MODEL THEORY OF THE MULTIPLICATIVE GROUPS OF FIELDS

1. Multiplicative group of an algebraically closed field

1.1. **Model theoretic description.** Let p be a prime number or 0. We will denote by $\mathbb{N}_p$ the set of nonzero natural numbers which are not divisible by p. If $p = 0$ it is the set of nonzero natural numbers. $(\mathbb{N}_p, \cdot)$ is a submonoide of $(\mathbb{N}, \cdot)$. Furthermore $\mathbb{N}_p$ inherits the divisibility relation from $\mathbb{N}$, since if $a \in \mathbb{N}_p$ and $b \in \mathbb{N}$ is such that $b \mid a$ then $b \in \mathbb{N}_p$.

In the language $\left\{\cdot, ^{-1}, 1\right\}$ of groups (denoted multiplicatively, for obvious reasons), let T_p be the theory of abelian group with the following set of axioms if p is prime

$$\{\forall x \exists^{=n} y \;\; y^n = x \mid n \in \mathbb{N}_p\} \cup \left\{\forall x \exists^{=1} y \; y^p = x\right\}$$

and if $p = 0$

$$\{\forall x \exists^{=n} y \;\; y^n = x \mid n \in \mathbb{N}_0\}.$$

We call a model of T_p an *M-divisible group of characteristic p.*

The following is a standard group theoretic fact, see for instance [Kap54]

{fact_tors}

Fact 1.1. *Let G be an abelian group, then $G[tor]$ the subgroup of torsion is a direct factor G, hence there exists F torsion-free such that*

1. $G = G[tor] \cdot F$
2. $G[tor] \cap F = \{1\}$

We will denote the previous two conditions by $G = G[tor] \odot F$.

Definition 1.2. Let $G \models T_p$ and $A \subseteq G$.

- We denote by A^{Mdiv} the set $\{g \in G, g^n \in \langle A\rangle$ for some $n \in \mathbb{N}\}$, with $\langle A\rangle$ the subgroup of G spanned by A.
- We denote by $\mu_n(x)$ the formula $x^n = 1$, and $\mu(G)$ for the set $\bigcup_{n\in\mathbb{N}_0} \mu_n(G)$.

Remark. If $G \models T_p$ then for all $n \in \mathbb{N}_p$ we have that $\mu_n(G)$ is a cyclic group of order n. Indeed we have that k divides n if and only if $\mu_k(G)$ is a subgroup of $\mu_n(G)$, and by the axioms $\mu_n(G)$ has one and only one subgroup of order k for all k dividing n, hence by the characterization of cyclic groups by the subgroups, $\mu_n(G)$ is cyclic. Note that if p is prime and $G \models T_p$ then

$$\mu(G) = \bigcup_{i<\omega} \mu_n(G) = \bigcup_{n\in\mathbb{N}_p} \mu_n(G)$$

as $\mu_{p^k n}(G) = \mu_n(G)$ for all $k, n \in \mathbb{N}$. Remark that the union is a directed as $(\mu_n(G))_{n\in\mathbb{N}_p}$ is a directed family of groups.

{lm_extend}

Lemma 1.3. *Let G and H be two models of T_p and $A \subseteq G$ and $B \subseteq H$ two subgroups such that there exists an isomorphism f between A and B. Then f extends to an isomorphism $\tilde{f}$ between A^{Mdiv} and B^{Mdiv}.*

Proof. Assume the hypothesis. By Fact 1.1, $A[tor]$ is a direct factor of A. Note that $A[tor] = \bigcup_{n\in\mathbb{N}_p} \mu_n(A) = \mu(G) \cap A$ and we will denote it $\mu(A)$. Now there is a torsion-free subgroup F_1 of G such that

