Generic expansion by a reduct.

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June 22, 2021

Generic constructions

1. **Generic Predicate.** [Chatzidakis-Pillay 98] T a model complete $\mathscr{L}$-theory which eliminate $\exists^\infty$. The $\mathscr{L} \cup \{P\}$-theory T with a unary predicate, admits a model-companion TP. If T is simple then so is TP.
2. **Generic expansion.** T a model complete $\mathscr{L}$-theory, $\mathscr{L} \subset \mathscr{L}'$, does T as an (incomplete) $\mathscr{L}'$-theory admits a model-companion $T_{\mathscr{L}'}$?
   - YES : if (and only if) T eliminate $\exists^\infty$ [Winkler 1975]
   - If T is $NSOP_1$ then so is $T_{\mathscr{L}'}$ [Kruckman-Ramsey 2018]. ([Jeřábek 2018] for $T^{\emptyset}_{\mathscr{L}'}$)

General setting

- T an $\mathscr{L}$-theory with infinite models;
- $\mathscr{L}_0 \subseteq \mathscr{L}$ and let $T_0 \subseteq T$ and $\mathscr{L}_0$-theory;
- S a new unary predicate symbol and $\mathscr{L}_S = \mathscr{L} \cup \{S\}$;
- T_S the $\mathscr{L}_S$-theory of $\mathscr{L}_S$-structures $(\mathscr{M}, \mathscr{M}_0)$ where $\mathscr{M} \models T$ and $S(\mathscr{M}) = \mathscr{M}_0 \models T_0$ is a substructure of $\mathscr{M} \restriction \mathscr{L}_0$.

Example

Let $p > 0$ and $T = \mathrm{ACF}_p$ in the language $\mathscr{L} = \{+, -, \cdot, 0, 1\}$.Let $T_0 = T \restriction \{+, -, 0\}$, then T_0 is the theory of $\mathbb{F}_p$-vector spaces. T_S is the theory of pairs (K, G) where K is a model of ACF_p and G is an additive subgroup, we call it ACF_G.

Example

Let $T = \mathrm{ACF}$ and $T_0 = T = \mathrm{ACF}$. Call T_S is the theory of pairs (K, k) of algebraically closed fields, we call it $\mathrm{ACF}_{\mathrm{ACF}}$.

Question

Can we axiomatise "generic" models of T_S?

General setting

Assume that T_0 is pregeometric, there is an independence relation $\mathop{\smile\!\!\!\!\!\!\mid}^0$ defined over every subsets of every models of T_0.

Definition

An extension $(\mathscr{N}, \mathscr{N}_0) \supset (\mathscr{M}, \mathscr{M}_0)$ of models of T_S is *strong* if $\mathscr{N}_0 \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathscr{M}_0} \mathscr{M}$.

Example

- In ACF_G, every extension $(K, G) \supset (F, H)$ is strong, as $G \cap F = H$ by definition of an extension.
- In $\mathrm{ACF}_{\mathrm{ACF}}$, a strong extension $(L, l) \supset (K, k)$ is such that $l \mathop{\smile\!\!\!\!\!\!\mid}^{ld}_k K$.

Theorem

Assume that the following holds:

1. (H_1) T *is model complete;*
2. (H_2) T_0 *is model complete and for all infinite* $A \subseteq \mathscr{M} \models T_0$, $\mathrm{acl}_0(A) \prec \mathscr{M}$*;*
3. (H_3) T_0 *is pregeometric;*
4. (H_4) *"Definability of* $\mathop{\smile\!\!\!\!\!\!|}^0$*-dimension in* T*"*

Then there is a theory TS *containing* T_S *such that*

- *every model of* T_S *has a strong extension which is a model of* TS*;*
- *if* $(\mathscr{M}, \mathscr{M}_0) \models TS$ *and* $(\mathscr{N}, \mathscr{N}_0) \models T_S$ *is a strong extension of* $(\mathscr{M}, \mathscr{M}_0)$ *then* $(\mathscr{M}, \mathscr{M}_0)$ *is existentially closed in* $(\mathscr{N}, \mathscr{N}_0)$*.*

