Minimal expansions of $(\mathbb{Z},+,0)$.

(Joint with E. Alouf)

Notes for a talk in Leeds at the BPGMT conf, summer 2017.

There has been quite few result about expansions of $(\mathbb{Z},+,0)$. The important ones, namely by Palacín & Sklinos, at first and then by Pillay & Conant; will be exposed in the first part.

The article of Conant about intermediate structures between $(\mathbb{Z},+,0)$ and $(\mathbb{Z},+,0,<)$ can be considerate as the starting point of all that has been done by E. Alouf and myself.

Three weeks ago an article of Conant came out on modnet which appears to be a huge breakthrough in the study of stable expansions of $(\mathbb{Z},+,0)$.

Although this note is a little more concerned with unstable ones.

I - Expansions of $(\mathbb{Z},+,0)$

II - Minimal expansions

III - An idea of the proof.

IV - Expectations & questions.

I - Expansions of $(\mathbb{Z},+,0)$.

Let $\mathcal{M}$ and $\mathcal{N}$ be two structures with the same domain. We say that **$\mathcal{M}$ is a reduct of $\mathcal{N}$** (or equivalently **$\mathcal{N}$ is an expansion of $\mathcal{M}$**) if every definable set in $\mathcal{M}$, maybe with parameters, is also definable in $\mathcal{N}$. (noted $\begin{smallmatrix}\mathcal{N}\\ |\\ \mathcal{M}\end{smallmatrix}$)

There is no need for inclusion of language.

Ex: $(\mathbb{R},<)$ is a reduct of $(\mathbb{R},+,\cdot)$

If $\mathcal{M}$ is a reduct of $\mathcal{N}$ and $\mathcal{N}$ is a reduct of $\mathcal{M}$, we say that $\mathcal{M}$ and $\mathcal{N}$ are **interdefinable**.

(not ω-stable)

- The structure $(\mathbb{Z},+,0)$ is superstable, of U-rk 1.

  It has QE in the language $\{+,-,0,1,(P_n)_{n\geq 2}\}$ ← $\mathbb{Z}$-gp rules

  (but no prob as $(\mathbb{Z},+,0)$ $(\mathbb{Z},+,0,1)$ are n def.)

without 1 QE and

This structure is quite well-understood. It is always interesting to ask how complicated can the expansions be. For instance, expanding $(\mathbb{Z},+,0)$ by a finite set doesn't change anything by definition as finite sets are definable in any language. However, expanding $(\mathbb{Z},+,0)$ by the graph of multiplication is the nastiest thing to do!

Here are some examples of interesting expansions (unstable)

- **Presburger arithmetic**: $(\mathbb{Z},+,0,<)$ (P_n)

  It has QE in $L=\{+,-,0,1,<,(P_n)_n\}$

  it is dp-minimal and unstable.

- [Alouf, d'Elbée] : $(\mathbb{Z},+,0,|_p)$ $v_p(x)\leq v_p(y)$ iff $x|_p y$

It has QE result, is unstable and dp-minimal.

- [E. Alouf] : $(\mathbb{Z}, +, 0, 1_{p_1}, \dots, 1_{p_n})$. It has a good QE result, is unstable of dp-rk $= n$ (for $p_i \neq p_j$ $i \neq j$).

- [Kaplan - Shelah] : $(\mathbb{Z}, +, 0, 1, Pr)$ where Pr is a predicate for primes and their opposite. It is supersimple unstable. (and decidable). (under some conj)

Now some stable expansions :