$$A = \mu(A) \odot F_1.$$

We similarly find $\mu(B)$ the torsion of B and F_2 torsion-free such that

$$B = \mu(B) \odot F_2.$$

Let $f_1 = f \restriction \mu(A)$. As f preserves the n-torsion, we have that $f_1 : \mu(A) \to \mu(B)$ is an isomorphisms. We show that f_1 extends to $\tilde{f_1} : \mu(G) \to \mu(H)$. It is sufficient to show that for all $n \in \mathbb{N}_p$, we can extend $f_1 \restriction \mu_n(A) : \mu_n(A) \to \mu_n(B)$ to $\mu_n(G) \to \mu_n(H)$. But as $\mu_n(G), \mu_n(H)$ are cyclic, so are $\mu_n(A), \mu_n(B)$ so there exists a generator of $\mu_n(A)$ and b generator of $\mu_n(B)$ such that $f_1(a) = b$. Now if g is a generator of $\mu_n(G)$ we can find $k \in \mathbb{N}$ such that $g^k = a$. We can also find $h \in H$ generator of $\mu_n(H)$ such that $h^k = b$. Now we can extend f_1 between $\mu_n(G)$ and $\mu_n(H)$ by $g \mapsto h$. By union we have extended f_1 to $\tilde{f_1} : \mu(G) \to \mu(H)$. This means that we may assume $\mu(G) \subseteq A$ and $\mu(H) \subseteq B$.

We extend f to A^{Mdiv}.

Now assume that $g^n = a \in A$ and assume that n is minimal such that $g^n = a \in A$.Let $b = f(a)$ and take $h \in H$ such that $h^n = b$. We want to show that n is minimal with the property $h^n \in B$. Assume not, then there is i with $1 \leq i < n$ such that $h^i \in B$. If $d = gcd(i,n)$ then $h^d \in B$. Write $n = dn'$, set $b' = h^d$ and $a' = f^{-1}(b')$. It follows that $a'^{n'} = a$. We also have $(g^d)^{n'} = a$ so finally there is $\xi \in \mu_{n'}(G) \subseteq A$ such that $g^d = a'\xi$, and this contradicts the minimality of n.

Any element in $\langle Ag \rangle$ can be uniquely written as ag^i for some i such that $0 \leq i < n$. We extend f by sending ag^i to $f(a)h^i$. It is well defined and is an isomorphism. □

{prop_pregeo_mult}

Proposition 1.4. *The theory T_p has quantifier elimination in the language $\{\cdot, ^{-1}, 1\}$. It is complete. It is strongly minimal and $\aleph_1$-categorical. Furthermore for any A in a model of T_p, the algebraic closure is given by $\mathrm{acl}_p(A) = A^{Mdiv}$. Every algebraically closed set is a model of T_p.*

Proof. We do the classical back and forth. Let $G, H \models T_p$ sufficiently saturated and $A \subseteq G$, $B \subseteq H$ two small subsets and f a partial isomorphism between A and B. First it is clear that we can extend f to the subgroup generated by A, so we may assume that A and B are subgroups of G, H respectively. Now by Lemma 1.3 we can also assume that $A = A^{Mdiv}$ and $B = B^{Mdiv}$. Now look at some $g \in G \setminus A$. The quantifier-free type $p(x, A) = qftp(g/A)$ is implied by disjunction of formulae of the form

$$\bigwedge_i x \neq a_i.$$

Now as the group H is infinite, we can easily find a corresponding $h \in H$ satisfying $f(p(x, A))$, by compactness. So the theory T_p has quantifier elimination. By Lemma 1.3, $\mu(G)$ and $\mu(H)$ are isomorphic for any two models of T_p, and they are models of T_p, so the theory is complete. It is clearly strongly minimal, $\aleph_1$-categorical. The description of algebraic closure follows easily from quantifier elimination. The last part is just checking the axioms. □

Let G, H be models of T_p. We saw that $\mu(G)$ and $\mu(H)$ are isomorphic, and this is $\mathrm{acl}_p(\emptyset)$. We will denote it μ. Now for any model G of T_p we have that G/μ is divisible and torsion-free, so this is a $\mathbb{Q}$-vector-space, and we write $G = \mu \odot G/\mu$.