If T_0 *is modular, then* TS *is the model-companion of* T_S *(*$(T, T_0, \mathscr{L}_0)$ *is called a suitable triple).*

Hypothesis (H_4)

(H_4) for all $\mathscr{L}$-formula $\phi(x, y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for $b \in \mathscr{M} \models T$,

$$\mathscr{M} \models \theta_\phi(b) \iff \begin{array}{l} \text{there exists } \mathscr{N} \succ \mathscr{M} \text{ and } a \in \mathscr{N} \text{ such that} \\ \phi(a, b) \text{ and } a \text{ is an} \\ \downarrow^0\text{-independent tuple over } \mathscr{M}. \end{array}$$

If $\mathscr{M}$ is $\aleph_0$-saturated, the following are equivalent:

1. there exists $\mathscr{N} \succ \mathscr{M}$ and some realisation a of $\phi(x, b)$ in $\mathscr{N}$ such that a is an $\downarrow^0$-independent tuple over $\mathscr{M}$;
2. there exists some realisation a of $\phi(x, b)$ in $\mathscr{M}$ such that a is $\downarrow^0$-independent over $\mathrm{acl}_T(b)$.

(H_4) in examples

Under $(H_1) - (H_3)$, condition (H_4) is equivalent to elimination of $\exists^\infty$ in the following cases:

- T_0 is the theory of infinite sets;
- $T = T_0$.

Example

- TS exists if T eliminates $\exists^\infty$ and T_0 is the theory of infinite sets: the generic predicate;
- Under $(H_1 - H_3)$, TS exists if $T = T_0$: lovely pairs of T.

Axioms

Definition

We say that a formula $\psi(t, y)$ is n-algebraic in t (or just algebraic in t) if for any tuple b the number of realisations of $\psi(t, b)$ is at most n. We say that a formula $\psi(t, x, y)$ algebraic in t is strict in y if whenever b is an $\mathop{\smile\!\!\!\!\!\mid}^0$-independent tuple over a, the set of realisations of $\psi(t, a, b)$ is in $\mathrm{acl}_0(a, b) \setminus \mathrm{acl}_0(a)$.

Example

In the language of vector spaces, the formula $t = \lambda x + \mu y$ is strict in y if and only if $\mu \neq 0$.

Lemma

Assume that T_0 is pregeometric. Then for u a singleton and tuples a and b, if $u \in \mathrm{acl}_0(a, b) \setminus \mathrm{acl}_0(a)$, there exists an $\mathscr{L}_0$-formula $\tau(t, x, y)$ algebraic in t and strict in y such that $u \models \tau(t, a, b)$.

Axioms

If T, T_0 satisfies $(H_1) - (H_4)$, the theory TS is axiomatised by adding to T_S the following axiom scheme: for each partition of tuple of variables $x = x^0 x^1$, for $\mathscr{L}$-formula $\phi(x, y)$, and $\mathscr{L}_0$-formulae $(\tau_i(t, x, y))_{i<k}$ which are algebraic in t and strict in x^1,

$$\forall y(\theta_\phi(y) \to (\exists x \phi(x, y) \wedge x^0 \subseteq S \wedge \bigwedge_{i<k} \forall t \ (\tau_i(t, x, y) \to t \notin S))).$$

In other words, it is given by expressing the randomness of the predicate S on $\mathrm{acl}_0(xy) \setminus \mathrm{acl}_0(x^0 y)$.

(H_4) in examples

$\mathbb{F}_q$ finite field. Let $\mathscr{L}_0 = \{(\lambda_\alpha)_{\alpha\in\mathbb{F}_q}, +, 0\}$, and $\mathscr{L} \supset \mathscr{L}_0$. Let T be a model-complete $\mathscr{L}$-theory which contains the $\mathscr{L}_0$-theory T_0 of infinite-dimensional $\mathbb{F}_q$-vector spaces. $\mathrm{acl}_0(A) = \langle A\rangle$ is the vector space spanned by A. For x, z tuples of variable with $|z| = q^{|x|} - 1$, there is a formula $z = \langle y\rangle_0$ holding when z enumerates all non-trivial linear combinations of x.