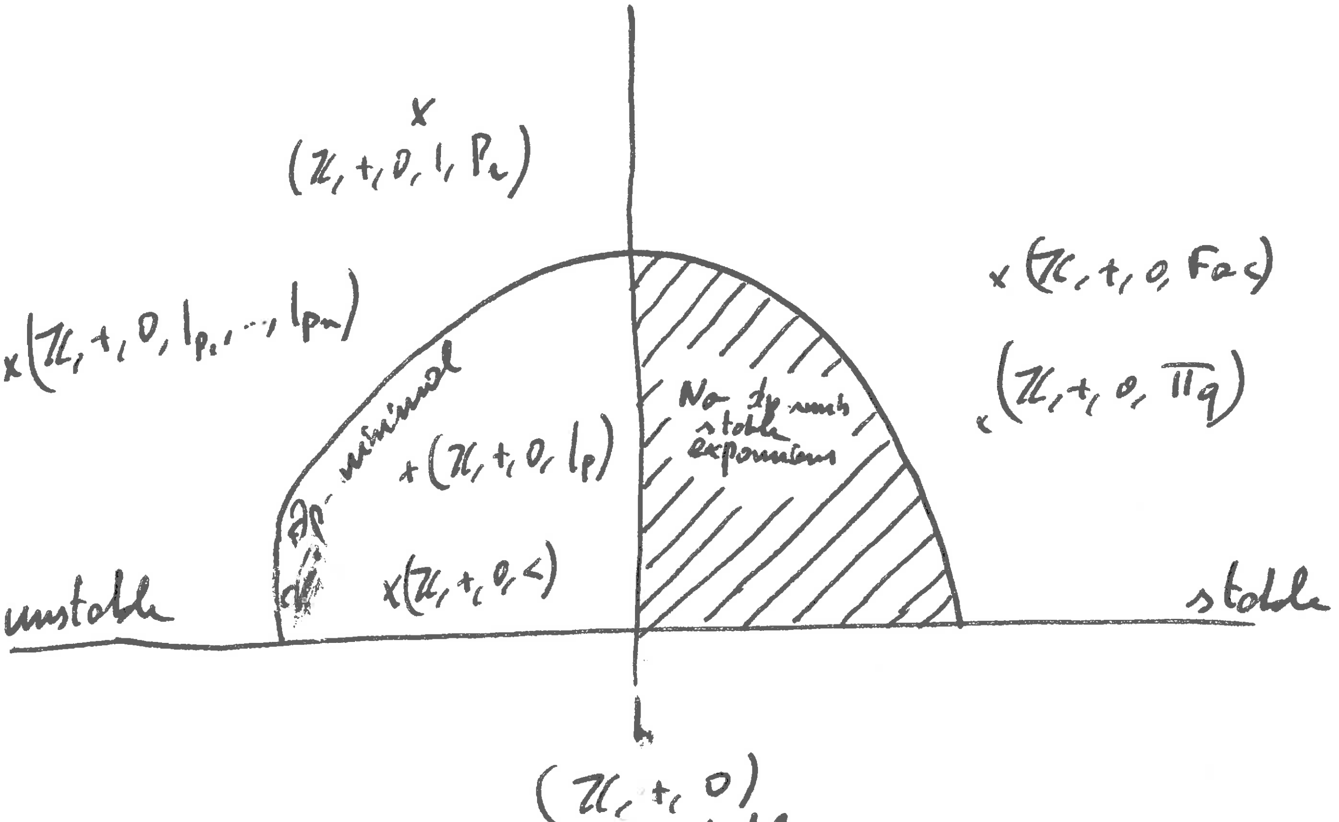
- [Palacin - Sklinos / Poizat] $(\mathbb{Z}, +, 0, \Pi_q)$ is superstable of U-rk ω. $\Pi_q = \{q^n, n \in \mathbb{N}\}$ with any $q \in \mathbb{N}^{\geq 2}$

- [Palacin - Sklinos] $(\mathbb{Z}, +, 0, Fac)$ is superstable of U-rk ω.

- [Palacin - Sklinos] There are no proper superstable [illegible] finite U-rk expansions of $(\mathbb{Z}, +, 0)$. (But there is a U-rk 1 expansion superstable of $(\mathbb{Z}^2, +, 0)$).

Later generalized by

- [Conant - Pillay] There are no finite dp-rk stable expansions of $(\mathbb{Z}, +, 0)$.



Unstable:

Question: Describe dp-minimal expansions of $(\mathbb{Z},+,0)$. Does it have a definable topology? (see section IV).

Stable:

Question: (Goodrick)

- Characterize the subsets $\Pi \subseteq \mathbb{Z}$ such that $(\mathbb{Z},+,0,\Pi)$ is (super) stable?
- $\underline{\qquad\qquad} \subseteq \mathbb{Z}^n$
  $\underline{\qquad\qquad}$?

