Institut Camille Jordan
UMR 5208 du CNRS

Expansions et néostabilité en théorie des modèles

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Thèse de doctorat

Université Claude Bernard Lyon 1

Université Claude Bernard Lyon 1
École doctorale InfoMaths (ED 512)
Spécialité : Mathématiques
n^o. d'ordre : 2019LYSE1076

Expansions et néostabilité en théorie des modèles

Thèse présentée en vue d'obtenir le diplôme de
Doctorat de l'Université de Lyon

Soutenue publiquement le 20 juin 2019 par
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À mes nièces, Lucille et Marie-Lys.

Remerciements

Ces remerciements sont un peu longs. La raison à cela est simple, pendant ces quatre années de thèse, j'ai rencontré des gens exceptionnels, j'ai été soutenu par des gens exceptionnels que je connaissais déjà, et je n'ai jamais pris le temps de dire à chacun d'eux, *merci d'être exceptionnel*, lorsqu'il l'était pour moi. J'ai donc accumulé beaucoup de retard que je vais maintenant rattraper.

Je vais tout d'abord remercier mes directeurs de thèse. Un directeur de thèse, c'est une bonne âme qui choisit de prendre sous son aile un jeune et naïf étudiant qui, tout frais émoulu, décide d'avoir tout à apprendre. J'ai eu la chance insolente d'en avoir une poignée. Zoé, une mer pour moi, tu m'as porté tandis que je naviguais dans un océan d'imaginaires, de génériques en tout genre, et que je sombrais dans des variétés et des corps instables. Merci pour ton acuité mathématique insondable qui m'impressionnera toujours, mais aussi –et surtout– pour ton oreille attentive et sincère quand c'était un peu la tempête dans ma vie. Thomas, tu étais dans le bateau avec moi, au quotidien. Tu as écouté mes idées farfelues, mes questions naïves, tu m'as procuré stabilité et simplicité. Tu étais là quand je débarquais en panique dans ton bureau car mon lemme était faux ! Merci pour ta voix apaisante et ton soutien sans faille. Pierre, cela n'a pas duré trois ans, mais tu es à l'origine de toute cette histoire. Par tes conseils, ton intuition et ton amitié, tu m'as initié au monde de la recherche, et, je dois l'avouer, tu m'as manqué lorsque l'Atlantique s'est mis en travers de nos routes. Il y aura toujours une place dans mon cœur pour les théories NIP. Enfin Amador, même si ce n'était pas officiellement acté, tu as été pour moi comme un directeur. Tu m'as couvé depuis mes débuts dans le milieu, du LMFI à l'ICJ, et à travers de nombreuses conférences. Tu as été sincère et bienveillant envers moi et je te suis reconnaissant pour tous les bons moments, les conseils, les anecdotes (et les potins).

I know my writing is not always easy to grasp, hence I am very grateful to the referees Artem Chernikov and Itay Kaplan for accepting the painful task of reading my thesis, making useful comments and corrections, and writing reports. Also, you have both a crucial role in the discovery of the many wonders that occurs in the world of NSOP$_1$ theories. Without your work, I would probably still be trying to convince myself that, despite all evidences, ACFG must be a simple theory !

Je voudrais remercier Itaï Ben Yaacov et Françoise Point pour avoir accepté de faire partie du jury. Itaï, c'était un honneur de partager avec toi la vie de labo à l'ICJ et les goûters pré-séminaires animés par tes anecdotes de la politique universitaire. Françoise, tu me connais depuis mes début dans le monde de la théorie des modèles, tu m'as enseigné le cours avancé de théorie des modèles au LMFI, tu m'as conseillé de nombreuses fois, à Paris, ou bien en conférence, merci.

Par la qualité de ses membres, l'Institut Camille Jordan est un lieu de travail privilégié.

Je pense d'abord à l'équipe de logique de Lyon, une bien belle famille. Frank, toujours le sourire aux lèvres, inébranlable pilier de l'équipe, tu m'as nourri de réponses à mes questions de couloir, et d'escargot. Merci aussi de m'avoir fait découvrir le joli théorème d'Ulm, qui n'a rien à voir avec la rue. Merci à Ludovic et Julien, mes proches voisins, pour avoir supporté le joyeux bazar qui se tramait dans le bureau d'à côté. Merci aussi à Lionel pour sa présence toujours appréciée. Tuna, je me souviens bien sûr des lokum et autres pistaches que tu nous ramenais depuis Constantinople. Mais à l'heure où je pose ces lignes, ce sont tes mots que je grave dans ma mémoire alors que tu es incarcéré de l'autre côté du Bosphore : « témoigner du passé et du présent pour lutter contre l'oubli ». Teşekkür ederim. Bruno, pour m'avoir montré qu'il faut un théoricien des modèles pour faire des va-et-vient à travers le monde, pour m'avoir fait voyager sans quitter ma bibliothèque, pour m'avoir initié à la science obtuse des rétroflexes,
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Mais il s'avère qu'il existe des personnes exceptionnelles en dehors de l'équipe de logique ! Merci à Maria C. pour avoir supporté mes nombreuses irruptions dans son bureau, à Ale et Léon pour leur dévouement, à Régis pour sa ténacité et à Francesco pour sa joie. Merci à Maria K. pour son efficacité et sa bonne humeur, et pour m'avoir sorti d'un sacré pétrin, une nuit, à Mexico. Claire, il n'y aura jamais assez de chocolats parisiens pour te remercier d'avoir régularisé tant de retards à la bibliothèque. Betty, toujours souriante, merci pour ta joie.

Une pensée sincère et reconnaissante pour mon co-auteur Eran Alouf, avec qui j'ai eu l'honneur de mener le front contre une armée de fromages en tout genre.

I would like to thank the amazing and living community of model theorists, that I had the chance to discover during the past four years. My dear Daoud (never forget, habibi : el beit beitak), Pedro, Erick, Rémi, Mohsen, Nadav, Alfonso, Arthur, Ulla, Nigel, Peter, Tim, Julide, Lotte, Sylvy, Jan, Leo, Pierre, Sylvain, Jinhe and many others... I will always be grateful to Daniel and Franzi for organizing the MMM in 2016, and I hope events such as this or the trimester at the IHP last year will continue to glue together many generations of happy model-theorists. I am very grateful to Minh-Chieu and to Eric for many fruitful mathematical discussions and ideas. I would like to thank Alex K. for his insights and his interests in my work. I have a special thanks for Gabe, for the quality of your jokes and stories as well as your mathematical help. Your influence is palpable in my work, in Chapter 7 and in Chapter 9. Finally, Nick, for your tremendous work in NSOP$_1$ theories, for many discussions, about mathematics, about life, thanks. No olvidaré al maravilloso equipo Colombiano, quien me hospedaba à Bogotá : Mariana, Sasha, Alex B., John, Alf, Luis Jaime. Dario, por aconsejarme muchas veces cuando era un doctorador jovén (y ahora también), por *intentar* a enseñarme salsa, por riéndote a mis chistes malas, por volverte en mi amigo y en uno de los mosqueteros, muchas gracias.

Parce qu'un doctorant esseulé est un doctorant malade, j'aimerais à présent remercier celles et ceux que j'ai eu la chance de côtoyer à l'ICJ. Je commencerai bien sûr par le couloir orange, et son bureau 109e. D'abord, les anciennes et ceux de passage Nadja, Sylvain, Simon B., Benjamin, Isa qui m'ont accueilli à bras ouverts quand j'ai décidé de migrer depuis le deuxième étage. Nadja, ma grande sœur, tant de souvenirs partagés, merci de m'avoir enseigné des proverbes allemands, l'idéal pour briller en société, en voilà un glané récemment, qui, je trouve, convient à ces temps de conclusion :« Alles hat ein Ende, nur die Wurst hat zwei ». Sylvain, le Toï-Toï, le pub Danois, les discussions jusqu'au bout de la nuit, l'exploration des recoins de notre bureau chéri, merci ! Isa, habibti, for so many laughs and cries, from the waves of Scotland to the northern shores of Norderney, the musketeers of Oaxaca and the blau gelb monsters, and of course the many beers at La passagère, thanks. Puis, la joyeuse compagnie, les gai-lurons : Chih-Kang, Octave (tu es pour moi l'âme du bureau !), Garry (Terii ! sur un air de Brassens...), Abed (Boaye boaye ! Al-ḥamdu lillāh !). Clément, qui a prouvé qu'il en était digne, hérite par la présente de mon bureau pour la durée de sa thèse, je lui fais confiance pour le transmettre à son tour à un cadet plein d'entrain. Martin, merci d'avoir partagé avec moi tant de choses, de la bonne chère aux chaînes youtubes à travers de nombreuses analyses de films que je n'avais, pour la plupart, pas vu ; sans oublier toutes ces anecdotes généreusement distillées alors que nous ripaillions gaiement. Simon, aaah Simon... Par où commencer, les edelweiss de la tête de la Maille ? Non, trop haut. La dent de Crolles ?... Glacial ! Les currys aux monts d'or, les soirées cuisines, musique et autres, l'escalade, le Rain... Tu as été là si souvent pour me sortir d'affaire, si présent quand j'en avais besoin, merci mon ami. Sam, membre du 109e par extension, toujours au parfum des bonnes astuces, les pieds dans les vases des cata ou la tête dans les algues de Miribel, merci pour toutes ces aventures, pour ton amitié et pour ton soutien, euuhhh... salut ! Martin, Simon, Sam, pour avoir subi mes humeurs passagères, dans une atmosphère d'arête de poisson, sous l'œil intrigué du Bon Pasteur et de Saint Bernard, je vous remercie. Je voudrais remercier celles et ceux avec qui j'ai pu fouler les paroies de Curis-au-Mont-d'Or, Doizieux ou encore du Gouffre d'Enfer, Coline, Ariane, Simon A., Julie, Vincent, et bien d'autres. Hugo, merci de susciter en moi une joie inexplicable, et pour nos blues endiablés. Ces derniers mois de rédaction provoquaient en moi une bougeotte insatiable et une humeur variable, ainsi, pour toutes celles et ceux qui m'ont vu squatter leurs bureaux, parfois jusqu'à des heures tardives, qui ont supporté mes distractions, et qui m'ont soutenu, par leur présence ou leurs paroles, merci. Je pense surtout à Gwladys et Marion (qui, curieusement, partagent ma hâte que tout cela soit fini !), Olga (strong independent Siberian woman !), mais aussi aux autres membres du bureau 111A, Felix, Jiao, Mickaël, Benoît et du bureau 100, Lola, Pan, Rémy, Marina, Mélanie, Jorge. À toutes celles et ceux qui m'ont apporté un petit quelque chose de Domus lorsque je n'avais pas le temps de m'alimenter... Merci ! Un grand merci à toi Caterina, pour ton enthousiasme communicatif, ta joie de vivre, ton soutien et tous les bons souvenirs (« This life is not just a read-through »). Et puis il y a ceux que je n'ai seulement connu de loin, Quentin (un garçon plein de potentiel !), Thomas, Benjamin, Khan, Théo, Michele, Daniel et bien d'autres. Je n'oublie pas mes premiers co-bureau, Tuba et Manaf, ni l'ancienne, Cécilia. Finally, because you have been there since the beginning, because I owe you sooo much (wàn coffees, among other things), because you laugh at my jokes even when you don't get them (or, more likely, because they are not funny), because you helped me in my thesis and in my life, because you gave me food and because you are my friend, I thank you, Tingxiang.

Une pensée particulière pour mes camarades du LMFI, Antoine, pour nos passionnantes discussions et pour partager ma fascination pour le médiator du destin ; Paul-Elliot, qui aura toujours une place spéciale dans mon cœur après nos aventures de la tour Verdi. Alex, tu ronfles comme une locomotive à vapeur, certes, mais pour tout ce que l'on a partagé, pour la richesse et la qualité de ta personne, un grand merci !

Jules, fier et militant membre du B.R.I.E. et de l'Académie de la Tête de Veau, mon cher ami. Pour tout ce que l'on partage, parce que l'« on entendait ronronner la Durance » depuis les fenêtres de mon appartement lyonnais, pour Sibélius, Elzéard Bouffier, et bien d'autres encore, merci, « C'était dans les ramures bleues, le chant léger du vent de mai ». Que Lucien Jacques t'ait en sa sainte garde. Magali, merci pour ton soutien, ton amitié et ta personnalité qui m'est chère. Je n'oublie pas non plus mes autres amis parisiens pour tous ces bons moments dans la capitale, Théo, Agnès, Nico, Alice.

Adrien, mon cher barde eichendorffien, pour ton illustre compagnie en conférences, en vadrouille sous les douches écossaises ou dans ton appartement parisien, peu de maths ensemble mais beaucoup de vie (et de whisky), merci.

Je tiens à remercier mes cousins Lyonnais, Marie-Claire et Yves-Armel, Silouane, Lilas, Marie-Christine et Jean-François, pour votre accueil et ces bons moments partagés.

Un abrazo muy fuerte para mi querido padrino Rolando, por sus consejos y su presencia a mi lado desde mi primera juventud, ¡ Gracias !

Un grand merci à mon très cher Philippe, qui a supporté ma distraction et mon indisponibilité grandissante durant ces années de service à l'Ordre de Malte, et dont je continue d'admirer le sens du service, et de la dévotion. Nos après-services à saucissoner et canonner gaiement avec Louis refont toujours surface lorsque je me ballade rue Mercière. Je n'oublie pas bien sûr mes camarades LAT Amélie et Tiphaine, pour leur joie et leur soutien (et les dîners sans salades de pâtes !).

Je tiens à exprimer toute ma reconnaissance à mes professeurs bordelais Michel Matignon et Jean Fresnel, qui ont su faire du châtelet de Lamartine un lieu si convivial pour préparer mon agrégation, et susciter en moi le goût de la recherche mathématique. Les murs de ce lieu privilégié feront longtemps raisonner les rires de son bourbakiste châtelain.

Diane, parce que tu as été à mes cotés durant cette thèse, parce que tu as cru en moi et parce que tu m'as soutenu, pour tout ce que tu m'as fait découvrir, la voile, le cheval, l'usage des intelliphone et bien d'autres... Pour toute la force que tu m'as donnée, parce que tu as été « ma lisse, un nid » et un agrume multivitaminé, infatigable et indispensable pour la réalisation de ce travail et parce que « le rire se pétrit dans la cuve de ta bouche », un immense merci.

« Certains hommes espéraient entrer dans l'Histoire, nous étions quelques-uns à préférer disparaître dans la géographie. » Je me tourne à présent vers mes compagnons de voyage, baladins des temps romantiques, quand nous nous échappions sur les « chemin noirs », lorsqu'il était temps de « quitter la ville et de tirer sur les discours le rideau des forêts ». Romain, tu m'as accueilli à l'ICJ, tu m'as fais faire mes premiers pas d'escalade, ton violon a bercé mon appartement d'un mugre céleste... Nous avons contemplé des voûtes étoilés depuis les collines enneigées du Pilat, les plaines altières de l'Aubrac, ou encore le mystique plateau du Gévaudan, merci pour tout ! Grégoire, mi carpintero rojo, nos pérégrinations au Chili, de Tantauco au sommet du Villarrica, et puis les premières aventures des trois ponchos perdus dans les hébrides, Mull, Tomsleibh, Iona et le Ardmore de l'aube... Tanguy, pour toutes ces heures de guitare acharnées, pour les systèmes brochatoires et les nuits dehors, pour notre interminable traversée du Black Cuillin et la nuit épique qui a suivi (tes chaussettes s'en souviennent certainement mieux que moi), parce que dormir avec les loups dans les neiges du Vercors ne t'effraie pas, pour ta joie et ton entrain, merci. Matthieu, tu as toujours répondu à l'appel de la forêt, tu as chaussé ton poncho de Dalcahue lorsque revenait « le temps des bivouacs ». Dans les lacs vaseux de la Dombes, les cromlechs des Causses Majeurs, les pieds dans la tourbe noir de Skye ou sur le machair des Hébrides extérieures, nous n'avions « ni riches buffets, ni verres de Venise », simplement le besoin de s'échapper, et de ragueneauder allègrement, entre terre et mer. Pour tous les imprévus qui font nos aventures, pour ces nuits de débat dans le septentrion, pour ces bothy coiffés par les nuits boréales, pour les rôtis braisés dans les flammes éthérées, aux haggis plus très frais, poêlées de cornichons, levons nos gobelets, mon cadet bien Gascon ! Alba gu bràth !

« Il est un coin de France où le bonheur fleurit, où l'on connaît d'avance les joies du paradis ». Parce que notre amitié traverse les années, parce que mon pays natal ne serait rien sans vous, parce que vous êtes toujours là pour moi malgré mes absences, pour tant de souvenirs et de rires partagés, tant de parties de mus et de joyeuses ripailles, JB, Julien, Virginie, Francis, Lorraine, Rémy, Isabeau, Boris, merci. Martin, pour les joyeuses bredouilles, les aventures sous-marines, le surf et la pala, parce que l'on ne supporte pas d'avoir quelque chose entre les yeux et l'horizon marin, parce que l'on restera toujours jusqu'à la fin du coucher, et pour ton amitié sans faille, merci. Merci aussi à Laetitia pour sa personnalité si attachante. Adrien, pour le Teink, l'O.P.A.P, pour les mots qui sortent de ta bouche quand ça devient sérieux, pour les blagues vaseuses quand ça ne l'est plus du tout, merci. Mon cher Bastou, pour tout ce que l'on a toujours partagé, la mer, la montagne, l'humour (?), pour tous les souvenirs, de la Sierra de Guara aux nuits avec les loups, le Teink, patx'and co, la liste est sans fin. Enfin, pour continuer d'appeler alors que je ne réponds jamais, pour ta personnalité qui m'est si chère, merci. Merci aussi à Justine pour sa joie et sa bienveillance. Ander, on se connaît depuis nos plus tendres enfances, devant le fronton ou derrière un bon verre de cidre du pays, sur les bancs de l'école (Araban bagare, sans crier !) ou dans les herbes hautes, tu partages avec moi l'amour de la nature et cette fascination pour ce qui l'habite (on l'aura eu notre *Anax imperator* !), milesker nere lagun onena !

Eneko, depuis qu'on a usé nos fond de culotte sur les bancs de Lamartine, tant de blagues et de vie, à Bordeaux, à Bayonne, dans le Gers et même à Lyon, « lau teilatu gainean », milesker ainitz !

Titi, mon couz', pour les traditions, les souvenirs à Ola, à Gueth, la gratte, et les joyeuses aventures des trois cornichons, enfin, pour ton soutien quand les temps sont à l'orage, merci !

Un grand merci bien sûr à toute ma grande famille, pour son soutien de près ou de loin, à tous mes cousins et cousines, oncles et tantes, à Pierre et Valou pour votre accueil parisien et tous les souvenirs à Argia, ma chère Marie-Hélène parce que la vielle ferme est toujours un peu trop loin, Philippe pour ta passion des flots et des alizés, Élisabeth et Santi, à ma très chère marraine et à Sybille, à tous les gascons de Château-Vieux, et à Robin et Marie-Pierre, pour des séjours inoubliables dans le septentrion français à grand renfort de gambettes, de potjevleesch, dans les vapeurs de houblon et les effluves lancinantes du vieux Lille, merci. Vlaanderen !

Enfin, je me tourne vers Maitasunean, le berceau qui m'a vu naître et qui porte si bien son nom. Un immense merci à Suzy, pour m'avoir élevé, accompagné et aimé comme un fils, merci aussi pour ta joie inaltérable et ton sourire. J'ai une reconnaissance infinie envers mes parents, vous m'avez fait grandir en liberté, vous m'avez fait confiance, vous avez nourri la maison familiale de votre amour indestructible, vous m'avez enseigné le goût de la contemplation et l'amour des belles choses, merci. Thomas merci de partager avec moi des goûts musicaux si incompris, et pour tous les barbecues menés de main de maître. Ione, tus sonrisas son cómo soles andaluces sobre aceitunas arbequinas, gracias !

Clairon, tu as toujours soutenu mes choix avec entrain, tu as bercé mon enfance de mélodies envoûtantes, qui m'ont donné le plaisir des notes. Pour toutes les fois ou j'ai trouvé refuge dans ta tendresse de sœur, tes sourires réconfortant et ta joie inébranlable, merci.