Fact (Chatzidakis-Pillay)

T eliminates $\exists^\infty$ if and only if for any formula $\phi(x, y)$ there is a formula $\theta_\phi(y)$ such that in any $\aleph_0$-saturated model $\mathscr{M}$ of T the set $\theta_\phi(\mathscr{M})$ consists of tuples b from $\mathscr{M}$ such that there exists a realisation a of $\phi(x, b)$ with $a \cap \mathrm{acl}_T(b) = \emptyset$.

Given some $\mathscr{L}$-formula $\phi(x, y)$, set $\psi(z, y) = \exists x\phi(x, y) \wedge (z = \langle x\rangle_0)$ and apply the fact to the formula $\psi(z, y)$, then the formula $\theta_\psi(b)$ holds if and only if there is an $\mathbb{F}_q$-independent realisation of $\phi(x, b)$ over $\mathrm{acl}_T(b)$.

(H_4) in examples

Let T_V be expansion of T by a predicate for a $\mathbb{F}_q$-vector-subspace V.

Theorem

If T is model-complete and eliminates $\exists^\infty$, then T_V admits a model companion. Further, TV eliminates $\exists^\infty$, hence we may iterate the expansion.

For a field of characteristic $p > 0$, the additive group is an $\mathbb{F}_p$-vector space.

Corollary

A model-complete theory T of field of characteristic $p > 0$ has a generic expansion by an additive subgroup if (and only if) T eliminates $\exists^\infty$.

- ACFG for $T = \mathrm{ACF}_p$;
- $\mathrm{Psf}\,G$ for $T = \mathrm{Psf}_{p,\overline{c}}$;
- Finite dp-rank model-complete theory of fields of positive characteristic, in particular ACVFG for $T = \mathrm{ACVF}_{p,p}$.

(H_4) in examples

Let $p \geq 0$ and $T_p = \mathrm{ACF} \restriction \{\cdot,^{-1},1\}$ the theory of the multiplicative group. T_p is given by adding to the theory of abelian groups the following sets of axiom:

- *If $p > 0$:* $\{\forall x\ \exists^{=n} y\ \ y^n = x \mid n \in \mathbb{N} \setminus p\mathbb{N}\} \cup \left\{\forall x \exists^{=1} y\ y^p = x\right\}$
- *If $p = 0$:* $\{\forall x\ \exists^{=n} y\ \ y^n = x \mid n \in \mathbb{N} \setminus \{0\}\}$.

By an affine variety, we mean an irreducible zariski-closed set. By a quasi-affine variety, we mean $V \cap \mathcal{O}$, where V is an affine variety and $\mathcal{O}$ is a zariski open set.

Fact (Johnson)

Let $\phi(x,y)$ be an $\mathscr{L}_{ring}$ formula, and $K \models \mathrm{ACF}_p$,

- *the set of c such that $\phi(K,c)$ is a quasi-affine variety is definable;*
- *if $\phi(K,c)$ defines a quasi-affine variety for all c, there exists $\tilde{\phi}(x,y)$ such that for all c, $\tilde{\phi}(x,y)$ is the zariski-closure of $\phi(K,c)$.*

(H_4) in examples

Lemma

Let p be a prime number or 0. For any formula $\phi(x, z)$ which defines a quasi-affine variety, there exists $\theta_\phi(z)$ such that for any model K of ACF_p and any tuple c from K, we have $K \models \theta_\phi(c)$ if and only if there exists $a \in L \succ K$ such that $\models \phi(a, c)$ and a is $\mathop{\smile\!\!\!\!\!\!\perp}^p$-independent over K.

Corollary

For $p \geq 0$, the expansion of ACF_p by a predicate for a multiplicative subgroup admits a model-companion, $\mathrm{ACFG}^\times$.

