^{M}, \mathop{\smile\!\!\!\!\!\!\mid}^{wM}$'. Note also that $\mathop{\smile\!\!\!\!\!\!\mid}^{M}$ in Adler or in [3] is $(\mathop{\smile\!\!\!\!\!\!\mid}^{a})^{M}$ for us, where $A \mathop{\smile\!\!\!\!\!\!\mid}^{a}_{C} B$ if and only if $\mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$.

Here is a list of updates on the consequences of the failure of (👻) in ACFG.

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- We still have $\mathop{\smile\hskip -0.9em ^| \ }^{f} = \mathop{\smile\hskip -0.9em ^| \ }^{K^M}$, where $\mathop{\smile\hskip -0.9em ^| \ }^{K}$ is Kim-independence, although the statement [4, Proposition 4.14] is now false. We expand on this just below.
- We have $\mathop{\smile\hskip -0.9em ^| \ }^{da} = \mathop{\smile\hskip -0.9em ^| \ }^{f}$, where $\mathop{\smile\hskip -0.9em ^| \ }^{da}$ is defined below. This is similar to [3, Subsection 5.3].
- We have $\mathop{\smile\hskip -0.9em ^| \ }^{d} \to (\mathop{\smile\hskip -0.9em ^| \ }^{a})^M$, simply because we already have $\mathop{\smile\hskip -0.9em ^| \ }^{K} \to (\mathop{\smile\hskip -0.9em ^| \ }^{a})^M$, see Remark 2.20.

One of the main consequences of the failure of (👻) is that if $\mathop{\smile\hskip -0.9em ^| \ }^{d} \to \mathop{\smile\hskip -0.9em ^| \ }$, then one does not necessarily have $\mathop{\smile\hskip -0.9em ^| \ }^{d} \to \mathop{\smile\hskip -0.9em ^| \ }^{M}$ but only $\mathop{\smile\hskip -0.9em ^| \ }^{d} \to \mathop{\smile\hskip -0.9em ^| \ }^{m}$. This simple observation make [4, Proposition 4.14] no longer true and thus the proof of $\mathop{\smile\hskip -0.9em ^| \ }^{f} = \mathop{\smile\hskip -0.9em ^| \ }^{d}$ obsolete. In Section 2 we give a correct account of [4, Proposition 4.14], Corollary 2.10 below which is essentially [4, Proposition 4.14] but replacing $\mathop{\smile\hskip -0.9em ^| \ }^{M}$ by $\mathop{\smile\hskip -0.9em ^| \ }^{m}$. Unfortunately, Corollary 2.10 cannot be applied in ACFG to conclude that $\mathop{\smile\hskip -0.9em ^| \ }^{d} = \mathop{\smile\hskip -0.9em ^| \ }^{f}$. Another variant of a correct statement of [4, Proposition 4.14] can be obtained with the following definition ([3, Subsection 2.4]):

$$A \mathop{\smile\hskip -0.9em ^| \ }^{da}_{C} B :\iff A \mathop{\smile\hskip -0.9em ^| \ }^{d}_{C} \operatorname{acl}(BC).$$

Now, $\mathop{\smile\hskip -0.9em ^| \ }^{d} \to \mathop{\smile\hskip -0.9em ^| \ }$ implies $\mathop{\smile\hskip -0.9em ^| \ }^{da} \to \mathop{\smile\hskip -0.9em ^| \ }^{M}$. By reading the proof of [4, Proposition 4.14], we obtain:

Correct version of [4, Proposition 4.14]**.** Let $\mathop{\smile\hskip -0.9em ^| \ }$ be an invariant relation satisfying left and right monotonicity such that

(1) $\mathop{\smile\hskip -0.9em ^| \ }^{da} \to \mathop{\smile\hskip -0.9em ^| \ }$
(2) $\mathop{\smile\hskip -0.9em ^| \ }$ satisfies $\mathop{\smile\hskip -0.9em ^| \ }^{h}$-amalgamation over models
(3) $\mathop{\smile\hskip -0.9em ^| \ }^{M}$ satisfies extension

then $\mathop{\smile\hskip -0.9em ^| \ }^{M} = \mathop{\smile\hskip -0.9em ^| \ }^{f} = \mathop{\smile\hskip -0.9em ^| \ }^{da}$.

In ACFG, $(\mathop{\smile\hskip -0.9em ^| \ }^{K})^M$ satisfies extension ([4, Corollary 4.18]), and $\mathop{\smile\hskip -0.9em ^| \ }^{K}$ satisfies (1) and (2) hence we conclude:

$$\mathop{\smile\hskip -0.9em ^| \ }^{f} = \mathop{\smile\hskip -0.9em ^| \ }^{da} = (\mathop{\smile\hskip -0.9em ^| \ }^{K})^M.$$

Recall the following notation from Adler, given an independence relation $\mathop{\smile\hskip -0.9em ^| \ }$:

$$A \mathop{\smile\hskip -0.9em ^| \ }^{*}_{C} B \iff \text{for all } D \supseteq B, \text{ there exists } A' \equiv_{BC} A \text{ with } A' \mathop{\smile\hskip -0.9em ^| \ }_{C} D.$$

It is in the current folklore on NSOP_1 theories that $\mathop{\smile\hskip -0.9em ^| \ }^{K^{M*}} = \mathop{\smile\hskip -0.9em ^| \ }^{f}$. We observe (Proposition 2.13) that either version of the monotonisation are enough to get this result:

$$(\mathop{\smile\hskip -0.9em ^| \ }^{K})^{M*} = (\mathop{\smile\hskip -0.9em ^| \ }^{K})^{m*} = \mathop{\smile\hskip -0.9em ^| \ }^{f}$$

In particular in ACFG we have $\mathop{\smile\hskip -0.9em ^| \ }^{K^M} = \mathop{\smile\hskip -0.9em ^| \ }^{K^{m^*}} \neq \mathop{\smile\hskip -0.9em ^| \ }^{K^m}$. The picture of independence relations in ACFG can be summarize as follows:

$$\begin{array}{ccccccccccccc}
\mathop{\smile\hskip -0.9em ^| \ }^{K^{m*}} & & & & & & & & & & & & \\
\| & & & & & & & & & & & & \\
\mathop{\smile\hskip -0.9em ^| \ }^{K^{M}} & & & & & & & & & & & & \\
\| & & & & & & & & & & & & \\
\mathop{\smile\hskip -0.9em ^| \ }^{f} & \longrightarrow & \mathop{\smile\hskip -0.9em ^| \ }^{d} & \overset{=?}{\Longrightarrow} & \mathop{\smile\hskip -0.9em ^| \ }^{K^m} & \longrightarrow & \mathop{\smile\hskip -0.9em ^| \ }^{K} & \longrightarrow & \mathop{\smile\hskip -0.9em ^| \ }^{a\,M} & \longrightarrow & \mathop{\smile\hskip -0.9em ^| \ }^{a} & & \\
\| & & & & & & & & & & & & \\
\mathop{\smile\hskip -0.9em ^| \ }^{da} & & & & & & & & & & & &
\end{array}$$

where every arrow is strict, except maybe the one between $\mathop{\smile\hskip -0.9em ^| \ }^{d}$ and $\mathop{\smile\hskip -0.9em ^| \ }^{K^m}$, for which we do not know.