Jean-Marie car j'ai tout appris de toi, des plongées aux Alcyons au vallon du Pouey Trenous, le canyoning sauvage, les bredouilles incessantes, la pêche à pied ou à la mouche, la chasse à l'arc, les vagues d'Elbée à Guéthary, et les palas au grand fronton, dans les pas de nos ancêtres... la liste est si longue, ma mémoire déborde de souvenirs qui fusent en couleurs dans toutes les directions, à travers le vert de nos sous-bois chéris, l'or de nos prés aux soirs d'automne, et le ciel embrasé qui coiffe le Jaizkibel, contemplé depuis les récifs des Alcyons... Pour avoir été le roc qui m'a soutenu, mon poisson préféré dans la mer, le sanglier qui m'a tant fait rire, merci beaucoup, mon frère.

Raphnoute, pour tes récits d'aventures sur les cimes, la pêche au gros et les champignons qui ont bercé mon enfance, pour ton esprit de baroudeur qui m'inspire, pour ta personnalité si attachante merci beaucoup, mon frère.

Marcou, parce que j'ai grandi avec toi à mon chevet, parce que l'on a passé le plus clair de notre enfance à *jouer aux frères* (et qu'on a jamais arrêté), à faire cramer des bergeries, manger des œufs, parce que l'on a redoublé d'imagination pour rendre nos vies propices aux plus belles aventures, merci beaucoup, mon frère.

Enfin, mes ultimes mercis, à Lucillette, parce que je serai toujours ton pagrain et à Marie-Lys, parce que j'ai la joie d'être ton tonton, mes deux distributeurs de joie dans ce monde de brutes.

« Ah çà ! mais vous ne pensez qu'à manger ?...
— Approche, Bertrandou le fifre, ancien berger ;
Du double étui de cuir tire l'un de tes fifres,
Souffle, et joue à ce tas de goinfres et de piffres
Ces vieux airs du pays, au doux rythme obsesseur,
Dont chaque note est comme une petite sœur,
Dans lesquels restent pris des sons de voix aimées,
Ces airs dont la lenteur est celle des fumées
Que le hameau natal exhale de ses toits,
Ces airs dont la musique a l'air d'être en patois !...

Le vieux s'assied et prépare son fifre.

Que la flûte, aujourd'hui, guerrière qui s'afflige,
Se souvienne un moment pendant que sur sa tige
Tes doigts semblent danser un menuet d'oiseau,
Qu'avant d'être d'ébène, elle fut de roseau ;
Que sa chanson l'étonne, et qu'elle y reconnaisse
L'âme de sa rustique et paisible jeunesse !...

Le vieux commence à jouer des airs languedociens.

Écoutez, les Gascons... Ce n'est plus, sous ses doigts,
Le fifre aigu des camps, c'est la flûte des bois !
Ce n'est plus le sifflet du combat, sous ses lèvre,
C'est le lent galoubet de nos meneurs de chèvres !...
Écoutez... C'est le val, la lande, la forêt,
Le petit pâtre brun sous son rouge béret,
C'est la verte douceur des soirs sur la Dordogne,
Écoutez, les Gascons : c'est toute la Gascogne ! »

Table des matières

Introduction en français

Avant propos

Ce manuscrit est organisé comme suit. Les Préliminaires exposent les résultats de base ainsi que quelques lemmes préparatoires. La partie A traite de l'expansion générique d'une structure par un prédicat pour un sous-modèle d'un réduit. Les résultats de la partie A sont issus de mes deuxième, troisième et quatrième années de doctorat, sous la supervision de Thomas Blossier et Zoé Chatzidakis. Deux articles sont disponibles en ligne, non encore publiés. Dans la partie B, nous étudions les expansions du groupe des entiers par des valuations p-adiques. Ce travail fait l'objet d'une publication [AE19], coécrite avec Eran Alouf, dans le Journal of Symbolic Logic. La partie B résulte de ma première année de doctorat, sous la direction de Pierre Simon. Les parties A et B peuvent être lue indépendamment.

On commence par introduire la partie A. Les § 1 et § 2 sont une mise en contexte autour des notions de structures existentiellement closes et d'expansions génériques. Le lecteur familier avec ces notions peut se rendre directement aux § 3 et § 4 où sont exposés nos premiers résultats. Les paragraphes § 5 et § 6 présentent les théories NSOP_1 dans leur contexte historique récent, et le § 7 fait le lien entre le § 3 et ces théories. Le § 8 présente nos résultats concernant ACFG, puis commence l'introduction de la partie B. Le § 9 expose la situation actuelle concernant les expansions du groupe des entiers, les § 10 et § 11 donnent nos résultats sur ce sujet.

§ 1 Structures existentiellement closes

La théorie des modèles étudie les structures mathématiques à travers le prisme de leur algèbre des ensembles définissables. Cette dernière étant en général difficile à appréhender, les théoriciens des modèles ont souvent été en recherche de structures dans lesquelles une description simple des ensembles définissables soit possible. À l'aube de la théorie des modèles, Tarski [Tar51] montre dans les années 30 que la théorie des corps algébriquement clos ACF dans le langage des corps et celle des corps réels-clos RCF dans le langage des corps ordonnés admettent une élimination totale des quantificateurs : étudier les ensembles définissables revient à étudier l'algèbre booléenne engendrée par des ensembles de bases. On déduit facilement du résultat sur ACF le théorème de Chevalley sur les ensembles constructifs ou encore le Nullstellensatz de Hilbert. Robinson donne une preuve élémentaire du dix-septième problème de Hilbert à partir du résultat sur RCF. Ceci marqua le début du développement de méthodes pour montrer l'élimination des quantificateurs, et la seconde moitié du vingt-et-unième siècle fut témoin de nombreux autres résultats similaires, la théorie DCF_0 des corps différentiellement clos de caractéristique nulle [Rob58] [Rob59a] ou encore la théorie $\mathrm{SCF}_{p,e}$ des corps séparablement clos de caractéristique p et de degré d'imperfection e [Ers67] [Del88] ont l'élimination des quantificateurs dans des langages naturels appropriés. Ces deux théories fourniront un cadre adéquat dans la preuve de la conjecture de Mordell-Lang par Hrushovski [Bou+98].

Une élimination complète des quantificateurs n'est pas toujours possible dans un langage naturel, cela mena Robinson à définir la notion de théorie *modèle-complète*, une forme plus faible de l'élimination des quantificateurs, l'élimination jusqu'aux formules *existentielles*. Wilkie [Wil96] montre que la théorie de $\mathbb{R}_{\exp}$, le corps des réels augmenté de la fonction exponentielle, est modèle-complète, ce qui entraîne l'o-minimalité $\mathbb{R}_{\exp}$ et répond partiellement à une question de Tarski [Tar51] : la théorie de $\mathbb{R}_{\exp}$ est décidable en admettant la conjecture de Schanuel. Il existe néanmoins un langage descriptible dans lequel $\mathbb{R}_{\exp}$ admet l'élimination des quantificateurs [DMM94], mais celui-ci est compliqué.

Intuitivement, l'élimination des quantificateurs pour une théorie correspond à ce que le langage impose un principe de transfert, un « Nullstellensatz » entre les modèles de la théorie et certaines de leurs extensions (toutes extensions pour l'élimination des quantificateurs, tout surmodèles pour la modèle-complétude). Forcer une structure à satisfaire des principes de transfert devrait résulter en une structure appréhendable. Une structure $\mathscr{M}$ est *existentiellement close* dans une extension $\mathscr{N}$ si toute formule existentielle à paramètres dans $\mathscr{M}$ qui est vraie dans $\mathscr{N}$ l'est aussi dans $\mathscr{M}$. Un corps algébriquement (resp. séparablement) clos est existentiellement clos dans toute extension (resp. extension séparable) de corps. Un modèle d'une théorie T est *existentiellement clos* s'il est existentiellement clos dans tout modèle de T qui l'étend. Si les modèles existentiellement clos d'une théorie T existent et forment une classe élémentaire, leur théorie — la *modèle-compagne* de T — est alors modèle-complète.

Les corps *pseudo algébriquement clos* (PAC) sont des purs corps, existentiellement clos dans toute extension régulière, mais, en général, il n'existe pas d'expansion naturelle du langage pour laquelle cette théorie soit modèle-complète[1]. Cependant, les corps PAC ont des invariants élémentaires [CDM81], [FJ05], et ont été étudiés à de nombreuses reprises [Hru02], [Cha99], [Cha02], [CH04]. Les corps PAC sont le théâtre de phénomènes complexes qui jouent un rôle important dans le développement récent de la théorie des modèles, [CR16], [KR17], [Cha19], [Ram18], nous y reviendront au § 6.

Les modèles existentiellement clos d'une théorie présentent en général un caractère aléatoire – ou *générique* – résultant de leur définition. De façon un peu informelle, nous appellerons *générique*[2] une théorie (ou un modèle d'une telle théorie) qui axiomatise des structures existentiellement closes dans une classe raisonnable d'extensions.

Dans de nombreuses théories familières, les modèles existentiellement clos ne forment pas une classe élémentaire : la théorie des groupes [ES70], des groupes nilpotents [Sar74], des groupes résolubles [Sar76], des anneaux commutatifs [Che73], des corps non commutatifs (Sabbagh, 1970, non publié). Les modèles existentiellement clos de ces théories interprètent $(\mathbb{Z}, +, \cdot)$. Cependant, les groupes existentiellement clos et les corps non commutatifs existentiellement clos ont été étudiés dans les années 70, menant à de surprenantes connexions entre la théorie des modèles, la théorie des groupes et la récursion, voir [Zie76], [SZ79], [Zie80], [HW75], [Bel74], [Bel74], [Bel78a], [Bel78b] et [Mac77, p. 5.2] pour un résumé sur le sujet. Plus récemment, Haykazyan et Kirby [HK18] ont étudié une autre classe de structures existentiellement closes qui n'admet pas de modèle-compagne, nous reviendrons là-dessus au § 6.

§ 2 Généricité et expansions aléatoires

Une première étude de structure inhabituelle, expansion d'une structure classique, a été initiée par Tarski [Tar51] lorsqu'il s'est interrogé sur la décidabilité de la structure $(\mathbb{R}, \mathbb{R}\cap\overline{\mathbb{Q}})$, le corps des réels augmenté d'un prédicat pour les réels algébriques. C'est Robinson qui répondra plus tard par l'affirmative, en montrant que la théorie de cette structure est modèle-complète, tout comme celle de $(\mathbb{C}, \overline{\mathbb{Q}})$ [Rob59b].

La structure $(\mathbb{C}, \overline{\mathbb{Q}})$, et plus généralement toute paire propre (K, k) de corps algébriquement clos, jouit de la propriété suivante : si (L, l) est une extension de (K, k) telle que l et K sont linéairement disjoints sur k alors (K, k) est existentiellement clos dans (L, l). Ainsi, l'expansion d'ACF par un prédicat pour un sous-modèle propre d'ACF présente naturellement une certaine généricité, ce qui est plutôt exceptionnelle. En générale, on étudie les modèles existentiellement clos d'une expansion, d'où le terme d'expansion générique.

Un exemple général d'expansions génériques est la construction de Winkler [Win75]. En considérant une $\mathscr{L}$-théorie T et $\mathscr{L}' \supseteq \mathscr{L}$, ont peut voir T comme une $\mathscr{L}'$-théorie qui n'impose rien sur les éléments de $\mathscr{L}' \setminus \mathscr{L}$. Si T est modèle-complète et élimine le quantificateur $\exists^\infty$ alors Winkler montre que les modèles existentiellement clos de T en tant que $\mathscr{L}'$-théorie, forment une classe élémentaire. Le cas particulier de cette construction où l'on étend génériquement le langage de T par un unique prédicat unaire, le *prédicat générique*, a été étudié par Chatzidakis et Pillay [CP98]. Ce prédicat présente alors un caractère très aléatoire, informellement, il réalise

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[1] La théorie des corps pseudo-finis, la théorie des corps PAC ω-libres sont des théories de corps PAC qui admettent une expansion naturelle du langage qui donne la modèle-complétude, cf. section 1.5.2.

[2] Le mot générique a eu un sens précis en théorie des modèles classique, il correspondait à un modèle d'une théorie dans lequel le forcing infini et la satisfaction modèle-théorique coïncide (cf. par exemple [Mac77] ou [Che76]). Si la théorie est modèle-complète, les notions d'existentiellement clos et de structures génériques coïncident. Aujourd'hui, le terme générique pour les structures est plutôt utilisé de manière imprécise dans un sens proche du nôtre.

dans une seule structure toutes les façons possibles de colorier les points de la structure dans une extension. Ces constructions génériques ont des connections avec la néostabilité, nous y reviendrons au §6.

Une avancée récente dans le domaine des expansions génériques est la *fusion interpolative* développée par Kruckman, Tran et Walsberg [KTW18]. Etant donné des théories modèle-complètes arbitraires, ils décrivent un cadre dans lequel la modèle-compagne de l'union de ces théories existe. Il apparait que la plupart des structures génériques connues sont bi-interprétables avec la fusion interpolative de théories plus simples.

Considérons à présent les expansions T_1 et T_2 de la théorie ACF : T_1 est l'expansion par un prédicat générique, T_2 est l'expansion par un prédicat pour un sous corps propre algébriquement clos. On peut alors voir T_2 comme l'expansion générique de la théorie ACF par un prédicat pour un sous-corps, ainsi T_1 et T_2 sont deux expansions génériques d'ACF par un prédicat pour un réduit (trivial dans chacun des cas), le premier est la théorie de l'ensemble infini, le deuxième est la théorie ACF elle-même. Le premier résultat de ce manuscrit, le théorème A, présente un cadre général pour l'expansion générique d'une théorie par un prédicat pour un réduit.

§ 3 Expansion générique par un réduit prégéométrique

Soit T une théorie dans un langage $\mathscr{L}$. Soit $\mathscr{L}_0 \subseteq \mathscr{L}$ et T_0 un réduit de T au langage $\mathscr{L}_0$, et soit acl_0 la clôture algébrique au sens de $\mathscr{L}_0$. Soit $\mathscr{L}_S$ l'expansion de $\mathscr{L}$ par un prédicat unaire S, et T_S la théorie dans le langage $\mathscr{L}_S$ des structures $(\mathscr{M}, \mathscr{M}_0)$ où $\mathscr{M}$ est un modèle de T et S est un prédicat pour un modèle $\mathscr{M}_0$ de T_0 qui est une sous-structure de $\mathscr{M}$. On décrit à présent un cadre dans lequel une classe de modèles génériques de T_S est axiomatisable. On suppose ce qui suit.

(H_1) T est modèle-complete ;

(H_2) T_0 est modèle-complète et pour tout ensemble A infini $\mathrm{acl}_0(A) \models T_0$;

(H_3) T_0 est prégéométrique (acl_0 satisfait l'échange) ;

(H_4) pour tout $\mathscr{L}$-formule $\phi(x,y)$, il existe une $\mathscr{L}$-formule $\theta_\phi(y)$ telle que pour tout modèle $\mathscr{M}$ de T et uplet b de $\mathscr{M}$,

$$\begin{aligned}\mathscr{M} \models \theta_\phi(b) \iff & \text{il existe } \mathscr{N} \succ \mathscr{M} \text{ et } a \in \mathscr{N} \text{ tels que} \\ & \phi(a,b) \text{ et } a \text{ est un uplet indépendant sur } \mathscr{M}, \\ & \text{au sens de la prégéométrie } \mathrm{acl}_0.\end{aligned}$$

On note $\mathop{\smile\!\!\!\!\!|}^0$ la relation d'indépendance associée à la prégéométrie acl_0. Une extension $(\mathscr{N}, \mathscr{N}_0)$ de $(\mathscr{M}, \mathscr{M}_0)$ est dite *forte* si $\mathscr{N}_0 \mathop{\smile\!\!\!\!\!|}^0_{\mathscr{M}_0} \mathscr{M}$.

Théorème A. *Il existe une théorie TS contenant T_S telle que*

- *tout modèle de T_S admet une extension forte qui est un modèle de TS ;*
- *si $(\mathscr{M}, \mathscr{M}_0) \models TS$ et $(\mathscr{N}, \mathscr{N}_0) \models T_S$ est une extension forte de $(\mathscr{M}, \mathscr{M}_0)$ alors $(\mathscr{M}, \mathscr{M}_0)$ est existentiellement close dans $(\mathscr{N}, \mathscr{N}_0)$.*

De plus, si la prégéométrie acl_0 *est* modulaire, *TS est la modèle compagne de T et la clôture algébrique au sens de TS coïncide avec la clôture algébrique au sens de T.*

Comme à l'accoutumée dans ce genre de résultat, l'axiomatisation donne une ligne directrice pour la preuve, elle est donnée au théorème 2.1.5. Pour un uplet donné b dans un modèle $\mathscr{M}$ de T, la formule $\theta_\phi(b)$ témoigne de l'existence d'une réalisation a de $\phi(x,b)$ dans une extension élémentaire $\mathscr{N}$ de $\mathscr{M}$ telle que pour toute partition $A_1 \cup A_2$ des coordonnées de a, il existe une sous-$\mathscr{L}_0$-structure $\mathscr{N}_0$ de $\mathscr{N}$ modèle de T_0 qui sépare A_1 de A_2, c'est-à-dire $A_1 \subseteq \mathscr{N}_0$ et $A_2 \cap \mathscr{N}_0 = \emptyset$. Un modèle existentiellement clos de T_S devra réaliser toutes ces attributions possibles de coordonnées pour toute formule du langage $\mathscr{L}$. L'élément essentiel ici est que c'est exprimable au premier ordre, à condition que les formules θ_ϕ existent.

L'hypothèse (H_1) est clairement nécessaire, et (H_3) fournit un cadre général pour exprimer les notions de base au premier ordre. De plus, cela nous permet de donner un traitement géométrique de la preuve, c'est-à-dire en utilisant $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{0}$ comme dans un calcul de déviation, pour préparer une éventuelle adaptation de la preuve à d'autres configurations. Dans (H_2), l'hypothèse que les ensembles infinis acl_0-clos soient modèles de T_0 peut certainement être affaibli en une condition proche de « tout ensemble acl_0-clos se plonge dans un modèle de T_0 », mais cela rendrait la preuve plus technique et ne produirait pas plus d'applications du théorème que celles que nous donnons dans ce manuscrit, dans l'état actuel de nos connaissances.

L'hypothèse (H_4) est, en pratique, une condition difficile à obtenir pour utiliser le théorème A. Cette hypothèse (H_4) est une généralisation de l'élimination du quantificateur $\exists^\infty$. Si $T_0 = T$ et T est prégéométrique, c'est d'ailleurs équivalent (cf. Fact 1.3.10), et il suffit de vérifier (H_2) afin d'appliquer le théorème A. La théorie obtenue dans ce cas est alors une paire générique de modèles de la théorie géométrique T. Si T_0 est la théorie de l'ensemble infini dans le langage vide, la condition (H_4) est encore équivalente à l'élimination du quantificateur $\exists^\infty$, (H_2) et (H_3) sont triviaux et le théorème A ne donne rien de plus que la construction du prédicat générique [CP98]. L'hypothèse (H_4) peut aussi être vue comme une condition de « définissabilité de la dimension », mais dans un sens fort puisque l'on considère la dimension au sens de la prégéométrie d'un réduit. Dans la section 2.2, on donne un énoncé équivalent à (H_4) en termes d'existence de bornes, ainsi qu'une réciproque faible du théorème A : en supposant (H_1, H_2, H_3), si TS existe, alors pour toute formule $\phi(x,y)$ et $k \leq |x|$, il existe $\theta_\phi^k(y)$ telle que pour tout uplet b dans un modèle $\mathscr{M}$ de T, $\mathscr{M} \models \theta_\phi^k(b)$ si et seulement s'il existe $\mathscr{N} \succ \mathscr{M}$ et $a \in \mathscr{N}$ tels que $\phi(a,b)$ et $a_k \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{0} \mathscr{M}(a_i)_{i \neq k}$. En particulier, T élimine $\exists^\infty$. En général, (H_4) n'est pas équivalent à l'élimination de $\exists^\infty$, nous verrons pourquoi dans le § 4.

La théorie TS du théorème A est la fusion interpolative de la théorie T avec la théorie des paires génériques de modèles de T_0, au sens de [KTW18].

Voyons à présent une première utilisation du théorème A. Soient $\mathbb{F}_{q_1}, \cdots, \mathbb{F}_{q_n}$ des corps finis et T une théorie dans le langage

$$\mathscr{L} = \left\{+, 0, (\lambda_\alpha)_{\alpha \in \mathbb{F}_{q_1}}, \ldots, (\lambda_\alpha)_{\alpha \in \mathbb{F}_{q_n}}, \cdots\right\},$$

telle que tout modèle de T admet, pour tout $1 \leq i \leq n$, une structure d'espace vectoriel infini sur $\mathbb{F}_{q_i}$ dans le langage $\{+, 0, (\lambda_\alpha)_{\alpha \in \mathbb{F}_{q_i}}\}$, où λ_α est la fonction de multiplication par le scalaire α.