Proof of the lemma

Let $K \models \mathrm{ACF}$, $V \subseteq K^n$ an affine variety, $\mathcal{O} \subset K^n$ a Zariski open set. The following are equivalent:

1. for all $k_1, \ldots, k_n \in \mathbb{N}$, $c \in K$ the quasi affine variety $V \cap \mathcal{O}$ is not included in the zero set of $x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} = c$
2. for all $k_1, \ldots, k_n \in \mathbb{N}$, $c \in K$ the variety V is not included in the zero set of $x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} = c$
3. there exist $L \succ K$ and a tuple a which is multiplicatively independent over K and with $a \in (V \cap \mathcal{O})(L)$

$1 \implies 2$ is clear. If 2 holds, any generic a of V over K is in $V \cap \mathcal{O}$ and is $\mathop{\smile\!\!\!\!\!\mid}^{p}$-independent over K (otherwise it would satisfy such an equation), so 3 holds. If 3 holds, then
$\exists x \in V \cap \mathcal{O} \wedge (x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} \neq c)$ holds in L for each $c \in K$, hence also in K, so 1 holds.

Proof of the lemma

Condition 2 is first order:

Fact (Bays-Gavrilovich-Hils/Tran)

Let $\phi(x, y)$ an $\mathscr{L}$-formula such that for all tuple b in a model of ACF_p, $\phi(x, b)$ defines an affine variety. Then there exists a formula $\theta_\phi(y)$ such that for $K \models \mathrm{ACF}_p$, we have $K \models \theta_\phi(b)$ if and only if for all $k_1, \ldots, k_n \in \mathbb{N}$, $c \in K$, the set $\phi(K, b)$ is not included in the zero set of $x_1^{k_1} \cdot \cdots \cdot x_n^{k_n} = c$.

Let $K \models \mathrm{ACF}_p$ and $\phi(x, z)$ defining quasi-affine varieties.There exists a formula $\tilde{\phi}(x, z)$ such that for all tuple c from K, the set $\tilde{\phi}(K, c)$ is the Zariski closure of $\phi(K, c)$.Now by the fact, there exists a formula $\theta(z)$ such that $K \models \theta(c)$ if and only if $\tilde{\phi}(K, c)$ is not included in the zero set of $x_1^{k_1} \cdot \cdots \cdot x_n^{k_n} = d$, for all $d \in K$, $k_1, \cdots, k_n \in \mathbb{N}$.Using the previous equivalent, $K \models \theta(c)$ if and only if there exist $L \succ K$ and a tuple a which is multiplicatively independent over K and with $a \models \phi(x, c)$.

Let T be an o-minimal expansion of the theory $T_0 = \mathrm{DOAG}$ of divisible ordered abelian groups in a language $\mathscr{L}$.

Definition

T has UEP (uniform endomorphisms property) if there is an $\mathscr{L}$-formula $\phi(x, \bar{y}, z)$ for which in every $M \models T$, there is an infinite definable set $J \subseteq M^{|\bar{y}|}$ such that for each $\bar{c} \in J$ there exists $\epsilon > 0$ such that the formula $\phi(x, \bar{c}, y)$ defines the graph of an endomorphism on a neighborhood of 0 with radius at least ϵ, and for no other $\bar{d} \in J$ does $\phi(x, \bar{d}, y)$ have the same germ at zero as $\phi(x, \bar{c}, y)$.

Theorem (Block-Gorman)

If T does not have UEP, then (H_4) holds.

Let $\mathrm{RCF}_{\mathcal{G}^\times}$ the expansion of RCF by a predicate for a dense divisible multiplicative subgroup of the positives, then $\mathrm{RCF}_{\mathcal{G}^\times}$ has a model-companion, $\mathrm{RCF}\mathcal{G}^\times$.

Non-examples

Example

Let T be the theory of a field of characteristic 0 in a language $\mathscr{L}$ containing , such that T is inductive. Let $\mathscr{L}_G = \mathscr{L} \cup \{G\}$ and let T_G be the $\mathscr{L}_G$-theory of models of T in which G is a predicate for an additive subgroup of the field.Let (K, G) be an existentially closed model of T_G. Then

$$S_K(G) := \{a \in K \mid aG \subseteq G\} = \mathbb{Z}.$$

In particular, the theory T_G does not admit a model-companion.

Example (Block-Gorman)

$(\mathbb{R}_{\exp}, \mathcal{G}^{\times})$ where G is dense divisible subgroup of $\mathbb{R}^{>0}$: no model-companion either ($\{x \mid \mathcal{G}^x = \mathcal{G}\} = \mathbb{Q}$).