Notations. We use A^{alg} for the field theoretic algebraic closure and $\mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{alg}}$ for the algebraic independence relation in the sense of fields. We use $A \mathop{\smile\hskip -0.9em ^| \ }^{h}_{C} B$ if and only if $\mathrm{tp}(A/BC)$ is a coheir of $\mathrm{tp}(A/C)$.

1. Forking and dividing differ for types

1.1. **Preliminaries.** Fix a prime number p. Let $\mathscr{L}_G$ be the expansion of the language of rings by a unary predicate G. Let $T_G = T_G(p)$ be the $\mathscr{L}_G$-theory of fields of characteristic p in which G is an additive subgroup. By [4, 6] T_G admits a model-companion, denoted ACFG, which is NSOP_1 and not simple. We recall some facts about the theory ACFG, see [4, 6] for details.

Fact 1.1. *Let (K, G) be a sufficiently saturated model of ACFG, where G is the generic subgroup of $(K, +)$. Let A, A', B, C be small subsets of K. We denote by $G(A)$ the set $G \cap A$.*

(1) $A \equiv_C A'$ if and only if there is an $\mathscr{L}_G$-isomorphism f over C between the two $\mathscr{L}_G$-structures

$$((AC)^{\mathrm{alg}}, G((AC)^{\mathrm{alg}}) \cong ((A'C)^{\mathrm{alg}}, G((A'C)^{\mathrm{alg}}))$$

which maps A to A'.

(2) $A \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{C} B$ if and only if $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{alg}}_{C} B$ and $G((AC)^{\mathrm{alg}} + (BC)^{\mathrm{alg}}) = G((AC)^{\mathrm{alg}}) + G((BC)^{\mathrm{alg}})$.

(3) $A \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{C} B$ if and only if $A \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{D} B$ for all $C \subseteq D \subseteq \mathrm{acl}(BC)$.

See [4, Subsection 2.1] for an explicit description on how to prove that types are consistent in ACFG.

1.2. $\mathop{\smile\!\!\!\!\!\!\mid}^{d} \neq \mathop{\smile\!\!\!\!\!\!\mid}^{f}$ **in the theory ACFG.** Fix a model M of ACFG and a bigger model (K, G) extending M. Let $d_1, d_2 \in K$ be algebraically independent over M with $G((Md_1d_2)^{\mathrm{alg}}) = G(M)$. Let $b = (d_1 + d_2, d_1d_2) \in K^2$, it is classical that the tuple b is a canonical parameter for the set $\{d_1, d_2\}$, in particular, $d_1, d_2 \in (\mathbb{F}_p(b))^{\mathrm{alg}}$ hence $(Md_1d_2)^{\mathrm{alg}} = (Mb)^{\mathrm{alg}}$. Let $a \in K$ be an element such that

$$G((Mab)^{\mathrm{alg}})) = G(M) \oplus \mathrm{Span}_{\mathbb{F}_p}(ad_2 - d_1). \quad (*)$$

We leave to the reader to check that $G((Ma)^{\mathrm{alg}}) = G((Md_1d_2)^{\mathrm{alg}}) = G((Mad_1)^{\mathrm{alg}}) = G((Mad_2)^{\mathrm{alg}} = G(M)$. We claim that $a \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{M} b$ whereas $a \not\mathop{\smile\!\!\!\!\!\!\mid}^{f}_{M} b$.

First, we check that $a \not\mathop{\smile\!\!\!\!\!\!\mid}^{f}_{M} b$. Assume otherwise, then as $d_1, d_2 \in (Mb)^{\mathrm{alg}}$, we have $a \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{Md_2} d_1 b$ using Fact 1.1 (3). Then $ad_2 - d_1 \in G((Mad_2)^{\mathrm{alg}} + (Mb)^{\mathrm{alg}})$ but $ad_2 - d_1 \notin G((Mad_2)^{\mathrm{alg}}) + G((Mb)^{\mathrm{alg}} = G(M)$, contradicting Fact 1.1 (2).

To prove that $a \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{M} b$, we need a small claim.

Claim 1. Let $(d^n)_{n<\omega}$ be an infinite sequence of pairs $d_n = (d_1^n d_2^n)$ which is indiscernible in some theory of $\mathbb{F}_p$-vector space. Then one (and only one) of the three following configurations occur:

(1) $(d_1^n)_{n<\omega}$ and $(d_2^n)_{n<\omega}$ are both infinite and $\{d_1^n, d_2^m \mid n, m < \omega\}$ is $\mathbb{F}_p$-linearly independent
(2) $(d_1^n)_{n<\omega}$ is constant and $\{d_2^n \mid n < \omega\}$ is $\mathbb{F}_p$-linearly independent over d_1^0.
(3) $(d_2^n)_{n<\omega}$ is constant and $\{d_1^n \mid n < \omega\}$ is $\mathbb{F}_p$-linearly independent over d_2^0.

Proof. Assume that the first case does not occur. Then there exists finite sets $I, J \subseteq \omega$ and $(\lambda_i)_{i\in I}$, $(\mu_j)_{j\in J} \subseteq \mathbb{F}_p \setminus \{0\}$ such that

$$\sum_I \lambda_i d_1^i + \sum_J \mu_j d_2^j = 0. \quad (\star)$$

Let $i_0 = \max(I \cup J)$. Assume that $i_0 \in I$. Then $d_1^{i_0} \in V := \mathrm{Span}_{\mathbb{F}_p}(d_1^i, d_2^j \mid i \in I \setminus \{i_0\}, j \in J)$ and by indiscernibility the set $\left\{d_1^k \mid k \geq i_0\right\}$ is included in the finite set V, which implies that $d_1^n = d_1^m$ for all $n \neq m$. In particular we may assume that $I = \{i_0\}$. If $J \neq \emptyset$, let $j_0 = \max J$. Reasoning as above, $(\star)$ implies that $\left\{d_2^k \mid k \geq j_0\right\}$ is included in a translate of $W = \mathrm{Span}_{\mathbb{F}_p}(d_1^{i_0}, d_2^j \mid j \in J \setminus \{j_0\})$ hence by indiscernibility $(d_2^n)_{n<\omega}$ is constant. This contradicts that $(d_1^n, d_2^n)_{n<\omega}$ is infinite hence (2) holds.