Théorème B. *Soient $V_1, \cdots, V_n$ des prédicats unaires, $T_{V_1 \ldots V_n}$ la théorie dont les modèles sont des modèles de T dans lesquels V_i est un prédicat pour un sous-espace vectoriel sur $\mathbb{F}_{q_i}$. Si T est modèle-complète et élimine le quantificateur $\exists^\infty$, alors $T_{V_1 \ldots V_n}$ admet une modèle-compagne.*

Le contexte décrit précédemment englobe les hypothèses (H_1, H_2, H_3). L'élimination du quantificateur $\exists^\infty$ donne, dans ce cas particulier, l'hypothèse (H_4), car la prégéométrie est uniformément finie, en utilisant un lemme classique de [CP98]. De plus, en appliquant une fois le théorème A, la théorie obtenue élimine aussi le quantificateur $\exists^\infty$, et on peut donc itérer et rajouter autant de sous-espaces vectoriels génériques que souhaité. Comme la prégéométrie associée aux espaces vectoriels est modulaire, la théorie obtenue est la modèle-compagne.

§ 4 Expansions génériques de corps

Sous-groupes additifs génériques en caractéristique positive. Soit p un nombre premier. Le groupe additif d'un corps de caractéristique p est un espace vectoriel sur $\mathbb{F}_p$, donc on peut appliquer le théorème B. Soit T l'une des théories suivantes.

- ACF_p la théorie des corps algébriquement clos de caractéristique p dans le langage des corps ;
- $\mathrm{SCF}_{p,e}$ la théorie des corps séparablement clos de caractéristique p et de degré d'imperfection $e \leq \infty$, dans le langage des corps augmenté de prédicats pour la p-indépendance ;
- Psf_c la théorie des corps pseudo-finis de caractéristique p dans le langage des corps augmentés de constantes pour les coefficients de polynômes irréductibles (cf. Section 1.5.2) ;
- ACFA_p la théorie des corps aux différences génériques de caractéristique p, dans le langage des corps aux différences.

L'expansion de T par un nombre arbitraire de sous-groupes additifs génériques existe. L'expansion de la théorie ACF_p par un sous-groupe additif générique sera notée ACFG, cette théorie n'existe qu'en caractéristique positive, comme nous allons le voir. Les chapitres 5, 6 et 7 de ce manuscrit sont consacrés à l'étude de la théorie ACFG, ces résultats sont décrits au § 8. Les corps PAC parfaits ont aussi une expansion générique similaire en caractéristique positive.

Théorème C. *Soit* PAC_G *la théorie dont les modèles sont des corps* PAC *parfaits de caractéristique p augmentés d'un prédicat G pour un sous-groupe additif, alors il existe une théorie* PACG *telle que :*

(1) tout modèle (F, G) de PAC_G *s'étend en un modèle (K, G) de* PACG *tel que K soit une extension régulière de F ;*

(2) tout modèle (K, G) de PACG *est existentiellement clos dans toute extension (F, G) modèle de* PAC_G *telle que F soit une extension régulière de K.*

De plus, il est possible d'itérer cette construction.

Tous les résultats précédents concernant les expansions génériques de corps de caractéristique positive sont vrais en remplaçant G par un $\mathbb{F}_q$-espace vectoriel V, pour tout sous-corps $\mathbb{F}_q$ du corps ambiant.

Sous-groupes additifs génériques en caractéristique nulle. Les résultats précédents n'ont pas d'analogue en caractéristique nulle. Soit T une théorie inductive *arbitraire* de corps de caractéristique nulle dans un langage $\mathscr{L}$ contenant le langage des anneaux. Soit G un nouveau prédicat unaire et T_G la théorie des modèles de T où G est un prédicat pour un sous-groupe additif, T_G est inductive. Un raisonnement simple (Proposition 3.2.7) montre que, si (K, G) est un modèle existentiellement clos de T_G, alors l'ensemble $\{a \in K \mid aG \subseteq G\}$ est égal à $\mathbb{Z}$. En particulier, T_G n'admet pas de modèle-compagne. De même, si l'on impose que G soit divisible, le stabilisateur de G est $\mathbb{Q}$. En outre, si T est la théorie de $\mathbb{R}$ ou $\mathbb{C}$, alors T satisfait les hypothèses (H_1, H_2, H_3) du théorème A (avec T_0 la théorie du groupe additif), donc par contraposée, (H_4)

n'est pas vérifié, alors que T élimine le quantificateur $\exists^\infty$.

Sous-groupes multiplicatifs génériques en toute caractéristique. L'expansion par un sous-groupe additif générique échoue en caractéristique nulle, cependant, nous avons le résultat suivant.

Théorème D. *Soit p un nombre premier ou nul. L'expansion de la théorie* ACF_p *par un sous-groupe multiplicatif générique existe.*

Les conditions (H_1, H_2, H_3) sont faciles à vérifier. Concernant l'hypothèse (H_4), on la montre seulement pour les formules définissant des variétés quasi-affines, et cela suffit pour l'existence de la modèle-compagne. L'hypothèse (H_4) découle d'un résultat de définissabilité en théorie de Kummer abstraite. Soit $W \subset K^n \setminus \{(0, \cdots, 0)\}$ une variété algébrique affine irréductible dans un corps algébriquement clos K de caractéristique $p \geq 0$. On dit que W est *libre* si elle n'est contenue dans aucun translaté d'un sous-groupe algébrique propre de $\mathbb{G}_m^n(K)$. Bays, Gavrilovitch et Hils montrent dans [BGH13] que W est libre si et seulement si tout élément de $\mathbb{G}_m^n(K)$ est le produit de $2n$ éléments de W, ce qui constitue une condition définissable[3].

§ 5 La classification de Shelah

Une part importante de la théorie des modèles consiste à classifier et comprendre les structures mathématiques, suivant l'idée directrice suivante, due principalement à Shelah : la complexité d'une structure est détectable dans les graphes bipartites associés aux ensembles définissables.

Par conséquent, la modération d'une structure est associée à l'absence de certaines configurations dans le graphe bipartite de toute formule. Par exemple, les structures stables sont celles qui ne définissent pas de demi-graphes infinis. Un fait remarquable en théorie de la stabilité est que cette définition en apparence combinatoire a un équivalent géométrique : l'existence d'une notion d'indépendance raisonnable (c'est à dire satisfaisant certaines propriétés) dans tout modèle, basée sur la déviation de Shelah. Pendant les vingt dernières années, les théoriciens des modèles se sont employés à adapter les méthodes et techniques provenant de la stabilité à des cadres instables, c'est ce qu'on appelle la *néostabilité*. Par exemple, la simplicité est une généralisation de la stabilité. Elle est aussi définie par une condition combinatoire sur les formules et est aussi caractérisée par un bon comportement de l'indépendance associée à la déviation, c'est le célèbre théorème de Kim-Pillay [KP97]. En utilisant ce théorème, c'est à dire en exhibant une relation d'indépendance qui vérifie une liste donnée d'axiomes, on montre que la théorie du graphe aléatoire, la théorie des corps PAC bornés ou encore ACFA, sont des théories simples. Cependant, d'autres théories, comme celle des corps PAC ω-libres [Cha99], celle des structures d'incidences génériques [CK17] ou encore la plupart des théories présentées au §4 ne sont pas simples, elles sont NSOP_1, une généralisation de la simplicité.

§ 6 Petite histoire récente des théories NSOP_1

Les théories NSOP_1, pour « not strong order property 1 », ont été définies par Džamonja et Shelah dans [DS04] (tout comme les théories NSOP_2) comme une extension de la hiérarchie $(\mathrm{NSOP}_n)_{n\geq 3}$. Dans [SU08], Shelah et Usvyatsov ont montré que T^*_{feq} (la modèle-complétion de la théorie d'une infinité de relations d'équivalences indépendantes paramétrées) est NSOP_1 et non simple. Pendant les trois dernières années, les théories NSOP_1 ont été intensément étudiées

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[3] Minh Chieu Tran [Tra17] a obtenu le même résultat de définissabilité pour les variétés qu'il appelle *multiplicativement large*, en utilisant le théorème des indécomposables de Zilber. Le résultat de Bays, Gavrilovitch et Hils [BGH13] est plus général, ils prouvent la définissabilité de la condition en remplaçant $\mathbb{G}_m^n(K)$ par n'importe quelle variété semi-algébrique.

à travers deux approches différentes (et non mutuellement exclusives) : l'approche abstraite, qui utilise la combinatoire et la théorie des modèles pure, et l'approche concrète, ou appliquée, qui consiste à étudier des exemples particuliers.

La première avancée, en ce qui concerne l'étude abstraite des théories NSOP$_1$ fut un critère à la Kim-Pillay développé par Chernikov et Ramsey [CR16], qui énonce qu'une théorie est NSOP$_1$ s'il existe une relation d'indépendance qui satisfait une certaine liste de propriétés. Ce résultat s'est avéré très utile pour montrer que certaines théories sont NSOP$_1$, la théorie des corps PAC ω-libres en est un bon exemple. Un corps PAC est simple si et seulement s'il est borné ([CP98],[Cha99]). Cependant, dans son étude des corps PAC ω-libres (qui sont non bornés), Chatzidakis [Cha02] définit une notion d'indépendance *faible* et montre que cette dernière satisfait quelques propriétés du critère de Kim-Pillay, en particulier le célèbre *théorème d'indépendance*. Il s'avère que toutes les propriétés du critère [CR16] furent prouvées à ce moment-là, sauf une. Chernikov et Ramsey ont ainsi déduit que les corps PAC ω-libres sont NSOP$_1$ en vérifiant que l'indépendance faible satisfait cette dernière propriété. Ils ont aussi montré que la théorie des formes bilinéaires génériques sur un espace vectoriel de dimension infinie sur un corps algébriquement clos, étudiée par Granger [Gra99], ainsi que la théorie des structures paramétrées généralisées [CR16, Example 6.3] sont NSOP$_1$.
La seconde avancée dans l'étude abstraite des théories NSOP$_1$ est le développement de la Kim-indépendance par Kaplan et Ramsey dans [KR17]. Ils introduisent des analogues de la division et de la déviation –la Kim-division et la Kim-déviation– qui se comportent bien dans les théories NSOP$_1$. La Kim-division est définie comme la division par rapport à une suite indiscernable bien particulière, une suite dans une extension globale invariante. De nombreuses propriétés de la déviation de Shelah dans les théories simples se retrouvent chez la Kim-déviation dans les théories NSOP$_1$. Par exemple, une théorie est NSOP$_1$ si et seulement si la Kim-indépendance est symétrique. Kaplan et Ramsey ont aussi complété le critère à la Kim-Pillay de [CR16] pour obtenir une caractérisation de la Kim-indépendance[4], de façon analogue au résultat classique de Kim-Pillay. Ils ont ensuite identifié la Kim-indépendance dans certaines théories NSOP$_1$. L'indépendance faible de Chatzidakis dans les corps PAC ω-libre s'est avérée être la Kim-indépendance. Dans l'exemple de Granger, la relation d'indépendance qui satisfait le critère de Chernikov et Ramsey est strictement plus forte que la Kim-indépendance.

En ce qui concerne l'approche appliquée, les structures d'incidences génériques (Conant et Kruckman [CK17]), les systèmes de triplets de Steiner (Barbina et Casanovas [BC18]) sont des nouveaux exemples de structures NSOP$_1$. Comme nous le verrons au § 7, ACFG et la plupart des théories décrites au § 4 sont des nouveaux exemples de théories NSOP$_1$. Ces exemples sont des constructions génériques, et ils partagent de nombreuses caractéristiques. Les théories simples ont longtemps été considérées comme des structures stables avec un « bruit aléatoire ». Un résultat encourageant fortement cette idée est la construction du prédicat générique [CP98], ajouter un prédicat générique à une théorie simple résulte en une théorie simple, cette expansion préserve la simplicité. Néanmoins, la simplicité n'est pas préservée si une généricité trop grande est présente. Par exemple, nous verrons au § 7 que la théorie ACFG n'est pas simple, et pourtant, il s'agit d'une expansion générique d'une théorie fortement minimale. Kruckman et Ramsey [KR18] montrent que l'expansion générique de Winkler [Win75] par un langage arbitraire (cf. § 2) préserve NSOP$_1$. Ils montrent aussi qu'une autre expansion générique due à Winkler préserve NSOP$_1$, l'expansion par des fonctions de Skolem génériques. Ils en déduisent que toute théorie NSOP$_1$ qui élimine le quantificateur $\exists^\infty$ admet une expansion NSOP$_1$ qui a des fonctions de Skolem définissables. Quant aux corps PAC, l'intuition générale est que leurs caractéristiques modèle-théoriques se réduisent à celles de leur groupe de Galois absolu. Les

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[4] Plus précisément, ils montrent qu'une relation $\mathop{\smile\hskip-0.9em ^{|}}$ satisfaisant [CR16, Proposition 5.3] renforce la Kim-indépendance. La propriété *Witnessing* assure que l'indépendance au sens de la Kim-division renforce $\mathop{\smile\hskip-0.9em ^{|}}$.

résultats récents de Chatzidakis et Ramsey encouragent cette idée : un corps PAC est NSOP$_n$ si et seulement si son groupe de Galois absolu l'est ([Cha19] pour $n \geq 3$, [Ram18] pour $n = 1, 2$).

Dernièrement, Haykazyan et Kirby [HK18] ont mis en lumière une nouvelle source de théories NSOP$_1$, au sens de la logique positive. Ils étudient la classe des corps exponentiels existentiellement clos (un corps exponentiel est un corps muni d'un homomorphisme entre le groupe additif et le groupe multiplicatif). Tout comme pour la classe des corps non commutatifs existentiellement clos, ou les corps munis d'un sous-groupe additif en caractéristique nulle, cette classe n'est pas élémentaire. Néanmoins, une idée qui remonte à Shelah [She75] consiste à étudier ce genre de classes en ne considérant seulement les formules existentielles. C'est l'ambition de la logique positive, qui fut développée de manière différente par Ben-Yaacov [Ben03a] et Pillay [Pil00]. Des analogues de la stabilité [Pil00] [Ben03b] et de la simplicité [Ben03b] dans ces cadres ont vu le jour par la suite. Haykazyan et Kirby [HK18] ont adapté les résultats de Chernikov et Ramsey [CR16] pour déduire de l'existence d'une relation d'indépendance raisonnable que les corps exponentiels existentiellement clos sont NSOP$_1$ au sens de la logique positive. Il est raisonnable de penser que la théorie développée par Haykazyan et Kirby peut être utilisée afin de montrer que la classe des corps algébriquement clos de caractéristique nulle munis d'un sous-groupe additif générique est NSOP$_1$ au sens de la logique positive.

§ 7 Preservation de NSOP$_1$

Le prochain résultat donne une condition pour que l'expansion générique du § 3 préserve NSOP$_1$. Comme dans le § 3, soit T une $\mathscr{L}$-théorie, $\mathscr{L}_0 \subseteq \mathscr{L}$ et T_0 le réduit de T au langage $\mathscr{L}_0$. On suppose que les hypothèses (H_1) à (H_4) sont satisfaites par T et T_0, donc la théorie TS existe. Soient respectivement acl_T, acl_0 les clôtures algébriques au sens de $\mathscr{L}$ et $\mathscr{L}_0$, et $\mathop{\smile\hskip -0.9em ^| \ }^0$ la relation d'indépendance au sens de la prégéométrie acl_0.

Théorème E. *On suppose que* acl_0 *définie une prégéométrie modulaire, que* T *est* NSOP$_1$ *et que* $\mathop{\smile\hskip -0.9em ^| \ }^T$ *est la Kim-indépendance au sens de* T*. Soit* $\mathscr{M}$ *un modèle de* T *et* A, B, C acl_T*-clos contenant* $\mathscr{M}$*, dans un modèle monstre. On suppose que*

(A) *Pour tout* $A, B, C, \mathscr{M}$ *comme au-dessus, si* $C \mathop{\smile\hskip -0.9em ^| \ }^T_{\mathscr{M}} A, B$ *et* $A \mathop{\smile\hskip -0.9em ^| \ }^T_{\mathscr{M}} B$ *alors*

$$(\mathrm{acl}_T(AC), \mathrm{acl}_T(BC)) \mathop{\smile\hskip -0.9em ^| \ }^0_{A,B} \mathrm{acl}_T(AB).$$

Alors la théorie TS *est* NSOP$_1$ *et la Kim-indépendance au sens de* TS *est donnée par la relation* $\mathop{\smile\hskip -0.9em ^| \ }^w$*, définie par*

$$A \mathop{\smile\hskip -0.9em ^| \ }^T_{\mathscr{M}} B \text{ et } S(\mathrm{acl}_0(\mathrm{acl}_T(A\mathscr{M}), \mathrm{acl}_T(B\mathscr{M}))) = \mathrm{acl}_0(S(\mathrm{acl}_T(A\mathscr{M})), S(\mathrm{acl}_T(B\mathscr{M}))).$$

Le théorème E est une conséquence de considérations plus générales. En partant d'une relation d'indépendance $\mathop{\smile\hskip -0.9em ^| \ }^T$ dans les modèles de T, on définit deux relations d'indépendance dans tout modèle de TS, une indépendance *forte* $\mathop{\smile\hskip -0.9em ^| \ }^{st}$ et une indépendance *faible* $\mathop{\smile\hskip -0.9em ^| \ }^w$. Ces deux relations étendent la relation $\mathop{\smile\hskip -0.9em ^| \ }^T$. Au chapitre 4, on analyse les propriétés de $\mathop{\smile\hskip -0.9em ^| \ }^T$ qui sont transmises à $\mathop{\smile\hskip -0.9em ^| \ }^w$ ou $\mathop{\smile\hskip -0.9em ^| \ }^{st}$. Si T est NSOP$_1$ et $\mathop{\smile\hskip -0.9em ^| \ }^T$ est la Kim-indépendance, alors toutes les propriétés satisfaites par $\mathop{\smile\hskip -0.9em ^| \ }^T$ qui caractérisent la Kim-indépendance et le fait que T soit NSOP$_1$ sont transférées à $\mathop{\smile\hskip -0.9em ^| \ }^w$, sauf le théorème d'indépendance, qui requiert (A). Cela donne le théorème E. On donne une analyse fine des propriétés conservées de $\mathop{\smile\hskip -0.9em ^| \ }^T$ à $\mathop{\smile\hskip -0.9em ^| \ }^w$ et $\mathop{\smile\hskip -0.9em ^| \ }^{st}$. Par exemple, si $\mathop{\smile\hskip -0.9em ^| \ }^T$ est stationnaire au-dessus de certains ensembles, il en est de même pour $\mathop{\smile\hskip -0.9em ^| \ }^{st}$, au-dessus de ces mêmes ensembles. Si $\mathop{\smile\hskip -0.9em ^| \ }^T$ satisfait $\mathop{\smile\hskip -0.9em ^| \ }'$*-amalgamation* (une version du théorème d'indépendance dans laquelle les paramètres sont choisis indépendants au sens d'une autre relation, $\mathop{\smile\hspace{-0.9em}\mid}\nolimits'$, cf. Section 1.2) alors $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{w}$ satisfait aussi $\mathop{\smile\hspace{-0.9em}\mid}\nolimits'$-amalgamation. On utilise ceci pour montrer que, lorsque c'est possible, si l'on itère l'expansion d'une théorie NSOP_1 par un réduit générique, on obtient toujours une théorie NSOP_1 (cf. Corollary 4.2.4).

Kaplan et Ramsey [KR17] donnent une condition nécessaire et suffisante géométrique pour qu'une théorie NSOP_1 soit simple : la Kim-indépendance doit satisfaire la propriété de *Monotonicité sur la base* (si $a \mathop{\smile\hspace{-0.9em}\mid}\nolimits_{C} bd$ alors $a \mathop{\smile\hspace{-0.9em}\mid}\nolimits_{Cb} d$, cf. Section 1.2). Dans notre cadre, cela se traduit en un critère sur l'intrication entre les clôtures algébriques acl_T et acl_0 (cf. Corollary 4.2.3). La condition (A) exprime également un certain « contrôle » de $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{T}$ sur $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}$. La « proximité » qu'entretient T_0 avec T joue un rôle intéressant en ce qui concerne la préservation des notions de néostabilité lors de l'expansion à TS, comme nous pouvons le voir dans la table suivante.

