Dp-Minimal Domains

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Motivations

How does a model-theoretic constraint on a structure (group, field, graph...) translate into an algebraic property of the structure?

1. Stable structure – "does not interpret an infinite half graph"
2. NIP structure – "does not code $\mathcal{P}(\mathbb{N})$"

1. Stable Fields Conjecture: Every infinite stable field is separably closed;
2. NIP Fields Conjecture: Every infinite NIP field is either real closed, separably closed or admits a non-trivial henselian valuation.

Fact 1

These conjectures are hard to prove.

Definition 2

1. Let T be a theory and $n \in \mathbb{N}$. An *inp-pattern of depth n* is a collection of formulae $(\phi_i(x, y))_{i<n}$, parameters $(b_{i,j})_{i<n,j<\omega}$ and integers $(k_i)_{i<n}$ such that
   - for every $i < n$, $\{\phi_i(x, b_{i,j}) \mid j < \omega\}$ is k_i-inconsistent;
   - for every function $\eta : n \to \omega$, $\left\{\phi_i(x, b_{i,\eta(i)}) \mid i < n\right\}$ is consistent.
2. T is *of burden n* if there are no inp-patterns of depth $n + 1$, and if there is an inp-pattern of depth n. T is *inp-minimal* if it is of burden 1.
3. T is NIP if there is no formula $\phi(x, y)$ and parameters $(a_i)_{i<\omega}$, $(b_I)_{I\subseteq\omega}$ such that $\phi(a_i, b_I) \iff i \in I$.
4. If T is NIP and of burden n, then we say that *T has dp-rank n*. If T is NIP and inp-minimal, then we say that *T is dp-minimal*.

Fact 3

[Joh18] Let F be a dp-minimal field, then F is algebraically closed, real closed, or admits a non trivial definable henselian valuation.

(This was recently extended by Johnson to finite dp-rank.)

What about dp-minimal integral domains?

Conventions

A domain means an integral domain, i.e. a subring of a field. Given a domain R and $a \in R$, we set $\langle a \rangle = Ra$. For an ideal I of R, $\sqrt{I} = \{a \in R \mid a^n \in I, n \in \mathbb{N}\}$ is the *radical* of I. An ideal I is *radical* if $\sqrt{I} = I$. A domain is *local* if it has only one maximal ideal.

A *valuation ring* is an integral domain whose principal ideals are linearly ordered by inclusion.

If R is a valuation ring and $K = Frac(R)$, then R induces a valuation map $v : K^\times \to \Gamma$ to some ordered abelian group (a group homomorphism satisfying $v(x+y) \geq \min\{v(a), v(b)\}$).

Examples of dp-minimal domains

- Every definable subring of a dp-minimal ring is dp-minimal.
- Thus $\mathbb{Z}_p$ (which is definable in $(\mathbb{Q}_p, v, +, \cdot, 0, 1)$) is a dp-minimal ring.
- $\mathbb{F}_p^{alg}((t^{\mathbb{Q}}))$ is the algebraically closed field whose elements are formal series
$$\sum_{r \in \mathbb{Q}} a_r t^r,$$
with well-ordered support, i.e. $\{r \in \mathbb{Q} : a_r \neq 0\}$ is well-ordered. We may define a valuation $v : \mathbb{F}_p^{alg}((t^{\mathbb{Q}})) \to \mathbb{Q}$ by
$$v\left(\sum_{r \in \mathbb{Q}} a_r t^r\right) = \min\{r \in \mathbb{Q} : a_r \neq 0\}.$$

A question

Dp-minimal domains: are they all valuation rings?

No.

Example in positive characteristic

Consider $K = \mathbb{F}_p^{alg}((t^{\mathbb{Q}}))$, the Hahn series over $\mathbb{F}_p^{alg}$ with value group $\mathbb{Q}$, together with the natural valuation v.
Consider $\mathbb{F}_p + \{x \mid v(x) \geq 1\}$, it is a definable subring of K and hence dp-minimal.
It is a local ring of equicharacteristic $p > 0$, it is *not* a valuation ring.
To see this directly, the ideals $\mathbb{F}_p t + \{x \in K \mid v(x) \geq 2\}$ and $\{x \in K \mid v(x) > 1\}$ are incomparable.