Classification

$NSOP_1$, NTP_2

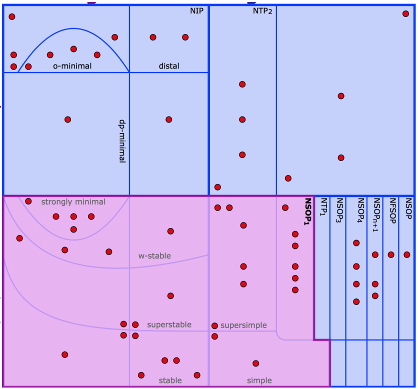

Figure: NSOP1

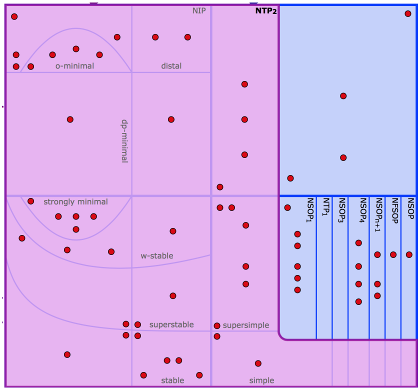

Figure: NTP2

Credits to G. Conant's wonderful interactive *map of the universe*
http://www.forkinganddividing.com

Classification

Independence relations and $NSOP_1$, Chernikov-Ramsey ; Kaplan-Ramsey

Let $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}$ be an invariant ternary relation in T.

- **Symmetry**.
  If $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{\mathscr{M}} B$ then $B \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{\mathscr{M}} A$.
- **Monotonicity**.
  If $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{\mathscr{M}} BD$ then $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{\mathscr{M}} B$.
- **Existence**.
  For any $\mathscr{M}$ and a we have $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{\mathscr{M}} \mathscr{M}$
- **Strong Finite Character**. For any model $\mathscr{M}$, if $a \not\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{\mathscr{M}} b$, then there is a formula $\phi(x, b, m) \in tp(a/b\mathscr{M})$ such that for all a', if $a' \models \phi(x, b, m)$ then $a' \not\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{\mathscr{M}} b$.
- **3-amalgamation**.
  For all model $\mathscr{M}$ if there exists tuples c_1, c_2 and sets A, B such that
  - $c_1 \equiv_{\mathscr{M}} c_2$
  - $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{E} B$
  - $c_1 \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{\mathscr{M}} A$ and $c_2 \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{\mathscr{M}} B$

Classification

Chernikov-Ramsey (2015) ; Kaplan-Ramsey (2017)

- **Symmetry**.
- **Monotonicity**.
- **Existence**.
- **Strong Finite Character**.
- 3-**amalgamation**.

} $NSOP_1$

- **Witnessing**.

} **Kim-Forking**

- **Base Monotonicity**.

} Simple

Independence relation in TS

Let $(T, T_0, \mathscr{L}_0)$ be a suitable triple ($(H_1 - H_4)$ and T_0 modular).
Assume that there is an independence relation $\mathop{\smile\!\!\!\!\!\!\,|}^T$ defined over subsets of models of T, and such that
$A \mathop{\smile\!\!\!\!\!\!\,|}^T_C B \implies \overline{AC} \cap \overline{BC} = \overline{C}$ for $\overline{A} = \mathrm{acl}_T(A)$.
We define the *weak independence* relation $\mathop{\smile\!\!\!\!\!\!\,|}^w$ by

$$A \mathop{\smile\!\!\!\!\!\!\,|}^w_C B \iff A \mathop{\smile\!\!\!\!\!\!\,|}^T_C B \text{ and } S(\mathrm{acl}_0(\overline{AC}, \overline{BC})) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC})).$$

Independence relations in TS

Theorem

*If $\mathop{\smile\!\!\!\!\!\!|}^{T}$ satisfies **Symmetry**, **Monotonicity**, **Existence**, **Strong Finite Character**, **Witnessing**, then so does $\mathop{\smile\!\!\!\!\!\!|}^{w}$.*
If further T satisfies the following:
(Condition $(*)$) for all A, B, C algebraically closed containing $E = \overline{E}$, if $C \mathop{\smile\!\!\!\!\!\!|}^{T}_{E} A, B$ and $A \mathop{\smile\!\!\!\!\!\!|}^{T}_{E} B$ then

$$(\overline{AC}, \overline{BC}) \mathop{\smile\!\!\!\!\!\!|}^{0}_{A,B} \overline{AB}.$$