If $i_0 \in J$, by a similar argument one conclude that (3) holds. $\square$

To prove that $a \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{C} b$, let $(b_n)_{n<\omega}$ be any infinite indiscernible sequence in $\mathrm{tp}(b/M)$. For each n there is d_1^n, d_2^n such that $b_n = (d_1^n + d_2^n, d_1^n d_2^n)$ and we may assume that $(b_n d_1^n d_2^n)_{n<\omega}$ is indiscernible over M. Note that $(d_1^n, d_2^n)_{n<\omega}$ is infinite. By the claim there are three configurations which reduce to the two following two cases:

- **Case I.** $(d_2^n)_{n<\omega}$ is infinite. This corresponds to configurations (1) and (2) from Claim 1.

- **Case II.** $(d_2^n)_{n<\omega}$ is constant, this is configuration (3) of Claim 1

Case I. Assume that we are in the configuration (2), i.e. $(d_1^n)_{n<\omega}$ is constant and $(d_2^n)_{n<\omega}$ is $\mathbb{F}_p$-independent over d_1^0. We now prove that there exists a' algebraically independent over $(M(b_n)_n)^{\mathrm{alg}}$ and such that

$$G((Ma'(b_n)_n)^{\mathrm{alg}}) = G(M) \oplus \bigoplus_{n<\omega} \mathrm{Span}_{\mathbb{F}_p}(a'd_2^n - d_1^0).$$

Assume that such a' exists, then it is an easy exercise to check that $G((Ma'b_n)^{\mathrm{alg}}) = G(M) \oplus \mathrm{Span}_{\mathbb{F}_p}(a'd_2^n - d_1^n)$ for each $n < \omega$. By Fact 1.1 (1), we conclude $a'b_n \equiv_M ab$ for all n, hence $a \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{M} b$. In order to check that such a' exists, it suffices to prove that if x algebraically independent over K, then

$$[G(K) \oplus \bigoplus_{n<\omega} \mathrm{Span}_{\mathbb{F}_p}(xd_2^n - d_1^0)] \cap K = G(K) \quad (**)$$

and then use that (K, G) is existentially closed in the $\mathscr{L}_G$ structure $(K(x)^{\mathrm{alg}}, H)$ where $H = G(K) \oplus \bigoplus_{n<\omega} \mathrm{Span}_{\mathbb{F}_p}(xd_2^n - d_1^0))$. We check $(**)$: assume that $g_K + \sum_n \lambda_n(xd_2^n - d_1^0) \in K$ for some $g_K \in G(K)$ and $\lambda_n \in \mathbb{F}_p$. Using that x is transcendental over K, we get $\sum_n \lambda_n d_2^n = 0$. As $(d_2^n)_n$ is linearly independent, we get $\lambda_n = 0$ for all n, hence $g_K + \sum_n \lambda_n(xd_2^n - d_1^0) \in G(K)$. The other inclusion is trivial.

Configuation (1) is treated similarly, with $(**)$ replaced by

$$[G(K) \oplus \bigoplus_{n<\omega} \mathrm{Span}_{\mathbb{F}_p}(xd_2^n - d_1^n)] \cap K = G(K).$$

Case II. In this case one easily sees[1] that there is no a' algebraically independent over $M(b_n)_n$ with

$$G((Ma'(b_n)_n)^{\mathrm{alg}}) = G(M) \oplus \bigoplus_{n<\omega} \mathrm{Span}_{\mathbb{F}_p}(a'd_2^0 - d_1^n).$$

However, interverting the roles of d_1^n and d_2^n and reasoning as in Case I, there is a' such that

$$G((Ma'(b_n)_n)^{\mathrm{alg}}) = G(M) \oplus \bigoplus_{n<\omega} \mathrm{Span}_{\mathbb{F}_p}(a'd_1^n - d_2^0)$$

Then $a'd_2^n d_1^n \equiv_M ad_1d_2$ for all $n < \omega$, which implies that $a'b^n \equiv_M ab$ for all $n < \omega$, so we conclude $a \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{M} b$.

1.3. $\mathop{\smile\!\!\!\!\!\!\mid}^{d} \neq \mathop{\smile\!\!\!\!\!\!\mid}^{f}$ **using a generic binary function in ACFG.** In $\mathrm{ACFG}^{\mathrm{eq}}$, a different witness of $\mathop{\smile\!\!\!\!\!\!\mid}^{d} \neq \mathop{\smile\!\!\!\!\!\!\mid}^{f}$ can be described, which seems very similar to the counter-example to $\mathop{\smile\!\!\!\!\!\!\mid}^{d} \neq \mathop{\smile\!\!\!\!\!\!\mid}^{f}$ in the generic binary function.

Imaginaries in ACFG have been described in [4, Section 3]. Assume that (K, G) is a model of ACFG and let $\pi : K \to K/G$ be the canonical projection and we consider the two sorted structure $(K, K/G, \pi)$ where K has the full field structure and K/G the group structure. We consider $(K, K/G, \pi)$ as a substructure of $(K, G)^{\mathrm{eq}}$. Then [4, Theorem 3.15] states that $(K, K/G, \pi)$ weakly eliminates imaginaries. The structure $(K, K/G, \pi)$ can be seen as a "generic forgetting" structure where the map $\pi : K \to K/G$ is a generic map only preserving the additive group structure in the field K. Precisely because π does not preserves the multiplicative structure of K, one sees that $(x, y) \mapsto \pi(xy)$ is a generic binary (and symmetric) map.

It is an easy exercise to prove that $A \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{C} B$ if and only if $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{alg}}_{C} B$ and $\pi((AC)^{\mathrm{alg}}) \cap \pi((BC)^{\mathrm{alg}}) = \pi(C^{\mathrm{alg}})$. Then $\mathop{\smile\!\!\!\!\!\!\mid}^{f}$ is given by forcing base monotonicity and algebraic extension on $\mathop{\smile\!\!\!\!\!\!\mid}^{K}$.

The second counterexample to $\mathop{\smile\!\!\!\!\!\!\mid}^{d} = \mathop{\smile\!\!\!\!\!\!\mid}^{f}$ in $\mathrm{ACFG}^{\mathrm{eq}}$ is the following. Start with an elementary substructure $(M, \pi(M))$ of $(K, K/G)$ and let u_1, u_2, d_1, d_2, a be as follows:

- $u_1, u_2 \in K/G$ are $\mathbb{F}_p$-linearly independent over $\pi(M)$.

[1] To get such an a', one would consider x algebraically independent over K and check that $[G(K) \oplus \bigoplus_{n<\omega} \mathrm{Span}_{\mathbb{F}_p}(xd_2^0 - d_1^n)] \cap K = G(K)$. Take $g_K + \sum_n \lambda_n(xd_2^0 - d_1^n) \in K$ with $g_K \in G(K)$. Then, as x is transcendental over K, we get $(\sum_n \lambda_n)d_2^0 = d_1^n$ which has a nontrivial solution, e.g. if $\sum_n \lambda_n = 0$ and $\lambda_n \neq 0$.