News for Stable expansions: [Conant 2017, Stability and sparsity in sets of natural numbers]

If $A \subseteq \mathbb{N}$ and "have a strong sparsity assumption" then $(\mathbb{Z},+,0\ A)$ is stable.

↳ encompass all examples above and more $(\mathbb{Z},+,0,\mathrm{Fib})$ etc

+ some reciprocity: For some $A \subseteq \mathbb{N}$, if $(\mathbb{Z},+,0,A)$ is stable then "a fair amount of sparsity for A".

II - Minimal expansions.

We are focusing now on the other side of the scheme.

An expansion $\mathcal{N}$ $\mathcal{M}$ is called minimal if for every structure $\mathcal{N}'$ (with same universe as $\mathcal{M}$ and $\mathcal{N}$) which is a reduct of $\mathcal{N}$ and a strict expansion of $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{N}'$ are interdefinable.

Theorem [Conant 2016]: $(\mathbb{Z},+,0,<)$ is a minimal expansion of $(\mathbb{Z},+,0)$.

Theorem [Alouf, d'Elbée] $(\mathbb{Z},+,0,|_p)$ is a minimal expansion of $(\mathbb{Z},+,0)$.

Corollaries:
- Let A be a definable set in Presburger, then if A is not $(\mathbb{Z},+,0)$-def, $(\mathbb{Z},+,0,A)$ defines the order.
- Let $A \subseteq \mathbb{Z}^n$ a definable set in $\{+,0,|_p\}$. Then if A is not $\{+,0\}$-def, $(\mathbb{Z},+,0,A)$ defines $|_p$.

III - Ideas of the proof

The reason I stated the main result as a corollary is because, even the two results are equivalent:

(I) $(\mathbb{Z},+,0,<)$ is a minimal expansion of $(\mathbb{Z},+,0)$

(II) For every $A \subseteq \mathbb{Z}^n$ $<$-definable and not [illegible]$,+,0$-definable, $(\mathbb{Z},+,0,A)$ defines $<$.

We prove more directly the result (I) than the result (II). Indeed Conant proves directly (II) by an induction on n, whereas our proof works by the language. It is actually half a proof mea from the results before, knowing that every reduct of $(\mathbb{Z},+,0,<)$ or $(\mathbb{Z},+,0,|_p)$ is dp-min, we already know:

- If Z is a reduct of $(\mathbb{Z},+,0,<)$ and an expansion of $(\mathbb{Z},+,0)$ which is <u>stable</u>, then Z is interdefinable with $(\mathbb{Z},+,0)$.

(as $(\mathbb{Z},+,0,<)$ is unstable and Z is dp-min).

The same holds for $(\mathbb{Z},+,0,|_p)$. I will present the ideas of the proof for $(\mathbb{Z},+,0,<)$ ~~both~~ the proof of $(\mathbb{Z},+,0,|_p)$ is similar but a bit more complicated.

So we will work in unstable reducts. The only crucial tool is the following easy fact, which allows us to work in 1-dimension, and avoid an induction.

Lemma: Let $\mathcal{L}$ be any language and T an unstable $\mathcal{L}$-theory. ~~with [illegible]~~
Assume that $\mathcal{L}^- \subseteq \mathcal{L}$ and that $T\restriction_{\mathcal{L}^-}$ is stable.
Then there exists an $\mathcal{L}$-formula $\phi(x,y)$ with $|x| = 1$ and $b \in \mathcal{M} \models T$ such that
$\psi(x,b)$ is not equivalent to an $\mathcal{L}^-$-fml.

The proof is easy and uses the standard fact:
T is unstable iff there is an formula $\psi(x,y)$ unstable with $|x| = 1$.

What we want to show:

Let $Z = (\mathbb{Z}, +, 0, \ldots)$ be an unstable reduct of $(\mathbb{Z}, +, 0, <)$, then $<$ is definable in Z.

(Formally we would have $\begin{matrix}(\mathbb{Z},+,0,<)\\ \mid \\ Z \\ \mid \\ (\mathbb{Z},+,0)\end{matrix}$ and $+, 0$ not in the language of Z but as it is an expansion of $(\mathbb{Z},+,0)$ they are def so we add them to the language.)