Configuration $T_0 \subseteq T$	Expansion générique TS
$T_0 = T$	Préserve la stabilité
$T_0 \subseteq T$	Préserve NSOP_1
$T_0 = \emptyset$	Préserve la simplicité

Si T est une théorie de corps de caractéristique positive et T_0 le réduit additif de T, la condition (A) se réduit à une condition beaucoup plus simple, la condition (B) du théorème F. Pour un corps A, on note $\overline{A}$ sa clôture algébrique au sens de la théorie des corps.

Théorème F. *Soit* T *une théorie modèle-complète et* NSOP_1 *de corps de caractéristique positive qui élimine le quantificateur* $\exists^\infty$. *Soit* A, B *des ensembles* acl_T*-clos et* $E \models T$ *contenu dans* A *et* B. *Soit* $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{T}$ *la Kim-indépendance dans* T. *On suppose que :*

(B) *pour tout* A, B *et* E *comme au-dessus, si* $A \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{T}_{E} B$ *alors* $\mathrm{acl}_T(AB) \subseteq \overline{AB}$.

Alors, l'expansion de T *par des sous-groupes additifs génériques* $TG_1 \ldots G_n$ *est* NSOP_1. *La Kim-indépendance dans* $TG_1 \ldots G_n$ *est donnée par*

$$A \mathop{\smile\hspace{-0.9em}\mid}\limits^{T}_{E} B \text{ et pour tout } i \leq n \ G_i(A+B) = G_i(A) + G_i(B),$$

pour tout A, B *et* E *comme au dessus. De plus,* $TG_1 \cdots G_n$ *n'est pas simple.*

En particulier, tous les exemples en caractéristique positive du § 4 sauf celui du théorème C sont NSOP_1 et non simple. En ce qui concerne la théorie PACG du théorème C, tous les corps PAC parfaits ne sont pas NSOP_1, mais satisfont tous l'hypothèse (B) (car ils sont *algébriquement bornés* [CH04]). Il suit de cela que l'expansion dans un language approprié d'un corps de Frobenius parfait, ou d'un corps PAC ω-libre parfait en caractéristique positive par des sous-groupes additifs génériques est NSOP_1. Remarquons que dans le théorème F, les G_i peuvent être remplacés par des $\mathbb{F}_{q_i}$-espaces vectoriels, pour tout sous-corps $\mathbb{F}_{q_i}$ d'un modèles de T.

La preuve du théorème F consiste à déduire (A) à partir de (B). Elle utilise une description de la Kim-indépendance dans toute théorie de corps, par Kaplan et Ramsey [KR17] qui est basée sur les travaux de Chatzidakis [Cha99]. On conclut par un mélange d'arguments de stabilité dans la clôture séparable du corps et de théorie galoisienne.

Enfin, pour tout p premier ou nul, l'expansion de la théorie ACF_p par un prédicat pour un sous-groupe multiplicatif générique est aussi NSOP_1 et non simple. On utilise le théorème E directement, la condition (A) découle d'un argument de cohéritier dans la théorie stable ACF_p.

§ 8 La théorie ACFG

Les chapitres 5, 6 et 7 sont dédiés à l'étude de la théorie ACFG, l'expansion de ACF_p par un sous-groupe additif générique, pour un p premier fixé.

Presque simple. Dans ACFG, aucune relation d'indépendance ne satisfait le théorème de Kim-Pillay pour la simplicité. Cependant, deux relations n'en sont pas loin.

Théorème G.

- *Dans* ACFG*, la Kim-indépendance satisfait toutes les propriétés du théorème de Kim-Pillay sur les théories simples sauf la* Monotonicité sur la base.
- *Dans* ACFG*, il existe une relation d'indépendance qui satisfait toutes les propriétés du théorème de Kim-Pillay sur les théories simples sauf le* Caractère local.

L'indépendance du deuxième point est l'indépendance forte, mentionnée au § 7.

Des modèles de la théorie ACFG. Soit (K, G) un modèle de la théorie ACFG, G est le sous-groupe additif générique du corps K. Le groupe G a des propriétés algébriques remarquables. Par exemple, il est dense et codense dans K pour la topologie de Zariski. De plus, tout élément de K est le produit de deux éléments de G, en particulier, G est stablement plongé dans K.

Soit $\overline{\mathbb{F}_p}$ la clôture algébrique au sens des corps du corps premier $\mathbb{F}_p$. En utilisant que le corps $\overline{\mathbb{F}_p}$ est localement fini, et l'élimination des quantificateurs dans ACF_p, on construit par union de chaîne un sous-groupe G de $\overline{\mathbb{F}_p}$ tel que $(\overline{\mathbb{F}_p}, G)$ soit un modèle d'ACFG. L'espace $\mathrm{Sg}(\overline{\mathbb{F}_p})$ des sous-groupes additifs de $\overline{\mathbb{F}_p}$ peut être munit de la topologie de Chabauty (cf. Section 1.6). On montre que presque tous (au sens de Baire) les sous-groupes additifs G de $\overline{\mathbb{F}_p}$ sont génériques.

Théorème H. *L'ensemble des sous-groupes G de $\overline{\mathbb{F}_p}$ tels que $(\overline{\mathbb{F}_p}, G) \models$ ACFG est un G_δ dense de $\mathrm{Sg}(\overline{\mathbb{F}_p})$, pour la topologie de Chabauty sur $\mathrm{Sg}(\overline{\mathbb{F}_p})$.*

Ce résultat se prouve de la même manière que le résultat analogue de Hrushovski concernant les modèles de la théorie ACFA dans $\overline{\mathbb{F}_p}$ [Hru04].

Les imaginaires dans ACFG. Soit (K, G) un modèle de la théorie ACFG. Il n'y a pas de paramètres canoniques pour les éléments du groupe quotient K/G. Une question naturelle s'impose alors : en rajoutant à (K, G) des paramètres canoniques pour le quotient K/G, peut-on éliminer tous les imaginaires de K/G ? Pour formuler la réponse à cette question de façon précise, on considère la structure à deux sortes $(K, K/G)$ qui consiste en une sorte pour le corps K algébriquement clos, une sorte pour le groupe K/G, et la surjection canonique $\pi : K \to K/G$.

Théorème I. *Pour tout modèle (K, G) de la théorie* ACFG*, la structure $(K, K/G)$ a l'élimination faible des imaginaires.*

L'élimination faible des imaginaires est un résultat optimal pour $(K, K/G)$. En effet, $(K/G, +)$ a une structure de pur espace vectoriel sur $\mathbb{F}_p$, donc les imaginaires finis ne peuvent pas être éliminés. En outre, rajouter des paramètres canoniques pour les imaginaires finis de K/G ne suffirait même pas à éliminer tous les imaginaires finis de la structure $(K, K/G)$, cf. Example 6.3.6. La preuve du théorème I suit le même schéma que les preuves classiques de l'élimination des imaginaires dans [Hru02], [CH99] ou encore [CP98], elle est basée sur le théorème d'indépendance. Cependant, dans notre cas la Kim-indépendance joue le rôle que l'indépendance de la déviation joue dans les preuves classiques. La Kim-indépendance dans (K, G) est donnée par $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} B$ et $G(A + B) = G(A) + G(B)$, pour $C = A \cap B$ et A, B, et C algébriquement clos. Dans $(K, K/G)$, la condition $G(A + B) = G(A) + G(B)$ se traduit en $\pi(A) \cap \pi(B) = \pi(C)$, une relation « modulaire stable » qui vient de la structure de pur $\mathbb{F}_p$-espace vectoriel de la sorte K/G. Donc dans $(K, K/G)$, la Kim-indépendance est la conjonction de l'indépendance au sens des corps algébriquement clos et cette indépendance « modulaire stable », qui se comporte bien mieux que l'indépendance dans (K, G). Par ailleurs, les éléments de K/G sont remplacés par des représentants spéciaux (*minimaux* ou *maximaux*) dans K afin d'obtenir une version du théorème d'indépendance et d'adapter l'argument classique. La preuve du théorème I devrait s'adapter telle quelle en remplaçant ACF_p par n'importe quelle théorie de corps stables qui élimine faiblement les imaginaires.

L'étude de la théorie de $(K, K/G)$ suggère une construction générique « duale » à celle que l'on présente au §3. Soit T une théorie et T_0 un réduit de T a un sous-langage $\mathscr{L}_0$ du langage de T. On considère la théorie T' dans un langage à deux sortes dont les modèles sont composés d'un modèle $\mathscr{M}$ de T dans une sorte, d'un modèle $\mathscr{M}_0$ de T_0 dans une autre sorte, et d'un $\mathscr{L}_0$-homomorphisme surjectif $\pi : \mathscr{M} \to \mathscr{M}_0$. On peut vérifier que, sous les hypothèses (H_1) à (H_4), on construit la modèle-compagne de la théorie T', dans laquelle l'étude des imaginaires devrait être facilitée.

Déviation et déviation épineuse dans ACFG. L'indépendance associée à la déviation n'est pas symétrique dans ACFG, car sinon, la théorie serait simple. Le théorème G donne l'intuition naïve que la Kim-indépendance et l'indépendance de la déviation ne diffèrent que par la propriété de monotonicité sur la base, et en effet, l'indépendance de la déviation est obtenue en « forçant » la Kim-indépendance à satisfaire la propriété de monotonicité sur la base. On montre aussi que la déviation, la division (et la déviation épineuse) coïncident pour les types.

Théorème J. *Soient respectivement* $\mathop{\smile\!\!\!\!\!\!\mid}^{f}$, $\mathop{\smile\!\!\!\!\!\!\mid}^{d}$, $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$, $\mathop{\smile\!\!\!\!\!\!\mid}^{K}$ *les indépendances au sens de la déviation, division, déviation épineuse*[5] *et la Kim-déviation dans* ACFG. *Alors*

$$a \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{C} b \iff a \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{C} b \iff a \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}_{C} b \iff \textit{pour tout } C \subseteq D \subseteq Cb,\ \ a \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{D} b.$$

Dans les théories NSOP_1, l'indépendance de la déviation et de la Kim-déviation sont des notions différentes, et aucune bonne description de la déviation n'existe en général, mais la plupart des exemples connus de théories NSOP_1 et non simples ont la même description de la déviation que dans ACFG [Cha02], [CK17], [KR18]. Une ligne directrice générale pour montrer que la déviation et la division coïncident pour les types semble émerger des différents exemples. Elle inclut un résultat de « transitivité mixte » entre une indépendance forte et la Kim-indépendance, qui permet de montrer que l'indépendance de la division satisfait la propriété d'extension et donc que c'est l'indépendance de la déviation. Une discussion à propos des similarités entre les différents exemples de théories NSOP_1 est proposée à la section 7.4, ainsi que les différences principales entre ACFG et les autres exemples (voir aussi la figure 7.2). On donne aussi une étude de quelques phénomènes qui apparaissent lorsque l'on force la propriété de monotonicité sur la base pour une relation arbitraire, cf. Section 7.1.

Notre étude de la déviation épineuse dans ACFG utilise la description géométrique de cette dernière, donnée par Adler [Adl09a]. De façon plus générale, la partie A et tout particulièrement le chapitre 7 regorge de ce traitement géométrique des relations d'indépendances, qui prend racine dans le théorème de Kim-Pillay, mais a été principalement développé par Adler [Adl08a], [Adl09a], [Adl09b] puis suivi par [CK12], [CR16], [CK17], entre autres.

Passons à la partie B.

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[5] Il s'agit en fait ici de la restriction de la déviation épineuse à la sorte réelle, l'indépendance épineuse étant définie en général dans T^{eq}.

§ 9 Expansions du groupe des entiers

D'un point de vue modèle théorique, la structure $(\mathbb{Z},+,\cdot)$ n'est pas appréhensible, cela suit des célèbres travaux de Godel sur l'arithmétique de Péano. Il est donc naturel d'étudier des réduits modérés de $(\mathbb{Z},+,\cdot)$. L'étude des structures $(\mathbb{Z},+,0)$ et $(\mathbb{Z},+,0,<)$ remonte à 1929 par Presburger, la théorie de $(\mathbb{Z},+,0,<)$, encore aujourd'hui, est appelée *l'arithmétique de Presburger*[6]. Ces deux structures admettent l'élimination des quantificateurs dans l'expansion du langage par la constante 1, et des prédicats pour les sous-groupes $n\mathbb{Z}$ pour tout $n \in \mathbb{N}$. La théorie de $(\mathbb{Z},+,0)$ est superstable de U-rang 1, celle de $(\mathbb{Z},+,0,<)$ est instable, mais NIP et même dp-minimale.

L'étude des expansions modérées de $(\mathbb{Z},+,0)$ est une sujet récent. Il y a peu, aucun exemples de telles structures n'avaient été étudiés, autre que $(\mathbb{Z},+,0,<)$. Les premiers exemples ont été donnés indépendamment par Palacín et Sklinos [PS18] d'une part, et par Poizat [Poi14] d'autre part. Plus spécifiquement, ils montrent, par des méthodes différentes que pour tout entier $q \geq 2$, la structure $(\mathbb{Z},+,0,\prod_q)$ est superstable de U-rang ω, où $\prod_q = \{q^n \mid n \in \mathbb{N}\}$. Palacín et Sklinos montrent le même résultat pour d'autres structures, par exemple $(\mathbb{Z},+,0,\mathrm{Fac})$, où $\mathrm{Fac} = \{n! \mid n \in \mathbb{N}\}$. Conant [Con17b], et Lambotte et Point [LP17] ont généralisé indépendamment ces résultats. Pour un ensemble $A \subseteq \mathbb{Z}$ majoré ou minoré, ils énoncent des conditions sur la répartition des points de A pour que la structure $(\mathbb{Z},+,0,A)$ soit superstable de U-rang ω. Conant donne aussi une condition nécessaire pour que la structure $(\mathbb{Z},+,0,A)$ soit stable.

Un exemple un peu différent a été produit par Kaplan et Shelah dans [KS17]. Ils montrent que pour $\mathrm{Pr} = \{p \in \mathbb{Z} \mid |p| \text{ est premier}\}$, la structure $(\mathbb{Z},+,0,\mathrm{Pr})$ a la propriété d'indépendance, et donc est instable. Néanmoins, en admettant la conjecture de Dickson [7], cette structure est supersimple de U-rang 1.

Par ailleurs, $(\mathbb{Z},+,0,<)$ restait la seule expansion instable dp-minimale de $(\mathbb{Z},+,0)$. Aschenbrenner, Dolich, Haskell, Macpherson, et Starchenko posent la question suivante [Asc+13, Question 5.32] : toute expansion dp-minimale de $(\mathbb{Z},+,0)$ est-elle un réduit de $(\mathbb{Z},+,0,<)$? $(\star)$. Dans [Asc+16], les mêmes auteurs montrent que $(\mathbb{Z},+,0,<)$ n'a pas d'expansion propre dp-minimales. Ils utilisent pour cela un résultat fort de théorie des automates dû à Michaux et Villemaire [MV96] qui peut s'énoncer comme suit : toute expansion propre de $(\mathbb{Z},+,0,<)$ définit un nouveau sous-ensemble de $\mathbb{Z}$. Ce fut plus tard généralisé par Dolich et Goodrick [DG17], ils obtiennent que $(\mathbb{Z},+,0,<)$ n'a pas d'expansion propre forte. Avec un résultat de Conant [Con18], que l'on décrit plus bas, toute autre expansion instable et dp-minimale de $(\mathbb{Z},+,0)$, si elle existe, n'est ni un réduit, ni une extension de $(\mathbb{Z},+,0,<)$, donc la question $(\star)$ devient

$$(\mathbb{Z},+,0,<) \text{ est-elle la seule expansion dp-minimale non-triviale de } (\mathbb{Z},+,0)\,? \qquad (\star\star)$$

§ 10 Une nouvelle expansion dp-minimale des entiers

On introduit une nouvelle famille d'expansions dp-minimales de $(\mathbb{Z},+,0)$, donnant une réponse négative à la question $(\star\star)$. Plus généralement, pour chaque $n \in \mathbb{N}$, on introduit une famille d'expansions de $(\mathbb{Z},+,0)$ qui ont dp-rang n. Après ces travaux, nous avons été informés

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[6] Pour certain logiciens, l'arithmétique de Presburger est la théorie de $(\mathbb{Z},+,0)$ et non celle de $(\mathbb{Z},+,0,<)$, et en effet, dans son papier de 1929, Presburger étudie principalement la théorie de $(\mathbb{Z},+,0)$. Néanmoins, dans ce même papier, il explique que ses résultats s'étendent à la théorie de $(\mathbb{Z},+,0,<)$, cf. [Haa18] pour une étude de l'arithmétique de Presburger des origines à nos jours.

[7] Une conjecture en théorie des nombres concernant la répartition des nombres premiers dans les suites arithmétiques, qui généralise le théorème de Dirichlet sur les nombres premiers.

qu'un résultat similaire avait été prouvé par François Guignot [Gui16], et par Nathanaël Mariaule [Mar17, Corollary 2.11].

Soit P un ensemble fini ou infini de nombres premiers. Pour chaque $p \in P$, soit $|_p$ le préordre sur $\mathbb{Z}$ définit par

$$a \mid_p b \iff v_p(a) \leq v_p(b)$$

où v_p est la valuation p-adique sur $\mathbb{Z}$.

Théorème K. *La structure* $(\mathbb{Z}, +, 0, (\cdot \mid_p \cdot)_{p \in P})$ *a l'élimination des quantificateurs dans le langage augmenté par* 1 *et des prédicats pour les sous-groupes* $n\mathbb{Z}$ *pour tout* $n \in \mathbb{N}$*. En outre, son dp-rang est égal à* $|P|$*, le cardinal de* P*.*

Chaque valuation p-adique induit sur $\mathbb{Z}$ une structure d'arbre, et une topologie d'arbre similaire à celle décrite en section 1.6. Chaque entier est représenté comme une branche dont les noeuds sont les coordonnées dans sa représentation p-adique. Cette structure d'arbre est préservée par extensions élémentaires et permet un traitement graphique des arguments. La preuve de l'élimination des quantificateurs est technique, mais n'utilise pas de résultats arithmétiques plus forts que le théorème des restes chinois, qui se traduit ici par la densité topologique des sous-groupes $n\mathbb{Z}$, pour n premier avec p.

Le calcul du dp-rang de $(\mathbb{Z}, +, 0, (\cdot \mid_p \cdot)_{p \in P})$ se fait en deux étapes. On montre d'abord que $(\mathbb{Z}, +, 0, |_p)$ est dp-minimale pour tout p, par l'élimination des quantificateurs et la dp-minimalité de la structure $(\mathbb{Q}_p, +, 0, |_p)$ [DGL11]. Puis on montre que le dp-rang de $(\mathbb{Z}, +, 0, (\cdot \mid_p \cdot)_{p \in P})$ ne peut excéder la somme des dp-rangs de chaque réduit $(\mathbb{Z}, +, 0, |_p)$, et on conclut en exhibant un ict-motif de taille $|P|$.

L'élimination des quantificateurs dans $(\mathbb{Z}, +, 0, |_p)$ implique que tout ensemble définissable est combinaison booléenne d'ensembles définissables sans paramètres et de boules (au sens de la valuation). La C-minimalité est une notion introduite par Macpherson et Steinhorn [MS96] pour être un analogue de l'o-minimalité dans le contexte des structures valuées ou arborescentes. Grossièrement, dans une structure C-minimale arborescente, tout ensemble définissable est une combinaison booléenne de boules. Cela suggère une définition de *quasi*-C-minimalité, en analogie avec la quasi-o-minimalité [BPW00], qui devrait impliquer la dp-minimalité, et devrait comprendre la structure $(\mathbb{Z}, +, 0, |_p)$. La même remarque s'applique à la notion encore plus générale de VC-minimalité introduite par Adler [Adl08b].

§ 11 Phénomènes de minimalité

Étant donné une class $\mathcal{C}$ de structures qui partagent le même domaine, et $\mathscr{M} \in \mathcal{C}$, on dit que $\mathscr{M}$ est *minimale* dans $\mathcal{C}$ s'il n'existe pas de réduit propre de $\mathscr{M}$ dans $\mathcal{C}$. De même, $\mathscr{M}$ est *maximale* dans $\mathcal{C}$ s'il n'existe pas d'expansions propres de $\mathscr{M}$ dans $\mathcal{C}$.