Example in mixed characteristic

Let $(\mathbb{Q}_p, v)$ be the p-adic numbers for $p \neq 2$, it is dp-minimal by [DGL11, Theorem 6.13]. Let $K := \mathbb{Q}_p(\sqrt{p})$ be the totally ramified finite extension given by adjoining the square root of p, it is also dp-minimal (together with the valuation v).
The ring $R := \{0, \ldots, p-1\} + \{x \in K : v(x) \geq 1\}$ is definable and hence dp-minimal.
It is a local ring of mixed characteristic, it is *not* a valuation ring: $\langle p \rangle$ and $\langle p\sqrt{p} \rangle$ are incomparable.

Lemma 4

Let R be an integral domain of burden $n \in \mathbb{N}$ and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$ be proper prime ideals of R. Then there exists $1 \leq i_0 \leq n+1$ such that $\mathfrak{p}_{i_0} \subseteq \bigcup_{j \neq i_0} \mathfrak{p}_j$. In particular R has at most n maximal ideals.

Corollary 5

Let R be an inp-minimal domain, then the prime spectrum is linearly ordered by inclusion. In particular R is a local domain. Further, all the radical ideals are prime and there exists $N \in \mathbb{N}$ such that for all $a, b \in R$ either $b^N \in \langle a \rangle$ or $a^N \in \langle b \rangle$.

Proof.

A radical ideal I is the intersection of all prime ideals containing I, as the intersection of a chain of prime ideals is prime, I is prime. The rest follows from the fact that $\sqrt{\langle a \rangle}$ and $\sqrt{\langle b \rangle}$ are comparable, and the uniformity on N is by compactness. □

Proof of Lemma 4

Proof.

Assume not, then for each $1 \leq i \leq n+1$ there exists $a_i \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j$. Also, since the ideals are prime, $a_i^k \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j$ for all $k \geq 1$. For each $1 \leq i \leq n+1$, and $k \geq 1$, we set

$$X_i^k := \langle a_i^k \rangle \setminus \langle a_i^{k+1} \rangle.$$

The latter is nonempty: assume that $a_i^k \in \langle a_i^{k+1} \rangle$, then for some $b \in R$, $a_i^k = a_i^{k+1} b$. Since R is an integral domain, it follows that a_i is a unit in R, which contradicts that $\mathfrak{p}_i$ is an ideal. We now conclude that $\{x \in X_i^k\}_{1 \leq i \leq n+1, k \geq 1}$ is an inp-pattern of depth $n+1$. Let $k_1, \ldots, k_{n+1} \geq 1$. We claim that $a_1^{k_1} \cdot \ldots \cdot a_{n+1}^{k_{n+1}} \in X_1^{k_1} \cap \cdots \cap X_{n+1}^{k_{n+1}}$. Indeed, if not, without loss of generality, $a_1^{k_1} \cdot \ldots \cdot a_{n+1}^{k_{n+1}} = a_1^{k_1+1} b$, for some $b \in R$, then $a_2^{k_2} \cdot \ldots \cdot a_{n+1}^{k_{n+1}} \in \langle a_1 \rangle \subseteq \mathfrak{p}_1$. Consequently, $a_j \in \mathfrak{p}_1$ for some $j \neq 1$, contradicting the choice of the a_i. To complete the argument, note that the rows are 2-inconsistent: as before, since R is an integral

Localisation

Let R be a domain and $K = \mathrm{Frac}(R)$. An *overring* of R is a subring of K containing R. If $S \subseteq R \setminus \{0\}$ is a multiplicatively closed set, we denote $S^{-1}R$ the overring of R consisting of elements of the form $\frac{a}{s} \in K$ with $a \in R$ and $s \in S$. For a prime ideal $\mathfrak{p}$ of R, $R_{\mathfrak{p}}$ denotes $(R \setminus \mathfrak{p})^{-1}R$.