- $d_1, d_2 \in K$ are algebraically independent over M with $\pi(d_i) = u_i$ and $\pi((Md_i)^{\mathrm{alg}} = \pi(M) \oplus \mathrm{Span}_{\mathbb{F}_p}(u_i)$, for $i = 1, 2$.
- a is algebraically independent over Md_1d_2 and $\pi(ad_1) = 0$, $\pi(ad_2) = u_1$ and $\pi(Mad_1d_2) = \pi(M) \oplus \mathrm{Span}_{\mathbb{F}_p}(u_1, u_2)$.

Then for $b = (d_1 + d_2, d_1d_2)$ we have $a \not\mathop{\smile\!\!\!\!\!\!|}^{f}_{M} b$ and $a \mathop{\smile\!\!\!\!\!\!|}^{d}_{M} b$. The proof is similar to the one above. Note however that the two examples differ greatly. In the first example we have $G((Mab)^{\mathrm{alg}})) = G(M) \oplus \mathrm{Span}_{\mathbb{F}_p}(ad_2 - d_1)$, so the group in $(Mab)^{\mathrm{alg}}$ is 'small' but in the second example we have $\pi(Mad_1d_2) = \pi(M) \oplus \mathrm{Span}_{\mathbb{F}_p}(u_1, u_2)$ hence the group in $(Mab)^{\mathrm{alg}}$ is 'big'.

2. Naive monotonisation

2.1. **Preliminary facts.** Here is a complete list of the axioms of independence relations, those are the same as in [5].

Definition (Axioms of independence relations).

(1) (finite character) If $a \mathop{\smile\!\!\!\!\!\!|}_C B$ for all finite $a \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$.
(2) (existence) $A \mathop{\smile\!\!\!\!\!\!|}_C C$ for any A and C.
(3) (symmetry) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ then $B \mathop{\smile\!\!\!\!\!\!|}_C A$.
(4) (local character) For all A there is a cardinal $\kappa = \kappa(A)$ such that for all B there is $B_0 \subseteq B$ with $|B_0| < \kappa$ with $A \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$.
(5) (right normality) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ then $A \mathop{\smile\!\!\!\!\!\!|}_C BC$.
(6) (right monotonicity) If $A \mathop{\smile\!\!\!\!\!\!|}_C BD$ then $A \mathop{\smile\!\!\!\!\!\!|}_C B$.
(7) (right base monotonicity) Given $C \subseteq B \subseteq D$ if $A \mathop{\smile\!\!\!\!\!\!|}_C D$ then $A \mathop{\smile\!\!\!\!\!\!|}_B D$.
(8) (right transitivity) Given $C \subseteq B \subseteq D$, if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_B D$ then $A \mathop{\smile\!\!\!\!\!\!|}_C D$.
(9) (anti-reflexivity) If $a \mathop{\smile\!\!\!\!\!\!|}_C a$ then $a \in \mathrm{cl}(C)$;
(10) (right closure) $A \mathop{\smile\!\!\!\!\!\!|}_C B \implies A \mathop{\smile\!\!\!\!\!\!|}_C \mathrm{cl}(B)$.
(11) (strong closure) $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff \mathrm{cl}(AC) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}(C)} \mathrm{cl}(BC)$.
(12) (strong finite character) If $a \not\mathop{\smile\!\!\!\!\!\!|}_C B$ there exists $\phi(x) \in \mathrm{tp}(a/BC)$ such that $a' \not\mathop{\smile\!\!\!\!\!\!|}_C B$ for all $a' \vDash \phi(x)$.
(13) (extension) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ then for any $D \supseteq B$ there is $A' \equiv_{BC} A$ with $A' \mathop{\smile\!\!\!\!\!\!|}_C D$.
(14) (full existence) For all A, B, C there exists $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B$.
(15) (the independence theorem over models) Let M be a small model, and assume $A \mathop{\smile\!\!\!\!\!\!|}_M B$, $C_1 \mathop{\smile\!\!\!\!\!\!|}_M A$, $C_2 \mathop{\smile\!\!\!\!\!\!|}_M B$, and $C_1 \equiv_M C_2$. Then there is a set C such that $C \mathop{\smile\!\!\!\!\!\!|}_M AB$, $C \equiv_{MA} C_1$, and $C \equiv_{MB} C_2$.
(16) (stationarity over models) Let M be a small model, and assume $C_1 \mathop{\smile\!\!\!\!\!\!|}_M A$, $C_2 \mathop{\smile\!\!\!\!\!\!|}_M A$, and $C_1 \equiv_M C_2$. Then $C_1 \equiv_{MA} C_2$.

Results in this section are classical and appear for instance in [5].

Definition 2.1. Let $\mathop{\smile\!\!\!\!\!\!|}$ be a ternary relation and cl a closure operator. We associate the *monotonisation* $\mathop{\smile\!\!\!\!\!\!|}^M$ *of* $\mathop{\smile\!\!\!\!\!\!|}$ which is defined as the following:

$$A \mathop{\smile\!\!\!\!\!\!|}^{M}_{C} B \iff A \mathop{\smile\!\!\!\!\!\!|}_{D} B \text{ for all } D \text{ with } C \subseteq D \subseteq \mathrm{cl}(BC).$$

We define the relation $\mathop{\smile\!\!\!\!\!\!|}^*$:

$$A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B \iff \text{for all } D \supseteq B, \text{ there exists } A' \equiv_{BC} A \text{ with } A' \mathop{\smile\!\!\!\!\!\!|}_{C} D$$

Proposition 2.2 ([5], Proposition 1.2.14)**.** *The relation* $\mathop{\smile\!\!\!\!\!\!|}^M$ *satisfies right base monotonicity.*

- *If* $\mathop{\smile\!\!\!\!\!\!|}$ *satisfies left or right monotonicity, left or right closure, left normality, so does* $\mathop{\smile\!\!\!\!\!\!|}^M$*. If* $\mathop{\smile\!\!\!\!\!\!|}$ *further satisfies right normality or left transitivity, then so does* $\mathop{\smile\!\!\!\!\!\!|}^M$*.*
- *If* $\mathop{\smile\!\!\!\!\!\!|}$ *satisfies left or right closure then so does* $\mathop{\smile\!\!\!\!\!\!|}^M$*.*

- *If $\mathop{\perp\!\!\!\perp}^0 \to \mathop{\perp\!\!\!\perp}$ and $\mathop{\perp\!\!\!\perp}^0$ satisfies the right-sided instance of: normality, monotonicity, closure and base monotonicity, then $\mathop{\perp\!\!\!\perp}^0 \to \mathop{\perp\!\!\!\perp}^M$.*

We always have $\mathop{\perp\!\!\!\perp}^* \to \mathop{\perp\!\!\!\perp}$. By definition, if $\mathop{\perp\!\!\!\perp}^0 \to \mathop{\perp\!\!\!\perp}$ and $\mathop{\perp\!\!\!\perp}^0$ satisfies extension, then $\mathop{\perp\!\!\!\perp}^0 \to \mathop{\perp\!\!\!\perp}^*$.