So we assume these hypotheses. We work in a very big, saturated model $\mathcal{M}$ of $(\mathbb{Z}, +, 0, <)$. Call $\mathcal{L}$ the language of Z. $\mathcal{M}\restriction_{\mathcal{L}}$ is a model for Z. We apply the lemma with $\mathcal{L}^- = \{+, 0\}$ and $\mathcal{L}$ and we can find a $b \in \mathcal{M}$ and an $\mathcal{L}$-fml $\psi(x,y)$ with $|x| = 1$ and that $\psi(x,b)$ is not $\{+,0\}$-def (later denoted $+$-def).

We will not do the all proof, but just consider the following example:

Ex: We assume that there is $a \in \mathcal{U}$ positive nonstandard such that

$$\varphi(\mathcal{U}, b) \equiv [0, a]$$

The aim is now to define in the language $\mathcal{L}$ the order, or equivalently (as we have $+, -$ in $\mathcal{L}$) the set of positives. What we are actually going to do is to make an $\mathcal{L}$-formula which, in $\mathbb{Z}$, will define $\mathbb{N}$.

- $\{+, 0\}$-characterization of intervals closed:

$$I(y, z) = \varphi(0, y) \wedge \varphi(z, y) \wedge \neg\varphi(-1, y) \wedge \neg\varphi(z+1, y)$$

$$II(y, z) = \forall x \left(x \neq z \rightarrow \left(\varphi(x, y) \rightarrow \varphi(x+1, y)\right)\right)$$

$$III(y, z) \quad \forall x \left(x \neq 0 \rightarrow \left(\varphi(x, y) \rightarrow \varphi(x-1, y)\right)\right)$$

Call $\theta(y)$ the formula $\exists z \; I(y, z) \wedge II(y, z) \wedge III(y, z)$

The formula $\theta(y)$ says that there is a bound such that φ is stable by $+1$ and -1 between 0 and that bound.

Claim: $S(x) = \exists y \; \theta(y) \wedge \varphi(x, y)$ defines $\mathbb{N}$.

$$S(\mathbb{Z}) = \mathbb{N}.$$

There is an $\mathcal{L}$ formula, ~~add~~ without parameters. (except 1, that we can consider in the language)

① for every $b \in \mathbb{Z}$, $\varphi(x, b) \wedge \theta(b) = [0, a]$ (or [illegible]) $a \in \mathbb{Z}^{\geq 0}$

(Otherwise there is an elt outside and by ± 1 get to -1 or $a+1$) It suffices to have an elt in $[0,a]$ so that $\varphi(x,b) \wedge \theta(b) = [0,a]$).

② For every $n \in \mathbb{N}$ there is $b \in \mathbb{Z}$ such that

$$[0,n] \subseteq \varphi(x,b) \wedge \theta(b).$$

indeed, as the a of the hypothesis is nonstandard we have that $\mathcal{M} \restriction \Sigma \models \exists y \; [0,n] \subseteq \varphi(x,y)$ (fine with param $1 \in \mathbb{Z} \cap \mathcal{M}$) (1 as a parameter is no problem)

[Actually for more convenience, we may consider $(\mathbb{Z},+,0,1)$ as the bottom structure, as it is $\emptyset$-definable with $(\mathbb{Z},+,0)$.]

So $\mathcal{M} \restriction \Sigma \models \exists y \; [0,n] \subseteq \varphi(x,y)$, so the same holds in Z, and thus for all n.

So we conclude that $S(\mathbb{Z}) = \mathbb{N}$ and as S is an Σ-formula, Z defines the order.

<u>Remark</u> 1: For any Σ-formula $\varphi(x,b)$, we may have change $\varphi(x,b)$; construct a 0-$\{+,0,\varphi(x,b)\}$ formula such that $\varphi(\mathcal{M},b) = [0,a]$.

<u>Remark 2</u>: We actually could have started with any unstable reduct $(N,+,0,\dots)$ of $(N,+,0,<) \equiv (\mathbb{Z},+,0,<)$ as we would again have $S((\mathbb{Z},+,0,<)\restriction\Sigma) = \mathbb{N}$ so $(\mathbb{Z},+,0,<) \models S(x) \Leftrightarrow x \geq 0$ so $(N,+,0,<) \models S(x) \Leftrightarrow x \geq 0$.