Lemma 6

Let R be an integral domain and S a multiplicatively closed subset of R.

1. *If S is definable then the burden of R is equal to the burden of $(S^{-1}R, R)$. In particular if R is NTP_2 then so is $(S^{-1}R, R)$.*
2. *If S is externally definable in R and R is NIP then $(S^{-1}R, R)$ is NIP and as a result, by (1),*
   $\textit{dp-rk}(R) < \kappa \iff \textit{dp-rk}(S^{-1}R, R) < \kappa$, *for any cardinal κ.*

Henselianity

A valuation ring is henselian if the valuation it defines on its field of fraction is henselian.

Fact 7

Let F be a valued field.

- *If* $char(F) > 0$ *and F is NIP, then F is henselian ([Joh20]).*
- *If F is dp-minimal, then it is henselian ([JSW17],[Joh16]).*

Corollary 8

Let R be valuation ring.

- *If* $char(R) > 0$ *and R is NIP then R is henselian.*
- *If R is dp-minimal then R is henselian.*

A criterion for being a valuation ring

Fact 9

[Sim15, Proposition 4.31] Let G be a inp-minimal group and H, N definable subgroups. Then either $|H/H \cap N| < \infty$ or $|N/H \cap N| < \infty$.

Lemma 10

Let R be an inp-minimal integral domain with maximal ideal $\mathscr{M}$. If R contains an infinite set F such that $F - F \subseteq R^\times \cup \{0\}$ then R is a valuation ring. In particular, if $R/\mathscr{M}$ is infinite then R is a valuation ring.

Proof.

Assume that such a set F exists, and let $(f_i)_{i<\omega} \subseteq F$ be such that $f_i \neq f_j$ for all $i \neq j$. Let $a, b \in R$ be nonzero elements. From Fact 9, without loss of generality, assume that $\langle a \rangle / (\langle a \rangle \cap \langle b \rangle)$ is finite. As $(f_i a)_i \subseteq \langle a \rangle$, there exists $i \neq j$ such that $(f_i a - f_j a) \in \langle b \rangle$. As $F - F \subseteq R^\times \cup \{0\}$, $a \in \langle b \rangle$. □

Prime ideals in dp-minimal domains

Let R be an inp-minimal domain with maximal ideal $\mathscr{M}$.

1. For each $a \in R \setminus \{0\}$ there exists a unique ideal P_a such that P_a is maximal with the property $P_a \cap \{a^n \mid n \in \mathbb{N}\} = \emptyset$. P_a is prime and externally definable.
   $P_a = \{x \in R \mid \forall n \in \mathbb{N}\ a^n \notin \langle x \rangle\}$.
2. For any prime ideal $\mathfrak{p}$, $\mathfrak{p} = \bigcap_{a \in R \setminus \mathfrak{p}} P_a$. In particular, every prime ideal is externally definable.
3. If R is dp-minimal and $\mathfrak{p}$ is a prime ideal, then $(R_{\mathfrak{p}}, R)$ is dp-minimal in the language of rings with a predicate for R.

Note that for any $a \in R$, $R_{P_a} = S^{-1}R$, where $S = \{a^n \mid n \in \mathbb{N}\}$. The ideals of the form P_a are the so-called *Goldman ideals* in the literature. $P_a \subsetneq \sqrt{\langle a \rangle}$.

Non-maximal prime ideals

In a local ring, every non-maximal radical ideal has infinite index in the maximal ideal $\mathscr{M}$ (as additive groups). Indeed, if $\mathfrak{r} \subsetneq \mathscr{M}$ is a radical ideal then for any $b \in \mathscr{M} \setminus \mathfrak{r}$ and $n \neq m \in \mathbb{N}$, b^n and b^m are in different classes modulo $\mathfrak{r}$.

Corollary 11

Let R be a dp-minimal domain with maximal ideal $\mathscr{M}$ and $\mathfrak{p}$ a non-maximal prime ideal. Then $R_{\mathfrak{p}}$ is a valuation ring.