*Then if $\mathop{\smile\!\!\!\!\!\!|}^{T}$ satisfies 3-**amalgamation**, so does $\mathop{\smile\!\!\!\!\!\!|}^{w}$.*

Independence relations in TS

Theorem

Assume that $\mathop{\smile\!\!\!\!\!\!|}^{T}$ *satisfies* ***Base Monotonicity***. *The following are equivalent.*

1. $\mathop{\smile\!\!\!\!\!\!|}^{w}$ *satisfies* ***Base Monotonicity***;
2. *For all algebraically closed sets* A, B, C, D *such that* A, B, D *contain* C *and* $A \mathop{\smile\!\!\!\!\!\!|}^{T}_{C} BD$, *the following holds*

$$\mathrm{acl}_0(A, \overline{BD}) \cup \overline{AD} = \mathrm{acl}_0(\overline{AD}, \overline{BD}).$$

In particular if acl_0 *is trivial or if* $\mathrm{acl}_0 = \mathrm{acl}_T$ *then* $\mathop{\smile\!\!\!\!\!\!|}^{w}$ *satisfies* ***Base Monotonicity***.

Preservation

Under assumption $(*)$, we may draw the following:

Configuration $T_0 \subseteq T$	Generic expansion TS
$T_0 = T$	Preserves stability
$T_0 \subseteq T$	Preserves NSOP_1
$T_0 = \emptyset$	Preserves simplicity

Back to the examples

1. Stable examples : pairs of geometric stable structures, $\mathrm{ACF}_{\mathrm{ACF}}$;
2. Simple examples : generic predicate;
3. NSOP_1 examples : ACFG, $\mathrm{ACFG}^{\times}$, $\mathrm{Psf}\,G$ algebraically bounded NSOP_1 theories of PAC_p fields;
4. SOP_1 and TP_2 examples : ACVFG, $\mathrm{RCF}\mathcal{G}^{\times}$.

Thanks

Thanks you !

Let $\langle a \rangle$ denote the $\mathbb{F}_p$-vector space spanned by a. The theory ACF_pG is obtained by adding to ACF_p the following axiom-schema; for all $\mathscr{L}$-formula $\phi(x, y)$, $x' \subseteq x$, $y' \subseteq y$:

$$\forall y \left(\theta_\phi(y) \wedge \langle y' \rangle \cap G = \{0\} \rightarrow \exists x(\phi(x, y) \wedge \langle xy' \rangle \cap G = \langle x' \rangle)\right)$$

Completions. Completions of ACF_pG are given by the $\mathscr{L}^G$-isomorphism type of $(\overline{\mathbb{F}}_p, G(\overline{\mathbb{F}}_p))$, i.e. for two models (K_1, G_1) and (K_2, G_2) of ACF_pG,

$$(K_1, G_1) \equiv (K_2, G_2) \iff (\overline{\mathbb{F}}_p, G_1(\overline{\mathbb{F}}_p)) \text{ and } (\overline{\mathbb{F}}_p, G_2(\overline{\mathbb{F}}_p)) \text{ are } \mathscr{L}^G - \text{isomorphic.}$$

Algebraic closure. The algebraic closure in ACF_pG is given by the field theoretic algebraic closure.

Proposition

For every $n \in \mathbb{N}$ and G_0 subgroup of $\mathbb{F}_{p^n}$ there exists $G_0 \subset G \subset \overline{\mathbb{F}}_p$ such that $(\overline{\mathbb{F}}_p, G) \models ACF_pG$.

Consider a non principal ultrafilter $\mathscr{U}$ on the set of prime numbers, and a model $(\overline{\mathbb{F}}_q, G_q)$ of ACF_qG, for each prime q. What is

$$\prod_{q \in \mathscr{U}} (\overline{\mathbb{F}}_q, G_q) \ ?$$

Remark (Characteristic 0?)