Un premier exemple de minimalité a été donné par Pillay et Steinhorn [PS87] : $(\mathbb{N}, <)$ n'a pas d'expansions propres o-minimales, donc $(\mathbb{N}, <)$ est maximale dans la classe des structures o-minimales de domaine $\mathbb{N}$. Un autre exemple fut donné par Marker, motivé par une question de Zilber : un corps algébriquement clos admet-il des expansions propres fortement minimales ? Hrushovski donna une réponse positive à la question [Hru92], néanmoins, Marker montra que $(\mathbb{C}, +, \cdot, 0, 1)$ n'admet *aucune* expansion propre qui soit un réduit propre de $(\mathbb{C}, +, \cdot, 0, 1, \mathbb{R})$, donc $(\mathbb{C}, +, \cdot, 0, 1, \mathbb{R})$ est minimale parmi les expansions propres de $(\mathbb{C}, +, \cdot, 0, 1)$.

L'étude des expansions de $(\mathbb{Z}, +, 0)$ a récemment produit divers résultats de minimalité et de maximalité. Nous l'avons vu plus haut, $(\mathbb{Z}, +, 0, <)$ est une structure maximale dp-minimale sur

$\mathbb{Z}$ [Asc+16]. Donnant suite aux résultats de Palacín et Sklinos [PS18], Conant et Pillay [CP18] ont montré que $(\mathbb{Z}, +, 0)$ n'a pas d'expansions propres stables de dp-rang fini. En d'autres termes, $(\mathbb{Z}, +, 0)$ est maximal parmi les structures stables de dp-rang fini sur $\mathbb{Z}$. Puisque $(\mathbb{Z}, +, 0, <)$ est dp-minimale, il suit de cela qu'il n'existe pas de structure stable sur $\mathbb{Z}$ qui soit une expansion propre de $(\mathbb{Z}, +, 0)$ et un réduit propre de $(\mathbb{Z}, +, 0, <)$. Dans [Con18], Conant renforce encore ce résultat en montrant qu'il n'existe pas de structures qui est une expansion propre de $(\mathbb{Z}, +, 0)$ et un réduit propre de $(\mathbb{Z}, +, 0, <)$, autrement dit $(\mathbb{Z}, +, 0, <)$ est minimale parmi toutes les expansions propres de $(\mathbb{Z}, +, 0)$. On montre le résultat analogue pour notre expansion de $(\mathbb{Z}, +, 0)$.

Théorème L. $(\mathbb{Z}, +, 0, |_p)$ *est une expansion propre minimale de* $(\mathbb{Z}, +, 0)$.

La preuve de Conant [Con18] n'utilise pas que $(\mathbb{Z}, +, 0)$ n'admet pas d'expansions propres stables qui soit un réduit de $(\mathbb{Z}, +, 0, <)$. Sa preuve consiste en une analyse détaillée des ensembles définissables en dimension arbitraire. Conant a demandé si son théorème pouvait être montré en utilisant des méthodes modèles théoriques qui incluaient le résultat [CP18]. C'est la stratégie que l'on a adopté pour montrer le théorème L : il n'existe pas de structure *instable* qui soit un réduit propre de $(\mathbb{Z}, +, 0, |_p)$ et une expansion propre de $(\mathbb{Z}, +, 0)$. On donne aussi une preuve plus rapide du résultat de Conant, par la même méthode que celle que l'on a utilisée pour montrer le théorème L, que nous décrivons maintenant. Par un résultat classique de Shelah, l'instabilité d'une théorie peut être témoignée par une formule dont l'un des uplets de variable est un singleton. Si $\mathcal{Z}$ est une expansion instable (et donc propre) de $(\mathbb{Z}, +, 0)$ et un réduit de $(\mathbb{Z}, +, 0, |_p)$, alors, le résultat de Shelah permet de montrer, quitte à travailler dans une extension élémentaire $\mathcal{Z}'$ de $\mathcal{Z}$, qu'une nouvelle formule du langage de $\mathcal{Z}$ définit un nouveau sous-ensemble du domaine dans $\mathcal{Z}'$. Par conséquent, le problème se réduit à une analyse des ensembles définissables unidimensionnels de $\mathcal{Z}'$, que l'on fait grâce à l'élimination des quantificateurs dans $(\mathbb{Z}, +, 0, |_p)$. Le reste de la preuve consiste à définir la relation $x \mid_p y$ en appliquant des transformations dans le langage $\{+, 0\}$ à la nouvelle formule. En étudiant les sous-groupes uniformément définissables de la structure, on montre qu'une telle nouvelle formule peut être transformée pour définir uniformément une chaîne de boules centrées en zéro (c'est à dire des sous-groupes) de rayons consécutifs strictement croissants, et uniquement des ensembles de cette forme. En considérant la formule dans $\mathbb{Z}$, elle définit alors un nombre cofini de sous-groupes de la forme $p^k\mathbb{Z}$, et seulement des ensembles de cette forme. Puisque $a \mid_p b$ si et seulement si pour tout $k \in \mathbb{N}$, $a \in p^k\mathbb{Z} \to b \in p^k\mathbb{Z}$, le résultat suit facilement.

Le théorème L et le résultat analogue de Conant ne sont plus vrais dans des extensions élémentaires. Nous donnons des contre-exemples dans la section 9.3. Néanmoins, pour des notions un peu plus fortes de réduits et d'expansions, les résultats de minimalité sont conservés dans des extensions élémentaires, voir le corollaire 9.1.9 et le théorème 9.2.12.

Au regard de la question $(\star\star)$ plus haut, on pourrait formuler la trichotomie suivante : une expansion dp-minimale de $(\mathbb{Z}, +, 0)$ est soit stable (et donc interdéfinissable avec $(\mathbb{Z}, +, 0)$), soit $(\mathbb{Z}, +, 0, <)$, soit elle définit une valuation. Cette conjecture est inspirée du résultat suivant concernant les corps, dû à Johnson [Joh15] : si K est un corps dp-minimal, alors il est soit algébriquement clos (donc stable), soit réel-clos, soit il admet une valuation henselienne définissable. Cependant, la conjecture pour $(\mathbb{Z}, +, 0)$ est fausse. En effet, Tran et Walsberg [TW17] ont trouvé une nouvelle famille d'expansions dp-minimales de $(\mathbb{Z}, +, 0)$, en ajoutant des ordres cycliques. Il serait intéressant de savoir si d'autres expansions dp-minimales de $(\mathbb{Z}, +, 0)$ existent, ou encore si ces expansions de $(\mathbb{Z}, +, 0)$ par des ordres cycliques satisfont la même propriété de minimalité que $(\mathbb{Z}, +, 0, <)$ et $(\mathbb{Z}, +, 0, |_p)$.

Introduction

Foreword

This thesis is organised as follows. First, the Preliminaries consist of basics, facts and preparatory results, to be used in the main text. Part A is devoted to the generic expansion of a structure by a predicate for a submodel of a reduct. The results in Part A come from my second, third and fourth years of Ph.D., under the supervision of Thomas Blossier and Zoé Chatzidakis, and two preprints are available online. In Part B, we study the expansions of the group of integers $(\mathbb{Z}, +, 0)$ by p-adic valuations. This work resulted in a paper [AE19], co-authored with Eran Alouf, published in the Journal of Symbolic Logic. Part B was done during the first year of my Ph.D., under the supervision of Pierre Simon. Part A and Part B can be read independently.

We start by introducing Part A. § 1 and § 2 focus on the notions of existentially closed structures and generic expansions. The reader familiar with these notions can skip those and jump directly to § 3 and § 4 where we present our first results. § 5 and § 6 are concerned with recent history and state of the art on classification and NSOP$_1$ theories, and § 7 links the construction of § 3 to this notion. § 8 presents our results on ACFG, then starts the introduction of Part B. § 9 introduces the current situation regarding expansions of the group of integers. § 10 and § 11 present our results on this subject.

§ 1 Existentially closed structures

Model theory is the study of mathematical structures through the prism of its algebra of definable sets. This algebra is in general hard to grasp, hence model theorists have always been in search of structures in which a reasonable description of definable sets is possible. Tarski [Tar51] shows in the 1930's that the theory of algebraically closed fields ACF in the language of fields and the theory RCF of real closed fields in the language of ordered fields have quantifier elimination: the study of the algebra of definable sets is reduced to the study of the boolean algebra spanned by basic sets. One easily deduce Hilbert's Nullstellensatz and Chevalley's theorem on constructible sets from the former. The latter allowed Robinson to give an elementary proof of Hilbert's 17th problem. This was the starting point of the development of methods for proving quantifier elimination, and the second half of the twenty-first century witnessed numerous other quantifier elimination results. The theories DCF$_0$ of differentially closed fields of characteristic 0 [Rob58] [Rob59a] and SCF$_{p,e}$ of separably closed fields of characteristic p and imperfection degree e [Ers67] [Del88] have quantifier elimination in suitable languages, these theories provided the adequate ambient structures in Hrushovski's celebrated proof of the Mordell-Lang conjecture [Bou+98].

A full quantifier elimination in a natural language is not always possible, this led Robinson to introduce the notion of *model-complete theory*, a weaker form of quantifier elimination, the elimination down to *existential* formulae. Wilkie [Wil96] proved that the theory of the real field with the exponential function $\mathbb{R}_{\exp}$ is model-complete, yielding o-minimality for $\mathbb{R}_{\exp}$, and answering partially a question asked by Tarski [Tar51]: the theory of $\mathbb{R}_{\exp}$ is decidable provided Schanuel's conjecture holds. Note that there is a comprehensible language in which $\mathbb{R}_{\exp}$ has quantifier elimination [DMM94], but it is rather complicated, and indicates that getting from model-completeness to quantifier elimination might be a hard step.

Intuitively, any form of quantifier elimination for a theory T witnesses when the language imposes a transfer principle, a "Nullstellensatz" between the models and some extensions (any superstructure for quantifier elimination, supermodels for model-completeness). Hence, forcing a tranfer principle for a structure would result in a well-behaved algebra of definable sets. A structure $\mathscr{M}$ is *existentially closed* in another structure $\mathscr{N}$ of the same language if every existential formula with parameters in $\mathscr{M}$ true in $\mathscr{N}$ is also true in $\mathscr{M}$. An algebraically (resp. separably) closed field is existentially closed in every field (resp. separable field)extension. A model of a theory T is *existentially closed* if it is existentially closed in every model of T extending it. If existentially closed models of a theory T exist and form an elementary class, their theory — the *model-companion* of T — is model-complete.

Pseudo algebraically closed (PAC) fields are pure fields which are existentially closed in every regular extension and, in general, there is no natural expansion of the language in which this theory is model-complete[8]. However, PAC fields have elementary invariants [CDM81], [FJ05]

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[8] The theory of pseudo-finite fields, the theory of ω-free PAC fields are theories of PAC fields in which a natural expansion of the language allows model-completeness, see Section 1.5.2.

and have been studied on several occasions [Hru02], [Cha99], [Cha02], [CH04]. PAC fields also provide examples of complex phenomena that play an important role in recent developments of model theory, see [CR16], [KR17], [Cha19], [Ram18], and also § 6.

Existentially closed models of a theory have in general some randomness — or *generic* — aspect, resulting from their definition. Informally, we will call *generic*[9] a theory (or a model of such theory) that axiomatises the structures that are existentially closed in a reasonable class of extension.

In many familiar theories, existentially closed models does not form an elementary class: the theories of groups [ES70], of nilpotent groups [Sar74], of solvable groups [Sar76], of commutative rings [Che73], of skew fields (Sabbagh, 1970, unpublished). Existentially closed models of these theories all interpret the structure $(\mathbb{Z}, +, \cdot)$. However, existentially closed groups and skew fields have been studied in the seventies, leading to striking connections between model theory, group theory and recursion theory, see [Zie76], [SZ79], [Zie80], [HW75], [Bel74], [Bel74], [Bel78a], [Bel78b] and [Mac77, p. 5.2] for a survey. More recently, Haykazyan and Kirby [HK18] have studied another class of existentially closed structures which admits no model-companion, we discuss it in § 6.

§ 2 Expansions and genericity

The first study of an unfamiliar expansion of a familiar structure was initiated by Tarski, when he asked in [Tar51] whether the theory of $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$, the real field structure with a unary predicate for the real algebraic numbers is decidable. Later, Robinson answered positively the question by proving the model-completeness of this theory in a natural expansion of the language, and the same result for the pair $(\mathbb{C}, \overline{\mathbb{Q}})$ [Rob59b].

The structure $(\mathbb{C}, \overline{\mathbb{Q}})$ or more generally any proper pair (K, k) of algebraically closed fields is existentially closed in any extension (L, l) such that l and K are linearly disjoint over k, hence the expansion of ACF by a predicate for a proper algebraically closed subfield enjoys some genericity property, which is rather exceptionnal. In general, one studies the existentially closed models of an expansion, hence the terminology “generic expansions”.

An important example of generic expansion is Winkler’s construction [Win75]. Consider an $\mathscr{L}$-theory T and a language $\mathscr{L}' \supset \mathscr{L}$. One can see T as an $\mathscr{L}'$-theory which does not impose any structure on elements of $\mathscr{L}' \setminus \mathscr{L}$. Winkler proves that as an $\mathscr{L}'$-theory, T has a model-companion, provided T is model-complete and eliminates $\exists^\infty$. The particular case in which the expansion is by a unary predicate –the *generic predicate*– has been studied by Chatzidakis and Pillay [CP98]. These generic expansions have connections with neostability theory, we investigate this direction in § 6.

A recent breakthrough in the area of generic expansions is the *interpolative fusion* construction [KTW18] by Kruckman, Tran and Walsberg. Given arbitrary many model-complete theories, they describe a general setting for the model-companion of the union of these theories to exists. It appears that many generic structures are bi-interpretable with an interpolative fusion of simpler structures.

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[9] The term generic has classically been used for models of a theory in which infinite forcing and model-theoretic satisfaction coincide (see for instance [Mac77] or [Che76]). If the theory is model-complete, then the notion of existentially closed model and generic model coincide. The term generic for a structure or a theory has nowadays an unclear meaning closer to ours.

Consider the following two expansions of ACF: T_1 is the expansion by a generic predicate, T_2 the expansion by a predicate for a proper algebraically closed subfield. These two theories are generic expansions of ACF by a predicate for a reduct (trivial in both cases), the first one is the theory of and infinite set in the trivial language, the second one is the full theory in the full language. The first result of this thesis, Theorem A, present a general setting for expanding a theory by a predicate for a reduct.

§ 3 Generic expansion by a pregeometric reduct

Let T be a theory in a language $\mathscr{L}$. Let $\mathscr{L}_0 \subseteq \mathscr{L}$ and T_0 a reduct of T to the language $\mathscr{L}_0$. Let acl_0 be the algebraic closure in the sense of $\mathscr{L}_0$. Let $\mathscr{L}_S = \mathscr{L} \cup \{S\}$, for S a new unary predicate symbol, and T_S be the $\mathscr{L}_S$-theory whose models $(\mathscr{M}, \mathscr{M}_0)$ consist in a model $\mathscr{M}$ of T in which S is a predicate for a model $\mathscr{M}_0$ of T_0 which is a substructure of $\mathscr{M}$. We present a setting in which we get partial results toward an axiomatisation of generic models of T_S. Assume the following.

(H_1) T is model complete;

(H_2) T_0 is model complete and $\mathrm{acl}_0(A) \models T_0$, for all infinite set A;

(H_3) T_0 is pregeometric (i.e. acl_0 satisfies exchange);

(H_4) for all $\mathscr{L}$-formula $\phi(x, y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for all $\mathscr{M} \models T$ and tuple b from $\mathscr{M}$,

$$\mathscr{M} \models \theta_\phi(b) \iff \begin{array}{l}\text{there exists } \mathscr{N} \succ \mathscr{M} \text{ and } a \in \mathscr{N} \text{ such that} \\ \phi(a, b) \text{ and } a \text{ is an independent tuple over } \mathscr{M}, \\ \text{in the sense of the pregeometry } \mathrm{acl}_0.\end{array}$$

We denote by $\mathop{\smile\!\!\!\!\!\!|}^0$ the independence relation in the sense of the pregeometry acl_0. We call an extension $(\mathscr{N}, \mathscr{N}_0)$ of $(\mathscr{M}, \mathscr{M}_0)$ *strong* if $\mathscr{N}_0 \mathop{\smile\!\!\!\!\!\!|}^0_{\mathscr{M}_0} \mathscr{M}$.

Theorem A. *There exists a unique theory TS containing T_S such that*

- *every model of T_S has a strong extension which is a model of TS;*
- *if $(\mathscr{M}, \mathscr{M}_0) \models TS$ and $(\mathscr{N}, \mathscr{N}_0) \models T_S$ is a strong extension of $(\mathscr{M}, \mathscr{M}_0)$ then $(\mathscr{M}, \mathscr{M}_0)$ is existentially closed in $(\mathscr{N}, \mathscr{N}_0)$.*

If acl_0 *defines a* modular *pregeometry, TS is the model-companion of T_S and in TS the algebraic closure is given by the algebraic closure in T.*

As usual in the proof of this kind of result, the axiomatisation gives an outline of the proof, it is given in Theorem 2.1.5. For a given tuple b in a model $\mathscr{M}$ of T, the formula $\theta_\phi(b)$ witnesses whenever the formula $\phi(x, b)$ have a realisation a in an elementary extension $\mathscr{N}$ of $\mathscr{M}$ such that for any partition $A_1 \cup A_2$ of the coordinates of a, there is an $\mathscr{L}_0$-substructure $\mathscr{N}_0$ of $\mathscr{N}$ model of T_0 which separates A_1 from A_2, by which we mean $A_1 \subseteq \mathscr{N}_0$ and $A_2 \cap \mathscr{N}_0 = \emptyset$. An existentially closed model of T_S should be able to realise all these possible attributions of coordinate for any such $\mathscr{L}$-formula. The main point here is that this is expressible in a first-order way, provided that formulae $\theta_\phi(y)$ exist.

Hypothesis (H_1) is obviously necessary, and (H_3) provides a general setup to express basic notions in a first-order way, and allows us to give a geometric treatment of the proof, by which we mean using $\mathop{\smile\!\!\!\!\!\!|}^{0}$ as in forking calculus, in order to prepare an adaptation of the proof to wider contexts. In (H_2), the fact that acl_0-closed infinite sets are models of T_0 can certainly be weakened to a condition close to "every acl_0-closed set embeds in a model of T_0", although this would increase the technicalities of the proof; and provide no more applications of the theorem than the ones we give in this thesis.

Hypothesis (H_4) is, in practice, a difficult condition to obtain. It can be thought of as a generalisation of elimination of $\exists^\infty$. If $T_0 = T$ and T is pregeometric, it is actually equivalent to elimination of $\exists^\infty$ (see Fact 1.3.10), hence for a geometric theory with $T = T_0$, one only need to check (H_2). The resulting theory is a generic pair of models of T. If T_0 is the theory of an infinite set in the emty language, condition (H_4) is again equivalent to elimination of $\exists^\infty$, (H_2) and (H_3) are trivial and Theorem A gives nothing more than the generic predicate construction of [CP98]. Condition (H_4) can also be seen as a "definability of the dimension" condition, although in a strong sense, as the condition involves independence in the sense of the reduct T_0 and not in the sense of T. An equivalent statement of (H_4) in terms of existence of bounds is given in Section 2.2, as well as a weak converse: assuming (H_1, H_2, H_3), if TS exists, then for each $\mathscr{L}$-formula $\phi(x, y)$ and $k \leq |x|$, there exists $\theta^k_\phi(y)$ such that for all $b \in \mathscr{M} \models T$, $\mathscr{M} \models \theta^k_\phi(b)$ if and only if there exists $\mathscr{N} \succ \mathscr{M}$ and $a \in \mathscr{N}$ such that $\phi(a, b)$ and $a_k \mathop{\smile\!\!\!\!\!\!|}^{0} \mathscr{M}(a_i)_{i \neq k}$. In particular T eliminates $\exists^\infty$. In general, (H_4) is not equivalent to elimination of $\exists^\infty$, this is discussed in § 4.

In the sense of [KTW18], the theory TS of Theorem A is the interpolative fusion of T with the theory of generic pairs of models of T_0.

We turn now to applications of Theorem A. Let $\mathbb{F}_{q_1}, \cdots, \mathbb{F}_{q_n}$ be finite fields, and T a theory in a language

$$\mathscr{L} = \left\{+, 0, (\lambda_\alpha)_{\alpha \in \mathbb{F}_{q_1}}, \ldots, (\lambda_\alpha)_{\alpha \in \mathbb{F}_{q_n}}, \cdots \right\},$$

such that every model of T carries, for all $1 \leq i \leq n$, a structure of infinite $\mathbb{F}_{q_i}$-vector space in the language $\{+, 0, (\lambda_\alpha)_{\alpha \in \mathbb{F}_{q_i}}\}$, where λ_α is the function interpreted as the multiplication by α.