Proof of Corollary 11

As $\mathfrak{p}$ is radical, $\mathfrak{p}$ has infinite index in R.
Observe that $|R/\mathfrak{p}| \leq |R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}|$. Hence the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ of $R_{\mathfrak{p}}$ has infinite index.
Since $\mathfrak{p}$ is externally definable, $(R_{\mathfrak{p}}, R)$ is dp-minimal. As the residue field is infinite, $R_{\mathfrak{p}}$ is a valuation ring.

Divided domains

Definition 12

([Aki67],[Dob76]) A ring R is *divided* if for all $a \in R$ and all prime ideal $\mathfrak{p}$ we have $\mathfrak{p} \subseteq \langle a \rangle$ or $\langle a \rangle \subseteq \mathfrak{p}$. Equivalently, for all prime ideals $\mathfrak{p} = \mathfrak{p} R_{\mathfrak{p}}$.

- Valuation ring : $\langle a \rangle$ and $\langle b \rangle$ are comparable;
- Divided ring : $\langle a \rangle$ and $\mathfrak{p}$ are comparable;
- Local treed domain : $\mathfrak{p}$ and $\mathfrak{q}$ are comparable.

Theorem 13

Every dp-minimal domain is divided. In particular, there exists $N \in \mathbb{N}$ such that for all a, b in the maximal ideal, either $a \in \langle b \rangle$ or $b^N \in \langle a \rangle$.

Dp-minimal valuation rings

A characterisation of dp-minimal valuation rings.

Theorem 14

Let R be a dp-minimal domain. R is a henselian valuation ring if and only if one of the following holds

- *R has infinite residue field;*
- *R has finite residue field and the maximal ideal is principal.*

Proof of Theorem 14

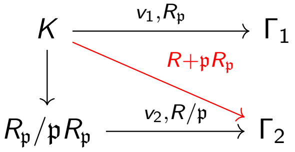
Assume first that R is a valuation ring and that R has finite residue field. As $K = \mathrm{Frac}(R)$ is dp-minimal, it follows from [Joh16] that $\mathrm{char}(K) = 0$ and $[0, v(p)]$ is finite. In particular the maximal ideal is principal.

Proof of Theorem 14

Conversely we already saw that if $R/\mathscr{M}$ is infinite, R is a valuation ring, so we may assume that $\mathscr{M} = \langle \pi \rangle$ and $R/\mathscr{M}$ is finite. Let $\mathfrak{p} = P_\pi$ be the maximal ideal not intersecting $\{\pi^n \mid n \in \mathbb{N}\}$.
Claim: $R/\mathfrak{p}$ is a discrete valuation ring.
We show that $R/\mathfrak{p}$ is Noetherian, it is standard that a local Noetherian domain whose maximal ideal is principal is a discrete valuation ring. By definition, for all $c \in R \setminus \mathfrak{p}$, there exists $n \in \mathbb{N}$ such that $\pi^n \in \langle c \rangle$. In particular for all $a \in R/\mathfrak{p}$, $\pi^n + \mathfrak{p} \in a(R/\mathfrak{p})$ for some $n \in \mathbb{N}$. This implies that $(\pi^n + \mathfrak{p})R/\mathfrak{p} \subseteq a(R/\mathfrak{p})$, hence, as $(\pi + \mathfrak{p})(R/\mathfrak{p})$ is of finite index in $R/\mathfrak{p}$, $a(R/\mathfrak{p})$ is of finite index in $R/\mathfrak{p}$. It follows that every ideal of $R/\mathfrak{p}$ has finite index, hence $R/\mathfrak{p}$ is Noetherian.

Compositions

Now we have that $R/\mathfrak{p}$ and $R_\mathfrak{p}$ are both valuation rings.
Let v_1 be the valuation on $K = \mathrm{Frac}(R)$ associated to $R_\mathfrak{p}$.
Let v_2 the valuation on the residue field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ of v_1 associated to the valuation ring $(R + \mathfrak{p}R_\mathfrak{p})/\mathfrak{p}R_\mathfrak{p} \cong R/\mathfrak{p}$
$(\mathrm{Frac}((R + \mathfrak{p}R_\mathfrak{p})/\mathfrak{p}R_\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$.
Then the composition of these two valuations give rise to a valuation on K whose valuation ring is $R + \mathfrak{p}R_\mathfrak{p}$.