Proposition 2.3 ([5], Proposition 4.1.17)**.** *If $\mathop{\perp\!\!\!\perp}$ is invariant and satisfies left and right monotonicity then $\mathop{\perp\!\!\!\perp}^*$ is invariant and satisfies left and right monotonicity, right normality, extension and right closure. If $\mathop{\perp\!\!\!\perp}$ satisfies one of the following property: right base monotonicity, left transitivity, left normality, anti-reflexivity then so does $\mathop{\perp\!\!\!\perp}^*$.*

Theorem 2.4 ([5], Theorem 4.1.24)**.** *Let $\mathop{\perp\!\!\!\perp}^0$ be an invariant relation satisfying right monotonicity, right base monotonicity and local character (for instance, take $\mathop{\perp\!\!\!\perp}^0 = \mathop{\perp\!\!\!\perp}^h$).*

Let $\mathop{\perp\!\!\!\perp}$ be an invariant relation satisfying left and right monotonicity and the following property ($\mathop{\perp\!\!\!\perp}^0$-amalgamation over models):

if $c_1 \equiv_M c_2$ and $c_1 \mathop{\perp\!\!\!\perp}_M a$, $c_2 \mathop{\perp\!\!\!\perp}_M b$ and $a \mathop{\perp\!\!\!\perp}^0_M b$ then there exists c with $c \mathop{\perp\!\!\!\perp}_E ab$ and $c \equiv_{Ma} c_1$, $c \equiv_{Mb} c_2$.

Then $\mathop{\perp\!\!\!\perp}^{M*} \to \mathop{\perp\!\!\!\perp}^f$.

The following is a standard use of Erdös-Rado and compactness.

Lemma 2.5 ([5], Lemma 3.2.18)**.** *Let $(b_i)_{i<\omega}$ be a C-indiscernible sequence and $\mathop{\perp\!\!\!\perp}$ be an invariant relation satisfying right monotonicity, right base monotonicity and local character. Then there exists a model M containing C such that $(b_i)_{i<\omega}$ is an M-indiscernible $\mathop{\perp\!\!\!\perp}$-Morley sequence over M.*

2.2. **Naive monotonisation.**

Definition 2.6. Let $\mathop{\perp\!\!\!\perp}$ be a ternary relation. We associate the *naive monotonisation* $\mathop{\perp\!\!\!\perp}^m$ *of* $\mathop{\perp\!\!\!\perp}$ which is defined as the monotonisation with respect to the trivial closure operator:

$$A \mathop{\perp\!\!\!\perp}^m_C B \iff A \mathop{\perp\!\!\!\perp}_D B \text{ for all } D \text{ with } C \subseteq D \subseteq BC.$$

Proposition 2.7. *The relation $\mathop{\perp\!\!\!\perp}^m$ satisfies right base monotonicity.*

- *If $\mathop{\perp\!\!\!\perp}$ satisfies left or right monotonicity, left normality, so does $\mathop{\perp\!\!\!\perp}^m$. If $\mathop{\perp\!\!\!\perp}$ further satisfies right normality or left transitivity, then so does $\mathop{\perp\!\!\!\perp}^m$.*
- *If $\mathop{\perp\!\!\!\perp}^0 \to \mathop{\perp\!\!\!\perp}$ and $\mathop{\perp\!\!\!\perp}^0$ satisfies the right-sided instance of: normality, monotonicity and base monotonicity, then $\mathop{\perp\!\!\!\perp}^0 \to \mathop{\perp\!\!\!\perp}^m$.*

Proof. The beginning and the first item are obtained by applying Proposition 2.2 with the trivial closure operator.

We prove the last item. If $A \mathop{\perp\!\!\!\perp}^0_C B$, then by normality, we have $A \mathop{\perp\!\!\!\perp}^0_C BC$. Then, by base monotonicity, for all D with $C \subseteq D \subseteq BC$ we have $A \mathop{\perp\!\!\!\perp}^0_D BC$ so $A \mathop{\perp\!\!\!\perp}^0_D B$ by monotonicity. As $\mathop{\perp\!\!\!\perp}^0 \to \mathop{\perp\!\!\!\perp}$, we get $A \mathop{\perp\!\!\!\perp}_D B$. We conclude that $A \mathop{\perp\!\!\!\perp}^m_C B$. □

Corollary 2.8. *If $\mathop{\perp\!\!\!\perp}$ is invariant and satisfies left and right monotonicity then $\mathop{\perp\!\!\!\perp}^{m*}$ is invariant and satisfies left and right monotonicity, right normality, right closure, right base monotonicity and extension. Further $\mathop{\perp\!\!\!\perp}^{m*} \to \mathop{\perp\!\!\!\perp}^m \to \mathop{\perp\!\!\!\perp}$.*

Proof. By Proposition 2.7, $\mathop{\perp\!\!\!\perp}^m$ satisfies left and right monotonicity and right base monotonicity. By Proposition 2.3, $\mathop{\perp\!\!\!\perp}^{m*}$ satisfies right monotonicity, right base monotonicity and extension. It is standard that extension implies right normality and right closure. As $\mathop{\perp\!\!\!\perp}^{m*} \to \mathop{\perp\!\!\!\perp}$ and $\mathop{\perp\!\!\!\perp}^{m*}$ satisfies the right-sided versions of normality, monotonicity and base monotonicity, Proposition 2.7 applies and $\mathop{\perp\!\!\!\perp}^{m*} \to \mathop{\perp\!\!\!\perp}^m$. □

Lemma 2.9. *Assume that $\mathop{\perp\!\!\!\perp}$ satisfies left and right normality and monotonicity then $\mathop{\perp\!\!\!\perp}^{M*} = \mathop{\perp\!\!\!\perp}^{m*}$.*

Proof. We have $\mathop{\perp\!\!\!\perp}^{M*} \to \mathop{\perp\!\!\!\perp}^{m*}$ by Propositions 2.7 and 2.3 because $\mathop{\perp\!\!\!\perp}^{M*}$ satisfies right base monotonicity and extension and $\mathop{\perp\!\!\!\perp}^{M*} \to \mathop{\perp\!\!\!\perp}$. By Corollary 2.8 $\mathop{\perp\!\!\!\perp}^{m*}$ satisfies left and right normality, monotonicity, right closure, right base monotonicity and extension. As $\mathop{\smile\hskip-0.9em^|\ }^{m*} \to \mathop{\smile\hskip-0.9em^|\ }$ and $\mathop{\smile\hskip-0.9em^|\ }^{m*}$ satisfies right closure, right base monotonicity, we get $\mathop{\smile\hskip-0.9em^|\ }^{m*} \to \mathop{\smile\hskip-0.9em^|\ }^{M}$. As $\mathop{\smile\hskip-0.9em^|\ }^{m*}$ satisfies extension, we conclude $\mathop{\smile\hskip-0.9em^|\ }^{m*} \to \mathop{\smile\hskip-0.9em^|\ }^{M*}$. □

We can already conclude the following correct version of [4, Proposition 4.14.].