Theorem B. *Let $V_1, \cdots, V_n$ be unary predicates and $T_{V_1 \ldots V_n}$ the theory of models of T where V_i is a predicate for a vector subspace over $\mathbb{F}_{q_i}$. If T is model-complete and eliminates $\exists^\infty$, then $T_{V_1 \ldots V_n}$ admits a model-companion.*

The framework described above encompasses the hypotheses (H_1, H_2, H_3), and elimination of $\exists^\infty$ gives, in that particular case, the condition (H_4), because of the uniform finiteness of the pregeometry and a classical lemma from [CP98]. Furthermore, applying once Theorem A results in a theory that also eliminates $\exists^\infty$, hence we may iterate and add as many generic vector subspaces as we want. The pregeometry associated to an $\mathbb{F}_{q_i}$-vector space is modular, hence the resulting theory is the model-companion.

§ 4 Generic expansions of fields

Generic additive subgroups in positive characteristic. Let p be a prime number. The additive group of a field of characteristic p is an $\mathbb{F}_p$-vector space, hence Theorem B applies. Let T be one of the following theory

- ACF_p the theory of algebraically closed fields of characteristic p in the language of fields;
- $\mathrm{SCF}_{p,e}$ the theory of separably closed fields of characteristic p and imperfection degree $e \leq \infty$, in the language of fields with predicates for p-independence;

- Psf_c the theory of pseudo-finite fields of characteristic p in the language of fields expanded by constants for coefficients of irreducible polynomials (Section 1.5.2);
- ACFA_p the theory of algebraically closed fields of characteristic p with a generic automorphism in the language of difference fields.

Then the expansion of T by finitely many generic additive subgroups exists. The expansion of ACF_p by a generic additive subgroup will be denoted by ACFG, and it only exists in positive characteristic, as we will see later. Chapters 5, 6 and 7 of this thesis are devoted to a study of ACFG, those results are described in § 8. We also get the following generic expansion of perfect PAC fields in positive characteristic.

Theorem C. *Let* PAC_G *be the theory whose models are perfect* PAC *fields of characteristic* p *with a predicate* G *for an additive subgroup. Then there exists a theory* PACG *such that*

(1) every model (F, G) *of* PAC_G *extends to a model* (K, G) *of* PACG *such that* K *is a regular extension of* F*;*

(2) every model (K, G) *of* PACG *is existentially closed in every extension* (F, G) *such that* F *is a regular extension of* K*.*

It is possible to iterate this construction.

All the previous results concerning generic expansions of fields of positive characteristic are also true when replacing G by an $\mathbb{F}_q$-vector space V, for some finite subfield $\mathbb{F}_q$ of the ambiant field.

Generic additive subgroups in characteristic zero. The previous results have no analogue in characteristic 0. Let T be *any* inductive theory of a field of characteristic 0 in a language $\mathscr{L}$ containing the language of rings. Let G be a new predicate and T_G be the $\mathscr{L} \cup \{G\}$-theory of models of T in which G is a predicate for an additive subgroup of the field, this is an inductive theory. An simple argument (Proposition 3.2.7) shows that if (K, G) is an existentially closed model of T_G, then $\{a \in K \mid aG \subseteq G\} = \mathbb{Z}$. In particular, the theory T_G does not admit a model-companion. Similarly, if one imposes G to be divisible, the stabiliser of the group is $\mathbb{Q}$. Furthermore, consider the case in which T is the theory of $\mathbb{R}$ or of $\mathbb{C}$, then hypotheses (H_1, H_2, H_3) hold, so by the contrapositive of Theorem A, condition (H_4) does not hold, even though T eliminate $\exists^\infty$.

Generic multiplicative subgroups in all characteristic. The generic expansion by an additive subgroup fails in characteristic zero, however, we have the following.

Theorem D. *Let* p *be a prime number or zero. The expansion of* ACF_p *by a generic multiplicative subgroup exists.*

Hypotheses (H_1, H_2, H_3) are easy to check. We prove hypothesis (H_4) only for formulae that define quasi-affine varieties, which is sufficient for proving that the model-companion exists. Hypothesis (H_4) follows from a definability result in abstract Kummer theory. Let $W \subset K^n \setminus \{(0, \cdots, 0)\}$ be an affine irreducible algebraic variety in an algebraically closed field K of characteristic $p \geq 0$. We say that W is *free* if it is not contained in any translate of a proper algebraic subgroup of the torus $\mathbb{G}_m^n(K)$. Bays, Gavrilovitch and Hils show in [BGH13] that W is free if and only if every element in $\mathbb{G}_m^n(K)$ is the product of $2n$ elements from W, which is a definable condition[10].

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[10] Note that Minh Chieu Tran [Tra17] also obtained the definability of the freeness of an affine irreducible variety (which he calls *multiplicatively largeness*), using Zilber's indecomposability theorem. As a matter of fact, the result of Bays, Gavrilovitch and Hils [BGH13] is more general, they prove it when replacing $\mathbb{G}_m^n(K)$ by any semiabelian variety.

§ 5 Shelah's Classification

An important part of model theory consists in defining various notions of "tameness" in order to classify and understand mathematical structures. A leading idea, initiated by Shelah, is that the "wildness" of a structure can be detected in the combinatorial complexity of bipartite graphs associated with definable sets. Thus, tameness is associated with the absence of some combinatorial configuration in the bipartite graph of any formula. For instance, the so-called stable structures are those that avoid defining an infinite half graph. One of the most striking facts in stability theory is that its combinatorial definition is equivalent to the existence of a well-behaved notion of independence in every model, based on Shelah's forking. During the last two decades, model theorists have tried to apply stability theoretic methods to unstable theories, this is called neostability theory. For instance, simplicity is a generalisation of stability. Simple theories are also defined by a combinatorial condition and are also characterized by the good behaviour of forking independence, this is known as Kim-Pillay theorem [KP97]. The theories of infinite random graphs, of bounded PAC fields, ACFA, are examples of simple theories. However, the theories of ω-free PAC fields [Cha99], of generic $K_{m,n}$-free bipartite graph [CK17] (for $n, m \geq 2$) are not simple, they are NSOP_1 theories, a generalisation of simplicity.

§ 6 A recent history of NSOP_1 theories

NSOP_1 theories, for "not strong order property 1", were defined by Džamonja and Shelah in [DS04] (together with NSOP_2) as an extension of the $(\mathrm{NSOP}_n)_{n\geq 3}$ hierarchy. In [SU08], Shelah and Usvyatsov proved that T^*_{feq} (the model completion of the theory of infinitely many independent parametrized equivalence relations) is NSOP_1 and not simple. For the past three years, NSOP_1 theories have been intensively studied through two different approaches (not mutually exclusive): the abstract one, in which combinatorics and pure model theory are involved; and the applied one, which consists in the study of particular examples.

The first breakthrough concerning the abstract study of NSOP_1 theories was made by Chernikov and Ramsey in [CR16]. They proved a Kim-Pillay style result [CR16] which states that a theory is NSOP_1 provided there exists an independence relation satisfying some specific properties. This result turned out to be a very useful tool to prove that a theory is NSOP_1. The ω-free PAC fields case is a good example. A PAC field is simple if [CP98] and only if [Cha99] it is bounded. Nonetheless, in her work [Cha02] on ω-free PAC fields (which are unbounded), Chatzidakis defined a weak notion of independence and showed that it satisfied some nice properties, in particular, the so-called *independence theorem*. It turned out that almost all the properties of the criterion [CR16] were proved at that time. Chernikov and Ramsey used this weak independence to deduce that the theory of ω-free PAC fields is NSOP_1. They also showed that Granger's example of generic bilinear form over an infinite dimensional vector space over an algebraically closed field is NSOP_1 (see [Gra99] or [CR16, Example 6.1]), as well as the combinatorial example of a generalised parametrized structure (see [CR16, Example 6.3]). The second breakthrough in the abstract study of NSOP_1 theories was the development of Kim-independence by Kaplan and Ramsey in [KR17]. They introduced analogues of forking and dividing –Kim-forking and Kim-dividing– which behave nicely in NSOP_1 theories. Kim-dividing is defined as dividing with respect to some particular indiscernible sequences, namely sequences in a global invariant type. Numerous properties of forking in simple theories appears for Kim-forking in NSOP_1 theories. For instance, a theory is NSOP_1 if and only if Kim-independence is symmetric. Kaplan and Ramsey also completed the Kim-Pillay style criterion in [CR16] to get a characterisation of Kim-independence[11] in terms of properties of a ternary relation, similarly to the Kim-Pillay

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[11] Actually they proved that if $\mathop{\smile}\limits^{|}$ satisfies the conditions of [CR16, Proposition 5.3] then $\mathop{\smile}\limits^{|}$ strengthens Kim-independence. The *Witnessing* condition ensures that Kim-dividing independence strengthens $\mathop{\smile}\limits^{|}$.

classical result. Using this tool, they identified Kim-independence in various NSOP$_1$ theories. Chatzidakis' weak independence in ω-free PAC fields turned out to be Kim-independence. In Granger's example, the independence relation that he studied which satisfied the Chernikov and Ramsey's criterion for NSOP$_1$ is strictly stronger than Kim-independence.

Concerning the applied approach, Conant and Kruckman's generic incidence structures [CK17], Barbina and Casanovas' Steiner triple system [BC18] are new examples of NSOP$_1$ theories. As we will see in § 7, ACFG and almost all examples in § 4 are new NSOP$_1$ theories. Most of these examples are generic constructions, and they share many common features. Simple theories have commonly been considered as stable ones with some "random noise". A strongly supporting fact for this is the construction of the generic predicate [CP98]. Adding a generic predicate preserves simplicity, by which we mean that if T is simple, then the theory of the generic predicate on T is also simple. However, if more complex genericity is involved, simplicity may not be preserved, even starting with a very tame theory. We will see in § 7 that ACFG is not simple, even though it is the generic expansion of a strongly minimal theory. Kruckman and Ramsey [KR18] prove that Winkler's generic expansion by an arbitrary language [Win75], discussed in § 2, preserves NSOP$_1$. They also show that another construction from Winkler [Win75], the expansion of a theory by generic Skolem functions, preserves NSOP$_1$, and deduce the following interesting fact: any NSOP$_1$ theory that eliminates $\exists^\infty$ can be expanded to an NSOP$_1$ theory that has built-in Skolem functions. Concerning PAC fields, the general intuition is that their model-theoretic features can be deduced from the model-theoretic features of their absolute Galois group. Recent results from Chatzidakis and Ramsey strongly support this idea: a PAC field is NSOP$_n$ if and only if its absolute Galois group is NSOP$_n$ ([Cha19] for $n \geq 3$, [Ram18] for $n = 1, 2$).

Recent work from Haykazyan and Kirby [HK18], highlights a new source of NSOP$_1$ theories, in the sense of positive logic. They study the class of existentially closed exponential fields (an exponential field is a field with a group homomorphism from the additive group to the multiplicative group of the field). As for the class of existentially closed skew fields or fields with an additive subgroup in characteristic 0, this class is not elementary. However, an idea which goes back to Shelah [She75], consists in dealing with those non-elementary classes by considering only *existential* formulae, this is called positive logic, and was developed in different ways by Ben-Yaacov [Ben03a] and Pillay [Pil00]. Suitable notions of stability [Pil00] [Ben03b] and simplicity [Ben03b] were further developed. Haykazyan and Kirby [HK18] adapted the result of Chernikov and Ramsey [CR16] to prove that the class of existentially closed exponential fields is NSOP$_1$ in the sense of positive logic, using the existence of a well-behaved independence relation. It is likely that the theory developed by Haykazyan and Kirby can be used to show that the class of algebraically closed fields of characteristic 0 with a generic additive subgroup is NSOP$_1$ in the sense of positive logic.

§ 7 Preservation of NSOP$_1$

Our next result gives a condition which makes the expansion by a generic reduct § 3 an NSOP$_1$-preserving construction. As in § 3, let T be an $\mathscr{L}$-theory, $\mathscr{L}_0 \subseteq \mathscr{L}$ and $T_0 \upharpoonright \mathscr{L}_0$. Assume that hypotheses (H_1) to (H_4) are satisfied. We denote by acl_T the algebraic closure in the sense of $\mathscr{L}$, acl_0 the algebraic closure in the sense of $\mathscr{L}_0$ and $\mathop{\smile\hskip -0.9em ^| \ }^{0}$ the independence relation associated with the pregeometry acl_0.

Theorem E. *Assume that* acl_0 *defines a modular pregeometry,* T *is* NSOP$_1$ *and* $\mathop{\smile\hskip -0.9em ^| \ }^{T}$ *is the Kim-independence relation in* T*. Let* $\mathscr{M} \models T$ *and* A, B, C *algebraically closed containing* $\mathscr{M}$*, in a monster model of* T*. Assume the following.*

(A) *For all such* $A, B, C, \mathscr{M}$*, if* $C \mathop{\smile\hspace{-0.9em}|}^{T}_{\mathscr{M}} A, B$ *and* $A \mathop{\smile\hspace{-0.9em}|}^{T}_{\mathscr{M}} B$ *then*

$$(\mathrm{acl}_T(AC), \mathrm{acl}_T(BC)) \mathop{\smile\hspace{-0.9em}|}\limits^{0}_{A,B} \mathrm{acl}_T(AB).$$

Then TS *is* NSOP$_1$ *and the Kim-independence relation in* TS *is given by the relation* $\mathop{\smile\hspace{-0.9em}|}^{w}$*, defined by*

$$A \mathop{\smile\hspace{-0.9em}|}\limits^{T}_{\mathscr{M}} B \text{ and } S(\mathrm{acl}_0(\mathrm{acl}_T(A\mathscr{M}), \mathrm{acl}_T(B\mathscr{M}))) = \mathrm{acl}_0(S(\mathrm{acl}_T(A\mathscr{M})), S(\mathrm{acl}_T(B\mathscr{M}))).$$

Theorem E is a consequence of more general considerations. Starting from an independence relation $\mathop{\smile\hspace{-0.9em}|}^{T}$ in models of T, we define two independence relations for models of TS, a *strong* independence $\mathop{\smile\hspace{-0.9em}|}^{st}$ and a *weak* independence $\mathop{\smile\hspace{-0.9em}|}^{w}$. Both relations $\mathop{\smile\hspace{-0.9em}|}^{w}$ and $\mathop{\smile\hspace{-0.9em}|}^{st}$ extend the relation $\mathop{\smile\hspace{-0.9em}|}^{T}$. In Chapter 4, we analyse properties of $\mathop{\smile\hspace{-0.9em}|}^{T}$ that are transferred to $\mathop{\smile\hspace{-0.9em}|}^{w}$ and $\mathop{\smile\hspace{-0.9em}|}^{st}$. If T is NSOP$_1$ and $\mathop{\smile\hspace{-0.9em}|}^{T}$ is Kim-independence, then all properties satisfied by $\mathop{\smile\hspace{-0.9em}|}^{T}$ that characterize Kim-independence and NSOP$_1$ theories are transferred to $\mathop{\smile\hspace{-0.9em}|}^{w}$, apart from the independence theorem, which requires hypothesis (A). This gives Theorem E. We give a fine analysis of the conservation of properties from $\mathop{\smile\hspace{-0.9em}|}^{T}$ to $\mathop{\smile\hspace{-0.9em}|}^{w}$ and $\mathop{\smile\hspace{-0.9em}|}^{st}$. For instance, if $\mathop{\smile\hspace{-0.9em}|}^{T}$ is stationnary over algebraically closed sets, so is $\mathop{\smile\hspace{-0.9em}|}^{st}$. If $\mathop{\smile\hspace{-0.9em}|}^{T}$ satisfies $\mathop{\smile\hspace{-0.9em}|}'$-*amalgamation* (a version of the independence theorem in which the parameters can be choosen independent in the sense of another independence relation $\mathop{\smile\hspace{-0.9em}|}'$, see Section 1.2) then $\mathop{\smile\hspace{-0.9em}|}^{w}$ also satisfies $\mathop{\smile\hspace{-0.9em}|}'$-amalgamation (Theorem 4.1.5). This is used to show that when it is possible, iterating the expansion by a generic reduct also preserves NSOP$_1$ (Corollary 4.2.4).

Kaplan and Ramsey [KR17] also give a necessary and sufficient geometric condition for an NSOP$_1$ theory to be simple: Kim-independence has to satisfy the so-called *Base monotonicity* property (if $a \mathop{\smile\hspace{-0.9em}|}_C bd$ then $a \mathop{\smile\hspace{-0.9em}|}_{Cb} d$, see Section 1.2). This translates in our context as a useful criterion to show when TS is not simple, this depends on the entanglement of acl_0 and acl_T (Corollary 4.2.3). Also, condition (A) expresses how $\mathop{\smile\hspace{-0.9em}|}^{T}$ controls $\mathop{\smile\hspace{-0.9em}|}^{0}$, the independence in the sense of the pregeometric reduct T_0. The "proximity" between T and T_0 plays an interesting role concerning the preservation of neostability notions for the expansion TS, as we can summarize in the following table.

Configuration $T_0 \subseteq T$	Generic expansion TS
$T_0 = T$	Preserves stability
$T_0 \subseteq T$	Preserves NSOP$_1$
$T_0 = \emptyset$	Preserves simplicity

If T is a theory of fields of positive characteristic and T_0 is the additive reduct of T, then condition (A) follows from a simpler assumption, (B) below. For a field A we denote by $\overline{A}$ the field theoretic algebraic closure of A.

Theorem F. *Let* T *be a model-complete theory of an* NSOP$_1$ *field of positive characteristic that eliminates* $\exists^\infty$*. Let* A, B *be* acl_T*-closed and* $E \models T$ *contained in* A *and* B*. Let* $\mathop{\smile\hspace{-0.9em}|}^{T}$ *be Kim-independence in* T *and assume that :*

(B) *for all such* A, B *and* E*, if* $A \mathop{\smile\hspace{-0.9em}|}^{T}_{E} B$ *then* $\mathrm{acl}_T(AB) \subseteq \overline{AB}$.

Then the expansion of T *by generic additive subgroups* $TG_1 \dots G_n$ *is* NSOP$_1$*. Kim-independence in* $TG_1 \dots G_n$ *is given by*

$$A \mathop{\smile\hspace{-0.9em}|}\limits^{T}_{E} B \text{ and for all } i \leq n \ G_i(A + B) = G_i(A) + G_i(B),$$

for A, B *and* E *as above. Furthermore,* $TG_1 \cdots G_n$ *is not simple.*

In particular, all examples in positive characteristic given in § 4 (except the one in Theorem C) are new NSOP_1 and not simple theories. Concerning the theory in Theorem C, not all perfect PAC fields are NSOP_1, however they always satisfy hypothesis (B) (because they are *algebraically bounded* [CH04]). It follows that the expansion in a suitable language of the theory of perfect Frobenius fields, or perfect ω-free PAC fields by generic additive subgroups, is NSOP_1. Note that in Theorem F, each G_i can be replaced by an $\mathbb{F}_{q_i}$-vector space, for a subfield $\mathbb{F}_{q_i}$ of any model of T.

The proof of Theorem F consists in deducing (A) from (B). It involves a description of Kim-independence in any theory of fields by Kaplan and Ramsey [KR17], following the work of Chatzidakis [Cha99]. The theorem follows from stability flavoured arguments in the separable closure of the field and some Galois theory.

Finally, for any p prime or zero, the expansion of ACF_p by a generic multiplicative subgroup is also NSOP_1 and not simple. We use Theorem E, condition (A) follows from a coheir argument in the stable theory ACF_p.

§ 8 The theory ACFG

Chapters 5, 6 and 7 are dedicated to the study of the theory ACFG, the expansion of ACF_p by a generic additive subgroup, for a fixed prime p.

Almost simple. In ACFG, there is no independence relation satisfying the Kim-Pillay criterion for simplicity. However, we have the following.

Theorem G.
- *In* ACFG*, Kim-independence satisfies all the properties of the Kim-Pillay characterisation of simple theories except* Base Monotonicity*.*
- *In* ACFG*, there is an independence relation which satisfies all the properties of the Kim-Pillay characterisation of simple theories except* Local Character*.*

The independence in the second item is the strong independence, mentioned in § 7.

Models of ACFG. Let (K, G) be a model of ACFG, G is the generic subgroup of the field K. The group G enjoys some interesting algebraic properties, for instance, it is dense and codense for the Zariski topology on K. Also, every element in K is the product of two elements of the group, which implies that G is stably embedded in K.