As R is divided, $R + \mathfrak{p}R_\mathfrak{p} = R$ hence R is a valuation ring.

Back to the example

Let $(\mathbb{Q}_p, v)$ be the p-adic numbers for $p \neq 2$, it is dp-minimal. Let $K := \mathbb{Q}_p(\sqrt{p})$ be the totally ramified finite extension given by adjoining the square root of p.
The ring $R := \{0, \ldots, p-1\} + \{x \in K : v(x) \geq 1\}$ is definable and hence dp-minimal.
It is a local ring of mixed characteristic, it is *not* a valuation ring: $\langle p \rangle$ and $\langle p\sqrt{p} \rangle$ are incomparable.
The maximal ideal is not principal, $\langle p \rangle$ is not the maximal ideal since it does not contain $p\sqrt{p}$. In fact, the maximal ideal is generated by p and $p\sqrt{p}$.
This shows that it is not a valuation ring and that for dp-minimal domain, finite residue field does not imply that the maximal ideal is principal.

Equicharacteristic

Theorem 15

Assume that R is of equicharacteristic $p \geq 0$. The following are equivalent.

1. *R is a henselian valuation ring;*
2. *R is integrally closed;*
3. *R is root-closed (in its field of fractions);*
4. *R has an infinite residue field;*
5. *R has an infinite subring which is a field (necessarily $\mathbb{Q}$ or $\mathbb{F}_p^{alg}$);*
6. *$(char(R) = p > 0)$ R is Artin-Shreier closed.*

A remark

Let $\mathcal{O}_p$ be the valuation ring of the valued field $\mathbb{F}_p^{alg}((t^\Gamma))$, for some divisible ordered abelian group Γ.
Let I_p any ideal of $\mathcal{O}_p$, then $R_p := \mathbb{F}_p + I_p$ is not a valuation ring.
Let $\mathcal{U}$ be a non-principal ultrafilter on the set of prime numbers.
The ultraproduct $\prod_\mathcal{U} \mathcal{O}_p$ is dp-minimal since it is the valuation ring of an algebraically closed valued field, however $\prod_\mathcal{U} R_p$ is not even inp-minimal.
Indeed, it is not a valuation ring (as none of the R_p are), but it has a pseudo-finite –hence infinite– residue field so it is not inp-minimal.

Externally definable domains

Theorem 16

Let R be a domain and $\mathcal{O}$ a valuation ring of $\mathrm{Frac}(R)$ such that $(R, \mathcal{O})$ is dp-minimal. Then one of the following holds:

- *$\mathcal{O} \subseteq R$, hence R is a valuation ring.*
- *$R \subseteq \mathcal{O}$ and*
  - *if $\mathcal{O}$ is dominant, $\mathrm{Spec}(R) \setminus \{\mathscr{M}\}$ is an initial segment of $\mathrm{Spec}(\mathcal{O}) \setminus \{\mathfrak{m}\}$;*
  - *if $\mathcal{O}$ is non-dominant, $\mathcal{O} = R_{\mathfrak{p}}$ for some non-maximal prime ideal $\mathfrak{p}$ of R.*

Three consecutive primes

Let ${}^*\mathbb{R}$ be the hyperreals (resp. ${}^*\mathbb{C}$ the hypercomplex) and ${}^b\mathbb{R}$ (resp. ${}^b\mathbb{C}$) the ring of bounded elements.

Fact 17

[EK19] In the prime spectrum of ${}^b\mathbb{R}$ and ${}^b\mathbb{C}$, there are no three consecutive elements.

We say that a domain with linearly ordered prime spectrum R has property $(\star)$ if it satisfies one of the following equivalent property:

1. $\{P_a \mid a \in \mathscr{M}\}$ is densely ordered by inclusion;
2. there are no three consecutive prime elements in $\mathrm{Spec}(R)$;
3. for all $a, b \in \mathscr{M}$ with $a, b \neq 0$, $P_a \neq \sqrt{\langle b \rangle}$.