Corollary 2.10. *Let $\mathop{\smile\hskip-0.9em^|\ }$ be an invariant relation satisfying left and right monotonicity such that*

(1) $\mathop{\smile\hskip-0.9em^|\ }$ is weaker than $\mathop{\smile\hskip-0.9em^|\ }^{d}$
(2) $\mathop{\smile\hskip-0.9em^|\ }$ satisfies $\mathop{\smile\hskip-0.9em^|\ }^{h}$-amalgamation over models
(3) $\mathop{\smile\hskip-0.9em^|\ }^{m}$ satisfies extension

then $\mathop{\smile\hskip-0.9em^|\ }^{m} = \mathop{\smile\hskip-0.9em^|\ }^{f} = \mathop{\smile\hskip-0.9em^|\ }^{d}$.

Proof. As $\mathop{\smile\hskip-0.9em^|\ }^{d} \to \mathop{\smile\hskip-0.9em^|\ }$ and $\mathop{\smile\hskip-0.9em^|\ }^{d}$ satisfies normality, monotonicity and base monotonicity we have $\mathop{\smile\hskip-0.9em^|\ }^{d} \to \mathop{\smile\hskip-0.9em^|\ }^{m}$ by Proposition 2.7. By Lemma 2.9 $\mathop{\smile\hskip-0.9em^|\ }^{M*} = \mathop{\smile\hskip-0.9em^|\ }^{m*}$, hence by Theorem 2.4 we have $\mathop{\smile\hskip-0.9em^|\ }^{m*} \to \mathop{\smile\hskip-0.9em^|\ }^{f}$. As $\mathop{\smile\hskip-0.9em^|\ }^{m}$ satisfies extension we have $\mathop{\smile\hskip-0.9em^|\ }^{m*} = \mathop{\smile\hskip-0.9em^|\ }^{m}$. We conclude that $\mathop{\smile\hskip-0.9em^|\ }^{f} \to \mathop{\smile\hskip-0.9em^|\ }^{d} \to \mathop{\smile\hskip-0.9em^|\ }^{m} \to \mathop{\smile\hskip-0.9em^|\ }^{f}$ i.e. $\mathop{\smile\hskip-0.9em^|\ }^{m} = \mathop{\smile\hskip-0.9em^|\ }^{f} = \mathop{\smile\hskip-0.9em^|\ }^{d}$. □

Remark 2.11. In Corollary 2.10, we may also conclude that $\mathop{\smile\hskip-0.9em^|\ }^{f} = \mathop{\smile\hskip-0.9em^|\ }^{d} = \mathop{\smile\hskip-0.9em^|\ }^{m} = \mathop{\smile\hskip-0.9em^|\ }^{M}$, simply because we always have $\mathop{\smile\hskip-0.9em^|\ }^{f} \to \mathop{\smile\hskip-0.9em^|\ }^{M} \to \mathop{\smile\hskip-0.9em^|\ }^{m}$. Indeed $\mathop{\smile\hskip-0.9em^|\ }^{M} \to \mathop{\smile\hskip-0.9em^|\ }^{m}$ is clear, and $\mathop{\smile\hskip-0.9em^|\ }^{f} \to \mathop{\smile\hskip-0.9em^|\ }^{M}$ follows from Propositions 2.2 and 2.3 because $\mathop{\smile\hskip-0.9em^|\ }^{f}$ always satisfies right closure and right base monotonicity.

For completeness, we include a proof of the following (which already follows from Theorem 2.4 and Lemma 2.9).

Theorem 2.12. *Let $\mathop{\smile\hskip-0.9em^|\ }^{0}$ be an invariant relation satisfying right monotonicity, right base monotonicity and local character (for instance, take $\mathop{\smile\hskip-0.9em^|\ }^{0} = \mathop{\smile\hskip-0.9em^|\ }^{h}$).*

Let $\mathop{\smile\hskip-0.9em^|\ }$ be an invariant relation satisfying left and right monotonicity and the following property ($\mathop{\smile\hskip-0.9em^|\ }^{0}$-amalgamation over models):

if $c_1 \equiv_M c_2$ and $c_1 \mathop{\smile\hskip-0.9em^|\ }_M a$, $c_2 \mathop{\smile\hskip-0.9em^|\ }_M b$ and $a \mathop{\smile\hskip-0.9em^|\ }^{0}_M b$ then there exists c with $c \mathop{\smile\hskip-0.9em^|\ }_E ab$ and $c \equiv_{Ma} c_1$, $c \equiv_{Mb} c_2$.

Then $\mathop{\smile\hskip-0.9em^|\ }^{m*} \to \mathop{\smile\hskip-0.9em^|\ }^{f}$.

Proof. By Corollary 2.8, the relation $\mathop{\smile\hskip-0.9em^|\ }^{m*}$ satisfies satisfies left and right monotonicity, right normality, right closure, right base monotonicity and extension. Note that we do not use right closure in this proof.

We show that $\mathop{\smile\hskip-0.9em^|\ }^{m*} \to \mathop{\smile\hskip-0.9em^|\ }^{d}$, the result follows from $\mathop{\smile\hskip-0.9em^|\ }^{f} = \mathop{\smile\hskip-0.9em^|\ }^{d\,*}$.

Assume that $a \mathop{\smile\hskip-0.9em^|\ }^{m*}_C b$, for some a, b, C. Let $(b_i)_{i<\omega}$ be a C-indiscernible sequence with $b = b_0$. By Lemma 2.5, there exists a model $M \supseteq C$ such that $(b_i)_{i<\omega}$ is an M-indiscernible $\mathop{\smile\hskip-0.9em^|\ }^{0}$-Morley sequence over M, i.e. $b_i \mathop{\smile\hskip-0.9em^|\ }^{0}_M b_{<i}$ for all $i < \omega$.

By extension there exists a' such that $a' \equiv_{Cb} a$ and $a' \mathop{\smile\hskip-0.9em^|\ }^{m*}_C bM$. It follows from base monotonicity and right monotonicity that

$$a' \mathop{\smile\hskip-0.9em^|\ }_M b.$$

For each $i \geq 0$ there exists an automorphism σ_i over M sending $b = b_0$ to b_i, so setting $a'_i = \sigma_i(a')$ we have: $a'_i b_i \equiv_M a'b$ hence by invariance $a'_i \mathop{\smile\hskip-0.9em^|\ }_M b_i$. Note that $a'b \equiv_C ab$.