Let $\overline{\mathbb{F}_p}$ be the field theoretic algebraic closure of the prime field $\mathbb{F}_p$. Using that $\overline{\mathbb{F}_p}$ is locally finite and quantifier elimination in ACF_p, we construct by union of chain a subgroup G of $\overline{\mathbb{F}_p}$ such that $(\overline{\mathbb{F}_p}, G)$ is a model of ACFG. The space $\mathrm{Sg}(\overline{\mathbb{F}_p})$ of additive subgroups of $\overline{\mathbb{F}_p}$ is endowed with the Chabauty topology (Section 1.6). We show that almost all (in the sense of Baire) additive subgroups G of $\overline{\mathbb{F}_p}$ are generic.

Theorem H. *The set of additive subgroups G of $\overline{\mathbb{F}_p}$ such that $(\overline{\mathbb{F}_p}, G) \models$ ACFG is a dense G_δ of $\mathrm{Sg}(\overline{\mathbb{F}_p})$, for the Chabauty topology on $\mathrm{Sg}(\overline{\mathbb{F}_p})$.*

This result is proved in the same way as the analogous result from Hrushovski about models of ACFA in $\overline{\mathbb{F}_p}$ [Hru04].

Imaginaries in ACFG. Let (K, G) be a model of ACFG. There are no canonical parameters for elements of the quotient group K/G. A natural question to ask is the following: if one adds to (K, G) canonical parameters for the quotient K/G, does one add canonical parameters for every definable equivalence class in (K, G)? In order to answer this question, denote by $(K, K/G)$ the 2-sorted structure consisting in one sort for the field K and one sort for the group K/G, together with the canonical projection $\pi : K \to K/G$.

Theorem I. *For any model* (K, G) *of* ACFG*, the structure* $(K, K/G)$ *has weak elimination of imaginaries.*

Weak elimination of imaginaries is optimal for $(K, K/G)$. Indeed, $(K/G, +)$ carries the structure of a pure $\mathbb{F}_p$-vector space, hence finite imaginaries from this sort cannot be eliminated. Furthermore, canonical parameters for finite imaginaries of K/G are not enough to code all finite imaginaries in $(K, K/G)$, see Example 6.3.6. The proof of Theorem I follows the same classical pattern as the proof of elimination of imaginaries in [Hru02], [CH99] or [CP98], and is based on the independence theorem. In our case, however, Kim-independence will play the role that forking independence plays in those classical proofs. Kim-independence in (K, G) is given by $A \mathop{\smile\hspace{-0.9em}|}\nolimits^{\mathrm{ACF}}_{C} B$ and $G(A+B) = G(A)+G(B)$, for $C = A \cap B$ and A, B, C algebraically closed. In $(K, K/G)$, the condition $G(A + B) = G(A) + G(B)$ translates to $\pi(A) \cap \pi(B) = \pi(C)$, a "stable modular" independence coming from the pure $\mathbb{F}_p$-vector space structure on the sort K/G. Hence in $(K, K/G)$, Kim-independence is given by the conjunction of the independence in the sense of ACF in the sort K and the "stable modular" independence in the sort K/G, which is easier to apprehend that Kim-independence in (K, G). Also, conditions involving elements of K/G are translated in terms of special representatives (*minimal* and *maximal*) in K, in order to deduce a version of the independence theorem and adapt the classical proofs. The proof of Theorem I should go through as it is by replacing the theory ACF_p by any stable theory of fields with elimination of imaginaries.

The study of the theory of $(K, K/G)$ suggests a "dual" generic expansion. Starting from a theory T and a reduct T_0 of T in the language $\mathscr{L}_0$. Consider the two sorted theory T' whose models consists in a model $\mathscr{M}$ of T in the first sort, a model $\mathscr{M}_0$ of T_0 in the second sort and a surjective $\mathscr{L}_0$-homomorphism $\pi : \mathscr{M} \to \mathscr{M}_0$. Then under hypotheses (H_1) to (H_4) one constructs the model-companion of T', in which imaginaries should be easier to deal with.

Forking and Thorn-forking in ACFG. Forking is not symmetric in ACFG, otherwise, the theory would be simple. Theorem G gives the loose intuition that Kim-independence and forking independence differs only by the property Base Monotonicity. Indeed, we prove that forking independence is obtained by "forcing" the property base monotonicity on Kim-independence. We show that forking equals dividing for types and that it also equals thorn forking.

Theorem J. *Let* $\mathop{\smile\hspace{-0.9em}|}\nolimits^{f}$, $\mathop{\smile\hspace{-0.9em}|}\nolimits^{d}$, $\mathop{\smile\hspace{-0.9em}|}\nolimits^{\text{þ}}$, $\mathop{\smile\hspace{-0.9em}|}\nolimits^{K}$ *be respectively the forking, dividing, thorn-forking*[12]*, and Kim independence relation in* ACFG*. Then*

$$a \mathop{\smile\hspace{-0.9em}|}\limits^{f}_{C} b \iff a \mathop{\smile\hspace{-0.9em}|}\limits^{d}_{C} b \iff a \mathop{\smile\hspace{-0.9em}|}\limits^{\text{þ}}_{C} b \iff \textit{for all } C \subseteq D \subseteq Cb,\ \ a \mathop{\smile\hspace{-0.9em}|}\limits^{K}_{D} b.$$

In NSOP_1 theories, forking independence and Kim independence are different notions, no good description of forking independence exists, but most of the known examples of NSOP_1 and not simple theories share the same description of forking as ACFG [Cha02], [CK17], [KR18]. A similar pattern for proving that forking equals dividing for types seems to emerge from different examples. It involves a "mixed transitivity" result between the strong independence and Kim-independence, from which one deduces that dividing independence satisfies the property *Extension*, hence equals forking. A discussion on the features shared by the main examples of NSOP_1 theories is given in Section 7.4, as well as the main differences between ACFG and

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[12] Here we mean the restriction of thorn forking to the home sort, as thorn forking in general is defined in T^{eq}.

the other examples (see also Figure 7.2). We also advertise some nice phenomena that appear when one forces the *Base Monotonicity* property on a given independence relation (Section 7.1).

Our treatment of thorn-forking in ACFG uses the geometric description of the thorn-forking independence given by Adler [Adl09a]. More generally, Part A and especially Chapter 7 make use of the geometric treatment of independence relations taking roots in Kim-Pillay theorem but mainly developed by Adler [Adl08a], [Adl09a], [Adl09b], then followed by [CK12], [CR16], [CK17] among others.

We now turn to part B.

§ 9 Expansions of the group of integers

From a model-theoretic point of view, the structure $(\mathbb{Z}, +, \cdot)$ is not understandable, this follows from Godel's celebrated work on Peano arithmetic. Starting from this fact, it is natural to study tame reducts of $(\mathbb{Z}, +, \cdot)$. The study of the structures $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, +, 0, <)$ dates back from 1929 by Presburger, the theory of $(\mathbb{Z}, +, 0, <)$ is still referred to as *Presburger arithmetic*[13]. Both structures admit quantifier elimination in the expansion of the language by the constant 1 and predicates for the subgroups $n\mathbb{Z}$, for all $n \in \mathbb{N}$. The theory of $(\mathbb{Z}, +, 0)$ is superstable of U-rank 1. Presburger arithmetic, however, is unstable but NIP and moreover dp-minimal.

The study of tame expansions of $(\mathbb{Z}, +, 0)$ is a recent subject. Until not long ago, no examples of such structures were studied, other than $(\mathbb{Z}, +, 0, <)$. The first examples were given independently by Palacín and Sklinos [PS18] and by Poizat [Poi14]. Specifically, they both proved, using different methods, that for any integer $q \geq 2$ the structure $(\mathbb{Z}, +, 0, \prod_q)$ is superstable of U-rank ω, where $\prod_q = \{q^n \mid n \in \mathbb{N}\}$. Palacín and Sklinos also showed the same result for other examples, such as $(\mathbb{Z}, +, 0, \mathrm{Fac})$, where $\mathrm{Fac} = \{n! \mid n \in \mathbb{N}\}$. Conant [Con17b] and Lambotte and Point [LP17] independently generalized these results. For a subset $A \subseteq \mathbb{Z}$ with either an upper bound or a lower bound, they give some sparsity conditions on A which are sufficient for the structure $(\mathbb{Z}, +, 0, A)$ to be superstable of U-rank ω. Conant also gives sparsity conditions which are necessary for the structure $(\mathbb{Z}, +, 0, A)$ to be stable.

A different kind of example was given recently by Kaplan and Shelah in [KS17]. They proved that for $\mathrm{Pr} = \{p \in \mathbb{Z} \mid |p| \text{ is prime}\}$, the structure $(\mathbb{Z}, +, 0, \mathrm{Pr})$ has the independence property (and even the n-independence property for all n) hence it is unstable. On the other hand, assuming Dickson's Conjecture [14], it is supersimple of U-rank 1.

In contrast to the above, $(\mathbb{Z}, +, 0, <)$ remained the only known unstable dp-minimal expansion of $(\mathbb{Z}, +, 0)$. In [Asc+13, Question 5.32], Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko ask the following question: is every dp-minimal expansion of $(\mathbb{Z}, +, 0)$ a reduct of $(\mathbb{Z}, +, 0, <) \quad (\star)$. In [Asc+16] the same authors prove that $(\mathbb{Z}, +, 0, <)$ has no proper dp-minimal expansions. They use a strong result from automaton theory of Michaux and Villemaire [MV96], that can be stated as follows: every proper expansion of $(\mathbb{Z}, +, 0, <)$ defines a new subset of $\mathbb{Z}$. This was later strengthened by Dolich and Goodrick [DG17], they obtain that $(\mathbb{Z}, +, 0, <)$ has no proper strong expansions. Together with a result of Conant [Con18]

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[13] Note that for some logicians, Presburger arithmetic is the theory of $(\mathbb{Z}, +, 0)$ and indeed, in his 1929 paper, Presburger studied mainly the theory of $(\mathbb{Z}, +, 0)$. However, in the same paper, he explains that his results extend to the theory of $(\mathbb{Z}, +, 0, <)$, see [Haa18] for a comprehensive survey on Presburger arithmetic.

[14] A strong number-theoretic conjecture about the distribution of prime numbers in arithmetic progressions, which generalizes Dirichlet's theorem on prime numbers.

which we describe below, any other unstable dp-minimal expansion of $(\mathbb{Z},+,0)$, if exists, is not a reduct, nor an expansion of $(\mathbb{Z},+,0,<)$, thus question $(\star)$ becomes:

$$\text{Is } (\mathbb{Z},+,0,<) \text{ the only dp-minimal non-trivial expansion of } (\mathbb{Z},+,0)? \qquad (\star\star)$$

§ 10 New dp-minimal expansions of the integers

We introduce a new family of dp-minimal expansions of $(\mathbb{Z},+,0)$, thus giving a negative answer to question $(\star\star)$ above. More generally, for every $n \in \mathbb{N} \cup \{\omega\}$ we introduce a family of expansions of $(\mathbb{Z},+,0)$ having dp-rank n. After proving this we were informed that a similar result has been proved independently by François Guignot [Gui16], and by Nathanaël Mariaule [Mar17, Corollary 2.11].

Let P be a (finite or infinite) set of prime numbers. For each $p \in P$ consider the preorder on integers given by

$$a \mid_p b \iff v_p(a) \leq v_p(b)$$

where v_p is the p-adic valuation on $\mathbb{Z}$.

Theorem K. *The structure* $(\mathbb{Z},+,0,(\cdot \mid_p \cdot)_{p\in P})$ *has quantifier elimination in the language expanded by the constant* 1 *and predicates for the subgroups* $n\mathbb{Z}$*, for all* $n \in \mathbb{N}$*. Furthermore, its dp-rank equals* $|P|$*, the cardinality of* P*.*

Each p-adic valuation on $\mathbb{Z}$ imposes a tree-like structure on the integers, and a tree topology similar to the one we describe in Section 1.6. Each integer is represented as a branch on which each node represents its coordinate in the p-adic representation. This tree structure is preserved in elementary extensions and allows a graphical treatment of the arguments. The proof of quantifier elimination is technical but does not use more complicated arithmetic result than the Chinese remainder theorem, which translates here as the topological density of each subgroup $n\mathbb{Z}$ in $\mathbb{Z}$, for n coprime with p.

The computing of the dp-rank of $(\mathbb{Z},+,0,(\cdot \mid_p \cdot)_{p\in P})$ is in two steps. First we prove dp-minimality for the case $P = \{p\}$, this involves quantifier elimination and the dp-minimality of the structure $(\mathbb{Q}_p,+,0,\mid_p)$ [DGL11]. Then we deduce from quantifier elimination that the dp-rank of $(\mathbb{Z},+,0,(\cdot \mid_p \cdot)_{p\in P})$ cannot grow more than the sum of the dp-ranks of each reduct $(\mathbb{Z},+,0,\mid_p)$, and we conclude by exhibiting an ict-pattern of length $|P|$.

Quantifier elimination in $(\mathbb{Z},+,0,\mid_p)$ implies that every definable set is a boolean combination of $\emptyset$-definable sets and balls. C-minimality is a notion introduced by Macpherson and Steinhorn [MS96] to give an analogue of o-minimality in the context of valued or tree-like structures which admits quantifier elimination. Morally in a C-minimal tree-like structure, every definable set is a boolean combination of balls. This suggests a definition of *quasi*-C-minimality, analogously to quasi-o-minimality [BPW00], which would hopefully imply dp-minimality. Similarly for the even more general notion of VC-minimality [Adl08b].

§ 11 Minimality phenomena

Given a class $\mathcal{C}$ of structures with the same underlying universe, and $\mathscr{M} \in \mathcal{C}$, we say that $\mathscr{M}$ is *minimal* in $\mathcal{C}$ if there is no proper reduct of $\mathscr{M}$ in $\mathcal{C}$. Similarly, $\mathscr{M}$ is *maximal* in $\mathcal{C}$ if there are no proper expansion of $\mathscr{M}$ in $\mathcal{C}$.

A first example of this phenomenon was given by Pillay and Steinhorn [PS87]: $(\mathbb{N}, <)$ has no proper o-minimal expansions, in other words, $(\mathbb{N}, <)$ is maximal in the class of o-minimal structures with underlying universe $\mathbb{N}$. Another example was given by Marker, motivated by the following question from Zilber: can an algebraically closed field have a proper strongly minimal expansion? Although this question was answered positively by Hrushovski [Hru92], Marker proved [Mar90] that $(\mathbb{C}, +, \cdot, 0, 1)$ has no proper expansion *at all* which is a proper reduct of $(\mathbb{C}, +, \cdot, 0, 1, \mathbb{R})$ i.e., $(\mathbb{C}, +, \cdot, 0, 1, \mathbb{R})$ is minimal among the proper expansions of $(\mathbb{C}, +, \cdot, 0, 1)$.

The study of expansions of the group $(\mathbb{Z}, +, 0)$ recently produced numerous minimality/maximality flavoured results. As we saw above, $(\mathbb{Z}, +, 0, <)$ is a maximal dp-minimal structure on $\mathbb{Z}$ [Asc+16]. Based on a result by Palacín and Sklinos [PS18], Conant and Pillay [CP18] proved the following: $(\mathbb{Z}, +, 0)$ has no proper stable expansions of finite dp-rank. In other words, $(\mathbb{Z}, +, 0)$ is maximal among the stable structures of finite dp-rank on $\mathbb{Z}$. As $(\mathbb{Z}, +, 0, <)$ is dp-minimal, an immediate consequence of the above is that there is no stable structures which is both a proper expansion of $(\mathbb{Z}, +, 0)$ and a proper reduct of $(\mathbb{Z}, +, 0, <)$. In [Con18] Conant strengthened this result by proving that there are no structures *at all* which are both proper expansions of $(\mathbb{Z}, +, 0)$ and proper reducts of $(\mathbb{Z}, +, 0, <)$, hence Conant proved that $(\mathbb{Z}, +, 0, <)$ is a minimal proper expansion of $(\mathbb{Z}, +, 0)$. We proved the corresponding result for our expansion of $(\mathbb{Z}, +, 0)$.

Theorem L. *$(\mathbb{Z}, +, 0, |_p)$ is a minimal proper expansions of $(\mathbb{Z}, +, 0)$.*

Conant's proof [Con18] does not use that $(\mathbb{Z}, +, 0)$ has no proper stable expansion which is a reduct of $(\mathbb{Z}, +, 0, <)$. His proof involves detailed analysis of definable sets in arbitrary dimension. Conant asked whether his theorem could be proved using model-theoretic methods which incorporate the result [CP18]. This is the strategy we adopt to prove Theorem L, hence the content of Theorem L is really that there is no *unstable* structure which is a reduct of $(\mathbb{Z}, +, 0, |_p)$ and a proper expansion of $(\mathbb{Z}, +, 0)$. We also give a shorter proof of the result of Conant, by the same method we used for the proof of Theorem L, which we describe now. By a classical result of Shelah, the unstability of a theory can be witnessed by a formula for which one of the tuples of variables have length one. If $\mathcal{Z}$ is an unstable (hence proper) expansion of $(\mathbb{Z}, +, 0)$ and a reduct of $(\mathbb{Z}, +, 0, |_p)$, then Shelah's result implies that, at the cost of working in an elementary extension $\mathcal{Z}'$ of $\mathcal{Z}$, there is a formula in the language of $\mathcal{Z}$ which defines a new subset of the domain in $\mathcal{Z}'$, hence the problem is reduced to an analysis of unidimensional definable subsets of $\mathcal{Z}'$, allowed by quantifier elimination in $(\mathbb{Z}, +, 0, |_p)$. The rest of the proof consists in defining the relation $x \mid_p y$ by applying transformations in the language $\{+, 0\}$ to the new formula. It uses an analysis of uniformly definable subgroups of the domain in elementary extensions of $(\mathbb{Z}, +, 0, |_p)$. We prove that any such new formula can be transformed to uniformly define (and only define) a chain of balls centered in 0 (hence subgroups) of strictly increasing consecutive radiuses. When considering this formula back in $\mathbb{Z}$, it uniformly defines cofinitely many subgroups of the form $p^k\mathbb{Z}$, and only sets of this form. This yields the result, as $a \mid_p b$ if and only if for all $k \in \mathbb{N}$, $a \in p^k\mathbb{Z} \to b \in p^k\mathbb{Z}$.

Theorem L and Conant's analogous result does not hold in elementary extensions. We give counter examples in Section 9.3. However, for stronger notions of expansions and reducts, the minimality results goes through to elementary extensions, see Corollary 9.1.9 and Theorem 9.2.12.

In regard of question $(\star\star)$ above, one would formulate the following trichotomy: a dp-minimal expansion of $(\mathbb{Z}, +, 0)$ is either stable (and hence interdefinable with $(\mathbb{Z}, +, 0)$ itself), either $(\mathbb{Z}, +, 0, <)$ or it defines a valuation. This conjecture is inspired by the following result for fields due to Will Johnson [Joh15]: if K is a dp-minimal field, then it is either algebraically closed, real-closed or admits a definable henselian valuation. However, the conjecture for $(\mathbb{Z}, +, 0)$ is false. Indeed, Tran and Walsberg present in [TW17] a new family of dp-minimal expansions of

$(\mathbb{Z}, +, 0)$ obtained by adding cyclical orderings. It would be interesting to know whether other dp-minimal expansions of $(\mathbb{Z}, +, 0)$ exist, or if the expansion of $(\mathbb{Z}, +, 0)$ by a cyclical ordering satisfies the same minimality property as $(\mathbb{Z}, +, 0, <)$ and $(\mathbb{Z}, +, 0, |_p)$ do.

« Et cependant, je le répète une fois de plus et à présent je le jure : il y a quelque chose de tout glorieux et gracieux dans le vent. Ces tièdes vents alizés qui soufflent calmement dans les cieux clairs ; ces vents vigoureusement doux et qui ne dévient point malgrès les puissants tournoiements des plus mauvais courants de la mer et les plus puissants Mississipi de la terre et qui ne savent finalement plus où aller après s'être tant tordus. Par les pôles éternels ces vents alizés poussent droitement mon navire vers son but ; des vents semblables aussi droits et aussi forts poussent la quille de mon âme devant eux. . . Allons-y. »

CHAPTER 1

Preliminaries

1.1 Generalities

We assume basic knowledge in model theory, concerning formulae, types, theories, and models. Unless otherwise stated, a type means a complete type. Throughout we will denote by x, y, x^i, y^i tuples of variables, the subscript x_i, y_i will be used to denote a coordinate inside a tuple. Also, t will often denote a single variable. Capital letters A, B, C stand for sets whereas small latin letters a, b, c designate either singletons, finite or infinite tuples. For any tuple a (of elements or of variables), we denote by $|a|$ the length of a. For a set, $|C|$ is the cardinality of C. As usual in model theory, we denote by juxtaposition AB the union of the set A and the set B. We also identify juxtaposition of tuples ab as the concatenation of a and b. When dealing with independence relations or closure operators, we do not distinguish between tuples, enumerations, and sets.