We generalize the result of Echi and Khalfallah.

Theorem 18

Let D be a κ-saturated domain and R a $\bigvee$-definable local subring whose prime ideals are linearly ordered. Then R has property $(\star)$.

Three consecutive primes

Example 19

Let Γ be an ω-saturated ordered abelian group (for instance, a non-principal ultrapower of $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}$). Then any $\bigvee$-definable subring of $\mathbb{F}_p^{alg}[[t^\Gamma]]$ (if Γ is p-divisible), $\mathbb{C}[[t^\Gamma]]$ or $\mathbb{Q}_p[[t^\Gamma]]$ has property $(\star)$.

Example 20

Any $\bigvee$-definable subring of a non-principal ultrapower of $\mathbb{Z}_p$ has property $(\star)$.

Question

1. Our only examples of dp-minimal domains are definable in a dp-minimal valued field, are there other kind of examples? If R is dp-minimal, so is $R[[X^\Gamma]]$?
2. Let R be an integral domain. Do the three conditions :
   - R is divided;
   - $R_{\mathfrak{p}}$ is a henselian valuation ring;
   - $\mathrm{Frac}(R)$ and $R/\mathscr{M}$ are dp-minimal;

   imply that R is dp-minimal?

Thank You Very Much !

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A remark on NIP commutative rings

The following was observed by Simon.

Observation by Simon

Let R be an NIP commutative ring and $(\mathfrak{p}_i)_{i<\omega}$ an infinite family of prime ideals. Then there is $i_0 < \omega$ such that $\mathfrak{p}_{i_0} \subseteq \bigcup_{j \neq i_0} \mathfrak{p}_j$. Equivalently, there is no infinite antichain of prime ideals and in particular, there is only a finite number of maximal ideals.

Proof.

Assume otherwise and let $a_i \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j$, and for each finite $I \subseteq \omega$, set $b_I = \prod_{i \in I} a_i$. Let $\phi(x, y)$ be the formula $y \in \langle x \rangle$, i.e. $\exists z(y = zx)$. Then for every $i < \omega$ and finite set $I \subseteq \omega$, I claim that $\phi(a_i, b_I)$ holds if and only if $i \in I$. If $i \in I$ then $b_I \in \langle a_i \rangle$ is clear. Since the ideals $\mathfrak{p}_i$ are prime, $b_I \notin \bigcup_{j \notin I} \mathfrak{p}_j$ ($R \setminus p_j$ is multiplicatively closed) so if $i \notin I$ then $b_I \notin \langle a_i \rangle$. □

In a commutative ring, the maximal length of an antichain of prime ideals is bounded by the VC-dimension of the formula $y \in \langle x \rangle$.

Proof of Theorem 13

Let R be a dp-minimal ring with maximal ideal $\mathscr{M}$ and let $\mathfrak{p} \subsetneq \mathscr{M}$ be a prime ideal. Since $R \cap \mathfrak{p}R_{\mathfrak{p}} = \mathfrak{p}$, it will be enough to show that $\mathfrak{p}R_{\mathfrak{p}} \subseteq R$.

By dp-minimality of $(R_{\mathfrak{p}}, R)$, either $|R/\mathfrak{p}| < \infty$ or $|\mathfrak{p}R_{\mathfrak{p}}/\mathfrak{p}| < \infty$. Since $\mathfrak{p}$ has infinite index in R it must be the latter. Let $y_1, \ldots, y_n$ be representatives for the cosets of $\mathfrak{p}$ in $\mathfrak{p}R_{\mathfrak{p}}$. $\{1, y_1, \ldots, y_n\}$ generate $R + \mathfrak{p}R_{\mathfrak{p}}$ as an R-module, i.e. it is a finitely generated R-module. By an application of Nakayama's Lemma, there are no non-trivial finitely generated CPI-extensions of R, hence $R = R + \mathfrak{p}R_{\mathfrak{p}}$.