The right statement of the theorem is :

Let $(N,+,0,<) \equiv (\mathbb{Z},+,0,<)$ and $(N,+,0,\dots)$ an unstable reduct. Then $<$ is definable in $(N,+,0,\dots)$ i.e. $(N,+,0,\dots)$ and $(N,+,0,<)$ are interdefinable.

Pf: Do the same as previously, with $\varphi(x,b)$ and if χ is the language of $(N,+,0,\dots)$ you get that

$$S\big((\mathbb{Z},+,0,<)\restriction\chi\big) = \mathbb{N},$$

so $(\mathbb{Z},+,0,<) \models S(x) \leftrightarrow x \geq 0$ ← and so $(N,+,0,<) \models S(x) \leftrightarrow x \geq 0$. ∎

⚠ here S is really a $\tilde{S}$ where the $\mathbb{Z}$ language has been translated to P_z

Remark 3: Conant has an example of an ordered group $(G,+,0,<) \equiv (\mathbb{Z},+,0,<)$ with proper stable reduct strictly between $(G,+,0)$ and $(G,+,0,<)$. This shows that our result is somehow optimal.

Remark 4: Remark 2 works just as well with $(\mathbb{Z},+,0,|_p)$ and the proof is very similar.

Remark 5: Our proof is not constructible. But it prove a constructible statement:

if $A \underset{\text{Pres. def}}{\subseteq} \mathbb{Z}^n$ then either A is $\{+,0\}$ def or there is an $\{+,0,A\}$-formula such that it defines $<$ (or $|_p$).

IV - Expectations & questions.

① Dp-minimal expansions.

Consider the following result from [Aschenbrenner, Dolich, Haskell, Macpherson, Starchenko : VC-density ... I]

|| No proper expansion of $(\mathbb{Z},+,0,<)$ is dp-minimal.

it is based on a similar result from [Belegradek-Peterzil-Wagner] on quasi-o-minimal expansions of $(\mathbb{Z},+,0,<)$. More recent : [Dolich-Goodrick] There are no proper strong expansions of Pres. arithmetic. The proof uses Michaux-Villemaire and a lemma from Simon.

A question previously asked by [Asch, Dol, ...] and by Conant is the following:

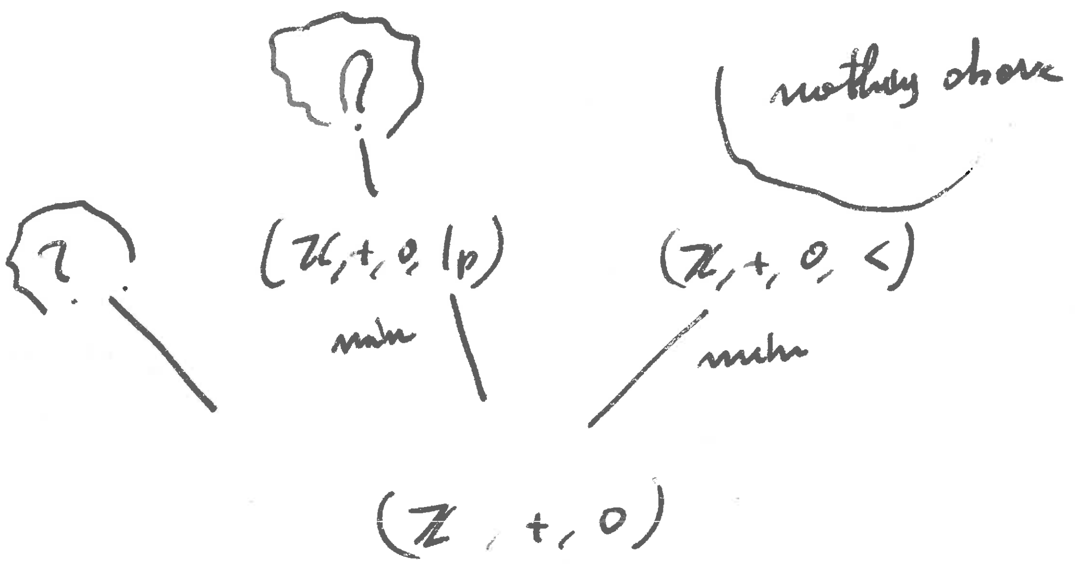
- Is every dp-minimal expansion of $(\mathbb{Z},+,0)$ a reduct of $(\mathbb{Z},+,0,<)$?