Lemma 1.5. *In a big model of T_p, we have that acl_p defines a pregeometry which is modular and the associated independence relation in T_p is given by*

$$A p_C B \iff \mathrm{acl}_p(AC) \cap \mathrm{acl}_p(BC) = \mathrm{acl}_p(C).$$

Proof. As T_p is strongly minimal, $\mathrm{acl}_p(.)$ defines a pregeometry. The dimension of some set in the sens of the pregeometry is given by the dimension of the associated $\mathbb{Q}$-vector space, i.e. $dim(A)$ is the dimension of $\mathrm{acl}_p(A)/\mu$ as $\mathbb{Q}$-vector space. Now this clearly satisfies the modular law, so the pregeometry is modular, see [TZ12, C.1.9]. □

1.2. **Models of T_p.** It is clear that given any field K of characteristic $p \geq 0$, the group $K^\times$ is a model of T_p. There is a natural question to ask: is every model of T_p isomorphic to the multiplicative group of some algebraically closed field of characteristic p? Let $G \models T_p$, then if $|G| \geq \aleph_1$, the mutiplicative group of any algebraically closed field K of characteristic p and same cardinality as G will be isomorphic to G by $\aleph_1$-categoricity.
Now we assume that $|G| = \aleph_0$. First, in the case of characteristic 0 if the acl_p-dimension of G (in the sense of the pregeometry acl_p) is equal to $\aleph_0$, then G is isomorphic to $\overline{\mathbb{Q}}^\times$. Indeed, in $\mathbb{Q}$, the set of prime numbers is multiplicatively independent, so the acl_0-dimension of $\overline{\mathbb{Q}}^\times$ is $\aleph_0$, so G and $\overline{\mathbb{Q}}^\times$ are isomorphic. It follows that the following are equivalent:

(1) G is a countable model of T_0;
(2) G is isomorphic to the multiplicative group of some model of ACF_0;
(3) the acl_0-dimension of G is $\aleph_0$;
(4) G is isomorphic to $\overline{\mathbb{Q}}^\times$.

As any field of characteristic zero contains $\mathbb{Q}$, it follows that if G has finite acl_0-dimension there exists no field k of characteristic 0 (algebraically closed or not) such that G is isomorphic to $k^\times$.

Finally we turn to the case $|G| = \aleph_0$ and prime characteristic. It is clear that $\overline{\mathbb{F}_p}^\times$ has dimension 0. Note that $\overline{\mathbb{F}_p(t)}^\times$ has dimension $\aleph_0$, since $(t-a)_{a\in\overline{\mathbb{F}_p}}$ is multiplicatively independent. It follows that if $G \models T_p$ and $|G| = \aleph_0$ then G is isomorphic to the multiplicative subgroup of an algebraically closed field of prime characteristic if and only if its dimension is either 0 of $\aleph_0$.

Let $\mathbb{Q}^{cyc}$ be the cyclotomic closure of $\mathbb{Q}$, i.e. the field $\mathbb{Q}(\mu)$. Then the multiplicative group of $\mathbb{Q}^{cyc}$ is a model of T_0, because it satisfies the axioms of T_0. Further, for every field extension K of $\mathbb{Q}^{cyc}$, $K^\times$ is a model of T_0. Similarly every field extension of $\overline{\mathbb{F}}_p$ is a model of T_p. It follows that the following are equivalent for any field K of characteristic 0

(1) $K^\times \models T_0$;
(2) K is an extension of $\mathbb{Q}^{cyc}$.

An the corresponding in positive characteristic:

(1) $K^\times \models T_p$;
(2) K is an extension of $\overline{\mathbb{F}}_p$.

In particular, by a famous result of Kaplan-Scanlon-Wagner, the multiplicative group of any NIP field of positive characteristic is a model of T_p.

Exercise 1. *Prove that for all $n \in \mathbb{N}$, there exists a "twisted addition" $\tilde{+}$ on $\overline{\mathbb{Q}}$ such that $(\overline{\mathbb{Q}}, \tilde{+}, \cdot)$ is a field of transcendence degree n. In particular $(\overline{\mathbb{Q}}, +, \cdot)$ embeds* strictly *in $(\overline{\mathbb{Q}}, \tilde{+}, \cdot)$ as a strict subfield.*

References

[Kap54] Irving Kaplansky. *Infinite Abelian Groups*. University of Michigan Publications in Mathematics vol. 2. University of Michigan Press, 1954.

[TZ12] Katrin Tent and Martin Ziegler. *A Course in Model Theory*, volume 40 of *Lecture Notes in Logic*. Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012.