Claim 2. There exists a'' such that $a''b_i \equiv_M a'b$ for all $i < \omega$.

Proof of the claim. By induction and compactness, it is sufficient to show that for all $i < \omega$, there exists a''_i such that for all $k \leq i$ we have $a''_i b_k \equiv_M a'b$ and $a''_i \mathop{\smile\hskip-0.9em^|\ }_M b_{\leq i}$. For the case $i = 0$ take $a''_0 = a'$. Assume that a''_i has been constructed, we have

$$a'_{i+1} \mathop{\smile\hskip-0.9em^|\ }_M b_{i+1} \text{ and } b_{i+1} \mathop{\smile\hskip-0.9em^|\ }^{0}_M b_{\leq i} \text{ and } a''_i \mathop{\smile\hskip-0.9em^|\ }_M b_{\leq i}.$$

As $a'_{i+1} \equiv_M a''_i$, by $\mathop{\smile\hskip-0.9em^|\ }^{0}$-amalgamation over models, there exists a''_{i+1} such that

(1) $a''_{i+1}b_{i+1} \equiv_M a'_{i+1}b_{i+1}$
(2) $a''_{i+1}b_{\leq i} \equiv_M a''_i b_{\leq i}$
(3) $a''_{i+1} \mathop{\smile\hspace{-0.9em}\mid}_M b_{\leq i+1}$.

By induction and compactness, there exists a'' such that $a''b_i \equiv_M ab$ for all $i < \omega$, which proves the claim. $\square$

Let a'' be as in the claim, then as $a'b \equiv_C ab$ we have $a''b_i \equiv_C ab$ for all $i < \omega$, hence $a \mathop{\smile\hspace{-0.9em}\mid}^d_C b$. $\square$

Another consequence of Theorem 2.12 (or Lemma 2.9) is the following, which implies that either version of the monotonisation are enough to get a description of forking in NSOP_1 theories.

Proposition 2.13. *Let T be an $NSOP_1$ theory and $\mathop{\smile\hspace{-0.9em}\mid}^K$ be Kim-independence in T. Then*

$$\mathop{\smile\hspace{-0.9em}\mid}^f = (\mathop{\smile\hspace{-0.9em}\mid}^K)^{M*} = (\mathop{\smile\hspace{-0.9em}\mid}^K)^{m*}$$

Proof. We already have $\mathop{\smile\hspace{-0.9em}\mid}^f \to (\mathop{\smile\hspace{-0.9em}\mid}^K)^{M*} \to (\mathop{\smile\hspace{-0.9em}\mid}^K)^{m*}$. In an NSOP_1 theory, $\mathop{\smile\hspace{-0.9em}\mid}^K$ satisfies the hypothesis of Theorem 2.12 with $\mathop{\smile\hspace{-0.9em}\mid} = \mathop{\smile\hspace{-0.9em}\mid}^0 = \mathop{\smile\hspace{-0.9em}\mid}^K$ thus $(\mathop{\smile\hspace{-0.9em}\mid}^K)^{m*} \to \mathop{\smile\hspace{-0.9em}\mid}^f$. $\square$

Remark 2.14. In ACFG, we have that $\mathop{\smile\hspace{-0.9em}\mid}^{K^{M*}} = \mathop{\smile\hspace{-0.9em}\mid}^{K^M} = \mathop{\smile\hspace{-0.9em}\mid}^f$ and $\mathop{\smile\hspace{-0.9em}\mid}^{K^m} \neq \mathop{\smile\hspace{-0.9em}\mid}^{K^M}$ (otherwise $\mathop{\smile\hspace{-0.9em}\mid}^{K^M} \to \mathop{\smile\hspace{-0.9em}\mid}^d \to \mathop{\smile\hspace{-0.9em}\mid}^{K^m}$ would yield $\mathop{\smile\hspace{-0.9em}\mid}^f = \mathop{\smile\hspace{-0.9em}\mid}^d$)). In particular $\mathop{\smile\hspace{-0.9em}\mid}^{K^m} \neq \mathop{\smile\hspace{-0.9em}\mid}^{K^{m*}}$ and $\mathop{\smile\hspace{-0.9em}\mid}^{K^m}$ fails extension. Note that in ACFG, we have $\mathop{\smile\hspace{-0.9em}\mid}^{K^{m*}} = \mathop{\smile\hspace{-0.9em}\mid}^{K^{M*}}$ but $\mathop{\smile\hspace{-0.9em}\mid}^{K^m} \neq \mathop{\smile\hspace{-0.9em}\mid}^{K^M}$.

2.3. **Forcing algebraicity.**
Now we expand a bit on ideas from [3].

Definition 2.15. Let $\mathop{\smile\hspace{-0.9em}\mid}$ be a ternary relation. We associate the *(right) closure extension* $\mathop{\smile\hspace{-0.9em}\mid}^c$ *of* $\mathop{\smile\hspace{-0.9em}\mid}$ which is defined as :

$$A \mathop{\smile\hspace{-0.9em}\mid}^c_C B \iff A \mathop{\smile\hspace{-0.9em}\mid}_C \mathrm{acl}(BC).$$

Remark 2.16. In [3], we have $\mathop{\smile\hspace{-0.9em}\mid}^{d\,c} = \mathop{\smile\hspace{-0.9em}\mid}^{da}$.

We naturally have the following:

Proposition 2.17. *Let $\mathop{\smile\hspace{-0.9em}\mid}$ be an invariant relation. Then $\mathop{\smile\hspace{-0.9em}\mid}^c$ satisfies right closure and right normality. If $\mathop{\smile\hspace{-0.9em}\mid}$ satisfies right monotonicity, then $\mathop{\smile\hspace{-0.9em}\mid}^c \to \mathop{\smile\hspace{-0.9em}\mid}$.*

- *If $\mathop{\smile\hspace{-0.9em}\mid}$ satisfies left or right monotonicity, left normality, so does $\mathop{\smile\hspace{-0.9em}\mid}^c$.*
- *If $\mathop{\smile\hspace{-0.9em}\mid}^0 \to \mathop{\smile\hspace{-0.9em}\mid}$ and $\mathop{\smile\hspace{-0.9em}\mid}^0$ satisfies right normality and right closure, then $\mathop{\smile\hspace{-0.9em}\mid}^0 \to \mathop{\smile\hspace{-0.9em}\mid}^c$.*
- *If $\mathop{\smile\hspace{-0.9em}\mid}$ satisfies right base monotonicity then so does $\mathop{\smile\hspace{-0.9em}\mid}^c$.*