Of course the answer is no as $(\mathbb{Z},+,0,|_p)$ is another dp-min expansion of $(\mathbb{Z},+,0)$. The case of $(\mathbb{Z},+,0,|_p)$ is actually new and last year nobody knew another example of a dp-min expansion of $(\mathbb{Z},+,0)$.

A first question to ask is the following:

- Are they other dp-minimal expansions of $(\mathbb{Z},+,0)$? if yes, are they minimal as well?

dp-minimal expansions



Following the result of W. Johnson:

> If K is a field of dp-rk 1, then K is algebraically closed, real closed or admits a henselian valuation

and the general idea that NIP structures are ~~[illegible]~~ either stable, or constructed around order or constructed around trees we may make the following conjecture:

> A dp-min expansion of $(\mathbb{Z},+,0)$ is either interdefinable with $(\mathbb{Z},+,0)$ or $(\mathbb{Z},+,0,<)$, or is an expansion of $(\mathbb{Z},+,0,|_p)$.

Other questions: • Does every proper dp-min exp of $(\mathbb{Z},+,0)$ have a "definable" topology?

② Other similar results:

Marker is, to my knowledge, the first to have proven a result of minimal expansion. In his 1990 paper Semialgebraic Expansions of $\mathbb{C}$. he proves that:

If $S \subseteq \mathbb{R}^{2n}$ is semi-algebraic, then either
$\tilde{S} = \{(z_1 \ldots z_n) \in \mathbb{C}^n, \exists a_1 \ldots a_{2n} \in S \ z_i = a_{2i-1} + i \cdot a_{2i}\}$
is constructible (i.e Def in $\mathbb{C}^n$), either

$$\mathbb{R} \in \mathrm{Def}((\mathbb{C}, +, \cdot, \tilde{S}))$$

which is another way of saying that

$$\begin{array}{c} (\mathbb{C}, +, \cdot, \mathbb{R}) \\ | \\ (\mathbb{C}, +, \cdot) \end{array} \quad \text{is a minimal expansion.}$$

The proof given by Marker is very similar to that of Conant in the sense that it is an induction on n. I am certain that this result is actually provable with our method.

It was actually Marker who first ask the question wether there was something between $(\mathbb{Z}, +, 0)$, and $(\mathbb{Z}, +, 0, <)$.

③ An other crucial lemma?

Consider the two results from Pierre Simon's NIP bible:

① $\varphi(x,y)$ has IP iff there is an indiscernible sequence $(a_i : i<\omega)$ and a tuple b such that $\models \varphi(a_i;b)$ iff i is even.

② If a theory T has IP then there is $\varphi(x,y)$ with $|y|=1$ that has IP.

From these we get:

> If T is a IP theory in language $\mathcal{L}$, and $\mathcal{L}^- \subseteq \mathcal{L}$ such that $T\restriction_{\mathcal{L}^-}$ is NIP then there is a $\mathcal{L}$-formula $\varphi(x,y)$ with $|x|=1$ such that for some b in some model of T, $\varphi(x,b)$ is not $\mathcal{L}^-$-def.

<u>Proof</u>: Same as the case of unstable/stable: take $\varphi(x,y)$ IP $\mathcal{L}$-fml with $|y|=1$ and $(a_i)_{i<\omega}$, b such the $\varphi(a_i;b) \iff i$ even.
Assume that there is $\mathcal{L}^-$-fm $\psi(c_0,y)$ equiv to $\varphi(a_0,y)$ and set $c_k = \sigma_k(c_0)$ with $\sigma_k(a_0) = a_k$.
As $\models \forall y\,(\varphi(a_0,y) \iff \psi(c_0,y))$ we get that
i even $\iff \models \varphi(a_i;b) \overset{\sigma \text{ with } k}{\iff} \models \psi(c_i;b)$ so ψ has IP ∎

This could be used to study the IP reduct of the full arithmetic that are above $(\mathbb{Z},+,0,<)$

$$\begin{tikzcd}
(\mathbb{Z},+,\cdot) \arrow[d, dash, "\exists \text{ IP!}"] \\
(\mathbb{Z},+,0,<)
\end{tikzcd}$$