If (K, G) is an existentially closed models of the the class of (K, G) with $char(K) = 0$, then $Stab(G) = \mathbb{Z}$. ***Not*** *axiomatisable.*

We define for A, B, C algebraically closed sets in a model of ACF_pG

$$A \mathop{\downfree}_C^{w} B \iff A \mathop{\downfree}_C^{ACF} B \text{ and } G(\overline{AC} + \overline{BC}) = G(\overline{AC}) + G(\overline{BC})$$

$$A \mathop{\downfree}_C^{st} B \iff A \mathop{\downfree}_C^{ACF} B \text{ and } G(\overline{ABC}) = G(\overline{AC}) + G(\overline{BC})$$

Theorem

- $\downfree^w$ *satisfies* ***Symmetry***, ***Monotonicity***, ***Existence***, ***Strong Finite Character***, ***3-amalgamation*** *so* ACF_pG *is* *NSOP1*. *It also satisfies* ***Witnessing***, *so* $\downfree^w$ *agrees with Kim-forking over models.* $\downfree^w$ *doesn't satisfy* ***Base Monotonicity***, *so* ACF_pG *is not simple, has* TP_2.
- $\downfree^{st}$ *satisfies every property except* ***Strong Finite Character*** *and* ***Witnessing***.

Remark (More properties for $\mathop{\smile\!\!\!\!\!\mid}^{w}$ and $\mathop{\smile\!\!\!\!\!\mid}^{st}$)

Actually, all properties listed in the last slide are satisfied over ***algebraically closed sets****. Furthermore both $\mathop{\smile\!\!\!\!\!\mid}^{w}$ and $\mathop{\smile\!\!\!\!\!\mid}^{st}$ satisfies* ***Finite Character****,* ***Extension*** *and* ***Transitivity****. $\mathop{\smile\!\!\!\!\!\mid}^{w}$ satisfies* ***Local Character*** *and $\mathop{\smile\!\!\!\!\!\mid}^{st}$ doesn't. $\mathop{\smile\!\!\!\!\!\mid}^{st}$ is stationnary over algebraically closed sets.*

Local Character. For all A countable, and B, there exists countable $B_0 \subseteq B$ such that

$$A \mathop{\smile\!\!\!\!\!\mid}_{B_0} B.$$

For $(K, G) \models ACF_pG$, consider the quotient map

$$\pi : K \to K/G.$$

By $(K, K/G)$ we mean the two sorted structure with one sort for the field K, one sort for the $\mathbb{F}_p$-vector space K/G, and the quotient map $\pi : K \to K/G$. This structure is interdefinable with (K, G) hence $NSOP_1$, and Kim-forking can be described. It also satisfies **3-amalgamation** over algebraically closed sets.

Theorem

$(K, K/G)$ *has weak elimination of imaginaries.*

Base Monotonicity and Forking

$$A \mathop{\smile\!\!\!\!\!\!|}_C^{wmon} B :\iff \forall D \subseteq \overline{BC}\ A \mathop{\smile\!\!\!\!\!\!|}_{CD}^{w} B.$$

$\mathop{\smile\!\!\!\!\!\!|}^{f}$ / $\mathop{\smile\!\!\!\!\!\!|}^{d}$ / $\mathop{\smile\!\!\!\!\!\!|}^{þ}$ = forking/dividing/thorn-forking independence relation.

Proposition

1. *$\mathop{\smile\!\!\!\!\!\!|}^{wmon}$ doesn't satisfy Local Character.*
2. *ACF_pG is not rosy ($\mathop{\smile\!\!\!\!\!\!|}^{þ} \to \mathop{\smile\!\!\!\!\!\!|}^{wmon}$).*
3. *Let A, B, C, D be algebraically closed, A, B, D containing C, $B \subseteq D$.*

   $$\text{if } A \mathop{\smile\!\!\!\!\!\!|}_C^{wmon} B \text{ and } A \mathop{\smile\!\!\!\!\!\!|}_B^{st} D \text{ then } A \mathop{\smile\!\!\!\!\!\!|}_C^{wmon} D.$$

4. $\mathop{\smile\!\!\!\!\!\!|}^{wmon} = \mathop{\smile\!\!\!\!\!\!|}^{f} = \mathop{\smile\!\!\!\!\!\!|}^{d} = \mathop{\smile\!\!\!\!\!\!|}^{þ}$.