Proof. For right closure: if $A \mathop{\smile\hspace{-0.9em}\mid}^c_C B$ we have $A \mathop{\smile\hspace{-0.9em}\mid}_C \mathrm{acl}(BC)$. As $\mathrm{acl}(\mathrm{acl}(BC)) = \mathrm{acl}(BC)$ we have $A \mathop{\smile\hspace{-0.9em}\mid}_C \mathrm{acl}(\mathrm{acl}(BC))$ i.e. $A \mathop{\smile\hspace{-0.9em}\mid}^c_C \mathrm{acl}(BC)$. For right normality: if $A \mathop{\smile\hspace{-0.9em}\mid}^c_C B$ then $A \mathop{\smile\hspace{-0.9em}\mid}_C \mathrm{acl}(BC)$ so $A \mathop{\smile\hspace{-0.9em}\mid}^c_C BC$. If $\mathop{\smile\hspace{-0.9em}\mid}$ satisfies right monotonicity, then $A \mathop{\smile\hspace{-0.9em}\mid}^c_C B$ implies $A \mathop{\smile\hspace{-0.9em}\mid}_C \mathrm{acl}(BC)$ which implies $A \mathop{\smile\hspace{-0.9em}\mid}_C B$.

For the first item: the left-sided properties are trivial. If $\mathop{\smile\hspace{-0.9em}\mid}$ satisfies right monotonicity, then $A \mathop{\smile\hspace{-0.9em}\mid}^c_C BD$ implies $A \mathop{\smile\hspace{-0.9em}\mid}_C \mathrm{acl}(CBD)$ which implies $A \mathop{\smile\hspace{-0.9em}\mid}_C \mathrm{acl}(BC)$ so $A \mathop{\smile\hspace{-0.9em}\mid}^c_C B$.

For the second item, if $A \mathop{\smile\hspace{-0.9em}\mid}^0_C B$ then by right normality and right closure we have $A \mathop{\smile\hspace{-0.9em}\mid}^0_C \mathrm{acl}(BC)$ which implies $A \mathop{\smile\hspace{-0.9em}\mid}_C \mathrm{acl}(BC)$ so $A \mathop{\smile\hspace{-0.9em}\mid}^c_C B$.

For the third item, assume that $A \mathop{\smile\hspace{-0.9em}\mid}^c_C B$ and $C \subseteq D \subseteq B$. Then $A \mathop{\smile\hspace{-0.9em}\mid}_C \mathrm{acl}(B)$ hence by base monotonicity $A \mathop{\smile\hspace{-0.9em}\mid}_D \mathrm{acl}(B)$ so $A \mathop{\smile\hspace{-0.9em}\mid}^c_D B$. $\square$

Proposition 2.18. *Assume that $\mathop{\smile\hspace{-0.9em}\mid}$ satisfies right normality and right monotonicity then*

$$(\mathop{\smile\hspace{-0.9em}\mid}^m)^c \to \mathop{\smile\hspace{-0.9em}\mid}^M$$

If $\mathop{\smile\hspace{-0.9em}\mid}$ further satisfies right closure then $(\mathop{\smile\hspace{-0.9em}\mid}^m)^c = \mathop{\smile\hspace{-0.9em}\mid}^M$.

Proof. First, $\mathop{\smile\hskip -0.9em ^| \ }^{m}$ satisfies the right-sided version of normality, monotonicity, base monotonicity, by Proposition 2.7. By Proposition 2.17 we have $\mathop{\smile\hskip -0.9em ^| \ }^{mc}$ satisfies the right-sided version of normality, monotonicity, base monotonicity and closure. Again by Proposition 2.17 $\mathop{\smile\hskip -0.9em ^| \ }^{mc} \to \mathop{\smile\hskip -0.9em ^| \ }$ hence by Proposition 2.2 we have $\mathop{\smile\hskip -0.9em ^| \ }^{mc} \to \mathop{\smile\hskip -0.9em ^| \ }^{M}$.

Assume that $\mathop{\smile\hskip -0.9em ^| \ }$ satisfies right closure, then by Proposition 2.2, $\mathop{\smile\hskip -0.9em ^| \ }^{M}$ satisfies the right-sided instances of normality monotonicity, base monotonicity and closure. As $\mathop{\smile\hskip -0.9em ^| \ }^{M} \to \mathop{\smile\hskip -0.9em ^| \ }$, from Proposition 2.7 we have $\mathop{\smile\hskip -0.9em ^| \ }^{M} \to \mathop{\smile\hskip -0.9em ^| \ }^{m}$. From $\mathop{\smile\hskip -0.9em ^| \ }^{M} \to \mathop{\smile\hskip -0.9em ^| \ }^{m}$ and Proposition 2.17 we get $\mathop{\smile\hskip -0.9em ^| \ }^{M} \to \mathop{\smile\hskip -0.9em ^| \ }^{mc}$. □

The essential difference between $\mathop{\smile\hskip -0.9em ^| \ }^{M}$ and $\mathop{\smile\hskip -0.9em ^| \ }^{m}$ is that the former preserves right closure and the other one does not a priori. However this distinction does not appear in a pregeometric theory:

Fact 2.19 ([3], Remark 2.19)**.** *If T is pregeometric then* $\mathop{\smile\hskip -0.9em ^{a} \ }^{M} = \mathop{\smile\hskip -0.9em ^{a} \ }^{m}$.

Remark 2.20. In ACFG, we have $\mathop{\smile\hskip -0.9em ^| \ }^{K} \to \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{alg}}$. As in ACFG, the algebraic closure coincide with the field theoretic algebraic closure (see [4, Section 2.1]), we have that $\mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{alg}} = (\mathop{\smile\hskip -0.9em ^{a} \ })^{m} = (\mathop{\smile\hskip -0.9em ^{a} \ })^{M}$. In particular:

$$\mathop{\smile\hskip -0.9em ^| \ }^{d} \to \mathop{\smile\hskip -0.9em ^| \ }^{K} \to (\mathop{\smile\hskip -0.9em ^{a} \ })^{M}.$$

References

[1] Hans Adler. Explanation of independence. Dissertation zur Erlangung des Doktorgrades der Fakultaet fuer Mathematik und Physik der Albert-Ludwigs-Universitaet Freiburg im Breisgau, 2005.

[2] Hans Adler. Theories controlled by formulas of vapnik-chervonenkis codimension 1. *Draft*, 2008.

[3] Gabriel Conant and Alex Kruckman. Three surprising instances of dividing, 2023.

[4] Christian d'Elbée. Forking, imaginaries, and other features of ACFG. *The Journal of Symbolic Logic*, 86(2):669–700, 2021.

[5] Christian d'Elbée. Axiomatic theory of independence relations in model theory, 2023.

[6] Christian d'Elbée. Generic expansions by a reduct. *Journal of Mathematical Logic*, 0(0):2150016, 2021.

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