Given a complete theory T in a language $\mathscr{L}$, a *monster model* $\mathbb{M}$ of T is a strongly κ-homogeneous and κ-saturated model of T, for some big enough κ. It is standard that $\mathbb{M}$ is κ^+-universal, in particular every model $\mathscr{M}$ of cardinality less than κ embeds in $\mathbb{M}$. Furthermore, we will assume that any reduct of $\mathbb{M}$ is also a monster model[1] for its theory in the reduct language. As usual in that context, a *small* model of T (or a small set) is a model of T (or a subset of some model of T) of cardinal less than κ and, by κ-universality, we consider them as elementary substructures (subsets) of $\mathbb{M}$. A small cardinal is a cardinal $\lambda < \kappa$. We will sometimes forget about the "small" adjective and even about the cardinal κ, as it will always be implied that, in every single proof, every set we consider has cardinality smaller than κ.

Given a theory T we use the notations tp^T, $\equiv^T$, acl_T and dcl_T to precise that we work in the language of T, and when the language is clear, we just use tp, $\equiv$, acl, dcl. By strong κ-homogeneity if $a \equiv_C^T b$ then there is an automorphism σ of $\mathbb{M}$ over C (i.e. fixing C pointwise) such that $\sigma(a) = b$ (i.e. $\sigma(a_i) = b_i$ for $0 \leq i < |a|$, $|a|$ may be infinite). Such an automorphism is also called a C-automorphism. For two sets A, A', we denote by $A \equiv_C A'$ if for all enumeration $(a_\alpha)_{\alpha<|A|}$ of A there exists an automorphism σ of the monster over C, and an enumeration $(a'_\alpha)_{\alpha<|A'|}$ of A' such that $\sigma(a_\alpha) = a'_\alpha$ for all $\alpha < |A|$ (in particular $|A| = |A'|$); equivalently, there is a C-automorphism of the monster such that $\sigma(A) = A'$ setwise. The restriction of σ to the set A is called a T-*elementary bijection* (or T-elementary isomorphism) between A and A'. This must not be confused with the notion of elementary equivalent models over C: $\mathscr{M} \equiv_C \mathscr{N}$

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[1] By [Hod08, Chapter 10], choose $\mathbb{M}$ to be κ-*big*, then it strongly κ-homogeneous and κ-saturated, and any reduct is also κ-big. Note that in general, a reduct of a strongly κ-homogeneous structure need not be strongly κ-homogeneous, see [Hod08, 10.1, Exercice 11].

if $C \subseteq \mathscr{M} \cap \mathscr{N}$ and for all $\mathscr{L}$-sentences θ with parameters in C, then $\mathscr{M} \models \theta$ if and only if $\mathscr{N} \models \theta$. If $\mathscr{M} \equiv_C \mathscr{N}$, in general there is no C-automorphism of the monster sending $\mathscr{M}$ on $\mathscr{N}$.

A theory T is *model-complete* if for all models $\mathscr{M}$ and $\mathscr{N}$ of T, if $\mathscr{M}$ is a substructure of $\mathscr{N}$ then $\mathscr{M}$ is an elementary substructure of $\mathscr{N}$. A model-complete theory need not be complete, see for instance ACF below. A *model-companion* of an $\mathscr{L}$-theory T is an $\mathscr{L}$-theory T^* such that:

- every model of T has an extension which is a model of T^*;
- every model of T^* has an extension which is a model of T;
- T is model-complete.

The model-companion of a theory need not exists, but if it does, it is unique (see for instance [Mar02, Exercise 3.4.13]).

An $\mathscr{L}$-structure $\mathscr{M}$ is *existentially closed* in some extension $\mathscr{N}$ if every existential formula with parameters in $\mathscr{M}$ that holds in $\mathscr{N}$ holds also in $\mathscr{M}$. An *existentially closed model of a theory* T is a model of T that is existentially closed in every extension which is a model of T. A theory is *inductive* if the union of any chain of models is still a model. Equivalently, it is $\forall\exists$-axiomatisable. Assume that T is inductive, then if the model-companion T^* exists, T^* is an axiomatization of the class of existentially closed models of T (see [Mar02, Exercise 3.4.13]). We say that a theory T has the *amalgamation property* if whenever $\mathscr{M}_0$, $\mathscr{M}_1$ and $\mathscr{M}_2$ are models of T such that there exists embedding $f_1 : \mathscr{M}_0 \to \mathscr{M}_1$ and $f_2 : \mathscr{M}_0 \to \mathscr{M}_2$ then there exists a model $\mathscr{N}$ of T and embeddings $g_1 : \mathscr{M}_1 \to \mathscr{N}$ and $g_2 : \mathscr{M}_2 \to \mathscr{N}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathscr{M}_0 \arrow[r, "f_1"] \arrow[d, "f_2"] & \mathscr{M}_1 \arrow[d, "g_1"] \\
\mathscr{M}_2 \arrow[r, "g_2"] & \mathscr{N}
\end{tikzcd}$$

If it exists, the model-companion of a theory which has the amalgamation property is called the *model-completion*.

Let $\mathbb{M}$ be a monster model of T and A, B small subsets of $\mathbb{M}$. Let p be a (complete) type over B. We say that p is *finitely satisfiable in* A if for all formula $\phi(x, b)$ in p, there exists a tuple a from A with $|a| = |x|$ and $\mathbb{M} \models \phi(a, b)$. We say that p is A*-invariant* if for all A-automorphism, $\sigma p = p$. If p is finitely satisfiable in A, then it is A-invariant (see [Sim15, Example 2.17]). A *global type* is a type over $\mathbb{M}$, a *global extension* q of p is a global type such that $q \restriction B = p$, where $q \restriction B$ is the type consisting of formulae in q with parameters in B. If a type p over B is finitely satisfiable in A then it has a global extension which is finitely satisfiable in A (see [Sim15, Example 2.17]). As every type over a model $\mathscr{M}$ is clearly finitely satisfiable in $\mathscr{M}$, it follows that every type over a model $\mathscr{M}$ has a global extension which is finitely satisfiable in $\mathscr{M}$, hence also $\mathscr{M}$-invariant. If p is a type over B and q is an extension of p which is finitely satisfiable in B, then q is called a *coheir* of p.

Let λ be a small cardinal and C a small set. A sequence $(b_i)_{i<\lambda}$ is C*-indiscernible* if for all $n < \omega$ and $\alpha_1 < \cdots < \alpha_n < \lambda$ we have $b_{\alpha_1}, \cdots, b_{\alpha_n} \equiv_C b_1, \cdots, b_n$. For all $\alpha < \lambda$, the sequence $(b_i)_{\alpha \leq i < \kappa}$ is $Cb_{<\alpha}$-indiscernible.

Given an $\mathscr{L}$-theory T, we recall the construction of T^{eq}, see for example [TZ12, Chapter 8]. To each model $\mathscr{M}$ of T is associated a structure $\mathscr{M}^{eq}$ consisting of the following: one sort (the *home* sort) for the structure $\mathscr{M}$ and for each definable equivalence class $E(x, y)$ in $\mathscr{M}$ without parameters (we will also say 0*-definable*) an associated *imaginary* sort S_E in the language of $\mathscr{M}^{eq}$, and a projection $\pi_E : \mathscr{M}^{|x|} \to S_E$ such that $\pi_E(x) = \pi_E(y)$ if and only if $E(x, y)$. Let T^{eq} be the theory of $\mathscr{M}^{eq}$. Basic facts about T^{eq} are first that it doesn't depend on the model $\mathscr{M}$ of T. Every automorphism of $\mathscr{M}$ extends to an automorphism of $\mathscr{M}^{eq}$ and every automorphism of $\mathscr{M}^{eq}$ restricted to $\mathscr{M}$ is an automorphism of $\mathscr{M}$. It follows that elements in the home sort $\mathscr{M}$ have the same type (over $\emptyset$) in $\mathscr{M}$ if and only if they have the same type in $\mathscr{M}^{eq}$. Also T^{eq} *eliminates imaginaries*, that is, for a monster model $\mathscr{N}$ of T^{eq} and formula $\phi(x,b)$ with parameters in $\mathscr{N}$, there exists a finite tuple d that is fixed (pointwise) by the same automorphisms which fix $\phi(\mathscr{N},b)$ setwise. Such a tuple is called a *canonical parameter* for $\phi(x,b)$. A theory T has elimination of imaginaries if and only if in T^{eq} every element of the home sort is interdefinable with an element of an imaginary sort. A theory T has *weak elimination of imaginaries* if in T^{eq}, for every element e in an imaginary sort, there is a tuple d from the home sort such that e is definable over d and d is algebraic over e. We denote by acl^{eq}_T and dcl^{eq}_T the algebraic closure and definable closure in the sens of the theory T^{eq}.

1.2 Independence relations

Let $\mathbb{M}$ be a monster model for a theory T.

Definition 1.2.1. For any a, b (finite or infinite) tuples from $\mathbb{M}$ and C small set in $\mathbb{M}$ we recall the following various ternary relations (see for instance [CK17]).

(1) *Algebraic independence.* $a \mathop{\smile\hskip -0.9em ^| \ }^{a}_{C} b$ if and only if $\mathrm{acl}(Ca) \cap \mathrm{acl}(Cb) = \mathrm{acl}(C)$

(2) *Imaginary algebraic independence.* $a \mathop{\smile\hskip -0.9em ^| \ }^{aeq}_{C} b$ if and only if $\mathrm{acl}^{eq}(Ca) \cap \mathrm{acl}^{eq}(Cb) = \mathrm{acl}^{eq}(C)$. This relation is defined over subsets of $\mathbb{M}^{eq}$. We write $\mathop{\smile\hskip -0.9em ^| \ }^{aeq} \restriction \mathbb{M}$ to specify the restriction to elements of the sort $\mathbb{M}$.

(3) *Kim-dividing independence.* $a \mathop{\smile\hskip -0.9em ^| \ }^{Kd}_{C} b$ if and only if $tp(a/Cb)$ does not Kim-divides over C if and only if for all global C-invariant extension p of $tp(b/C)$ and sequences $(b_i)_{i<\omega}$ such that $b_i \models p \restriction Cb_{<i}$ for all $i < \omega$, there exists a' such that $a'b_i \equiv_C ab$ for all $i < \omega$;

(4) *Kim-(forking)independence.* $a \mathop{\smile\hskip -0.9em ^| \ }^{K}_{C} b$ if and only if $tp(a/Cb)$ does not Kim-fork over C if and only if for any $b' \supseteq b$ there exists $a' \equiv_{Cb} a$ such that $a' \mathop{\smile\hskip -0.9em ^| \ }^{Kd}_{C} b'$;

(5) *Dividing independence.* $a \mathop{\smile\hskip -0.9em ^| \ }^{d}_{C} b$ if and only if $tp(a/Cb)$ does not divide over C if and only if for any C-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 = b$ there exists a' such that $a'b_i \equiv_C ab$ for all $i < \omega$;

(6) *Forking independence.* $a \mathop{\smile\hskip -0.9em ^| \ }^{f}_{C} b$ if and only if $tp(a/Cb)$ does not fork over C if and only if for any $b' \supseteq b$ there exists $a' \equiv_{Cb} a$ such that $a' \mathop{\smile\hskip -0.9em ^| \ }^{d}_{C} b'$;

(7) *Coheir independence.* $a \mathop{\smile\hskip -0.9em ^| \ }^{u}_{C} b$ if and only if $tp(a/Cb)$ is finitely satisfiable in C.

As usual, we extend these notions to sets by the following: $A \mathop{\smile\hskip -0.9em ^| \ }_{C} B$ if and only if for all enumeration a of A and b of B, then $a \mathop{\smile\hskip -0.9em ^| \ }_{C} b$. The following is a list of properties for a ternary relation $\mathop{\smile\hskip -0.9em ^| \ }$ defined over small subsets of $\mathbb{M}$, sometimes relatively to another ternary relation $\mathop{\smile\hskip -0.9em ^| \ }'$, also defined over small subsets of $\mathbb{M}$.

- INVARIANCE. If $ABC \equiv A'B'C'$ then $A \mathop{\smile\hskip -0.9em ^| \ }_{C} B$ if and only if $A' \mathop{\smile\hskip -0.9em ^| \ }_{C'} B'$.
- FINITE CHARACTER. If $a \mathop{\smile\hskip -0.9em ^| \ }_{C} B$ for all finite $a \subseteq A$, then $A \mathop{\smile\hskip -0.9em ^| \ }_{C} B$.
- SYMMETRY. If $A \mathop{\smile\hskip -0.9em ^| \ }_{C} B$ then $B \mathop{\smile\hskip -0.9em ^| \ }_{C} A$.
- CLOSURE $A \mathop{\smile\hskip -0.9em ^| \ }_{C} B$ if and only if $A \mathop{\smile\hskip -0.9em ^| \ }_{\mathrm{acl}(C)} \mathrm{acl}(BC)$.
- MONOTONICITY. If $A \mathop{\smile\hskip -0.9em ^| \ }_{C} BD$ then $A \mathop{\smile\hskip -0.9em ^| \ }_{C} B$.
- BASE MONOTONICITY. If $A \mathop{\smile\hskip -0.9em ^| \ }_{C} BD$ then $A \mathop{\smile\hskip -0.9em ^| \ }_{CD} B$.

- Transitivity. If $A \mathop{\smile\hspace{-0.9em}\mid}_{CB} D$ and $B \mathop{\smile\hspace{-0.9em}\mid}_{C} D$ then $AB \mathop{\smile\hspace{-0.9em}\mid}_{C} D$.
- Existence. For any C and A we have $A \mathop{\smile\hspace{-0.9em}\mid}_{C} C$.
- Full Existence. For all A, B and C there exists $A' \equiv_C A$ such that $A' \mathop{\smile\hspace{-0.9em}\mid}_{C} B$.
- Extension. If $A \mathop{\smile\hspace{-0.9em}\mid}_{C} B$, then for all D there exists $A' \equiv_{CB} A$ and $A' \mathop{\smile\hspace{-0.9em}\mid}_{C} BD$.
- Local Character. For all finite tuple a and infinite B there exists $B_0 \subset B$ with $|B_0| \leq \aleph_0$ and $a \mathop{\smile\hspace{-0.9em}\mid}_{B_0} B$.
- Strong Finite Character over E. If $a \not\mathop{\smile\hspace{-0.9em}\mid}_{E} b$, then there is a formula $\Lambda(x, b, e) \in tp(a/Eb)$ such that for all a', if $a' \models \Lambda(x, b, e)$ then $a' \not\mathop{\smile\hspace{-0.9em}\mid}_{E} b$.
- $\mathop{\smile\hspace{-0.9em}\mid}'$-amalgamation over E. If there exists tuples c_1, c_2 and sets A, B such that
    - $c_1 \equiv_E c_2$
    - $A \mathop{\smile\hspace{-0.9em}\mid}'_{E} B$
    - $c_1 \mathop{\smile\hspace{-0.9em}\mid}_{E} A$ and $c_2 \mathop{\smile\hspace{-0.9em}\mid}_{C} B$

  then there exists $c \mathop{\smile\hspace{-0.9em}\mid}_{E} A, B$ such that $c \equiv_A c_1$, $c \equiv_B c_2$, $A \mathop{\smile\hspace{-0.9em}\mid}^{a}_{Ec} B$, $c \mathop{\smile\hspace{-0.9em}\mid}^{a}_{EA} B$ and $c \mathop{\smile\hspace{-0.9em}\mid}^{a}_{EB} A$.
- Stationnarity over E. If $c_1 \equiv_E c_2$ and $c_1 \mathop{\smile\hspace{-0.9em}\mid}_{E} A$, $c_2 \mathop{\smile\hspace{-0.9em}\mid}_{E} A$ then $c_1 \equiv_{EA} c_2$.
- Witnessing. Let a, b be tuples, $\mathscr{M}$ a model and assume that $a \not\mathop{\smile\hspace{-0.9em}\mid}_{\mathscr{M}} b$. Then there exists a formula $\Lambda(x, b) \in tp(a/\mathscr{M}b)$ such that for any global extension $q(x)$ of $tp(b/\mathscr{M})$ finitely satisfiable in $\mathscr{M}$ and for any $(b_i)_{i<\omega}$ such that for all $i < \omega$ we have $b_i \models q \restriction \mathscr{M}b_{<i}$, the set $\{\Lambda(x, b_i) \mid i < \omega\}$ is inconsistent.

If $A \mathop{\smile\hspace{-0.9em}\mid}_{C} B$, the set C is called the *base set.*

Definition 1.2.2. Let $\mathop{\smile\hspace{-0.9em}\mid}$, $\mathop{\smile\hspace{-0.9em}\mid}^{0}$ be two ternary relations. We say that $\mathop{\smile\hspace{-0.9em}\mid}$ *is stronger than* $\mathop{\smile\hspace{-0.9em}\mid}^{0}$ (or $\mathop{\smile\hspace{-0.9em}\mid}^{0}$ *is weaker than* $\mathop{\smile\hspace{-0.9em}\mid}$) if for all a, b, C we have $a \mathop{\smile\hspace{-0.9em}\mid}_{C} b \implies a \mathop{\smile\hspace{-0.9em}\mid}^{0}_{C} b$. We denote it by $\mathop{\smile\hspace{-0.9em}\mid} \to \mathop{\smile\hspace{-0.9em}\mid}^{0}$.

Assume that $\mathop{\smile\hspace{-0.9em}\mid} \to \mathop{\smile\hspace{-0.9em}\mid}^{0}$, then if $\mathop{\smile\hspace{-0.9em}\mid}$ satisfies Full Existence or Local Character, so does $\mathop{\smile\hspace{-0.9em}\mid}^{0}$. Similarly, if a relation satisfies $\mathop{\smile\hspace{-0.9em}\mid}^{0}$-amalgamation then it also satisfies $\mathop{\smile\hspace{-0.9em}\mid}$-amalgamation.

Fact 1.2.3. *The following are standard facts more or less obvious from the definition.*

(1) $\mathop{\smile\hspace{-0.9em}\mid}^{a}$ *satisfies* Invariance, Monotonicity, Transitivity, Existence, Extension *and* Full Existence*;*

(2) $\mathop{\smile\hspace{-0.9em}\mid}^{d}$ *satisfies* Invariance, Monotonicity, Base Monotonicity, Transitivity*;*

(3) $\mathop{\smile\hspace{-0.9em}\mid}^{f}$ *satisfies* Invariance, Monotonicity, Base Monotonicity, Transitivity *and* Extension*;*

(4) $\mathop{\smile\hspace{-0.9em}\mid}^{u}$ *satisfies* Invariance, Monotonicity, Base Monotonicity, Transitivity, Extension, Existence *over models,* Full Existence *over models;*

(5) $\mathop{\smile\hspace{-0.9em}\mid}^{d} \to \mathop{\smile\hspace{-0.9em}\mid}^{aeq} \restriction \mathbb{M}$*;*

(6) $\mathop{\smile\hspace{-0.9em}\mid}^{u} \to \mathop{\smile\hspace{-0.9em}\mid}^{f} \to \mathop{\smile\hspace{-0.9em}\mid}^{d} \to \mathop{\smile\hspace{-0.9em}\mid}^{aeq} \restriction \mathbb{M} \to \mathop{\smile\hspace{-0.9em}\mid}^{a}$*;*

(7) $\mathop{\smile\hspace{-0.9em}\mid}^{f} \to \mathop{\smile\hspace{-0.9em}\mid}^{K}$ *and* $\mathop{\smile\hspace{-0.9em}\mid}^{d} \to \mathop{\smile\hspace{-0.9em}\mid}^{Kd}$*.*

Proof. *(1)* is [Adl09a, Proposition 1.5]. *(2)* and *(3)* are [Adl09b, Proposition 1.3]. *(4)* is [CK12, Remark 2.16], Base Monotonicity is trivial. For *(5)*, it is clear that if $a \mathop{\smile\kern-0.75em\mid}^d_C b$ in $\mathbb{M}$, then $a \mathop{\smile\kern-0.75em\mid}^d_C b$ in $\mathbb{M}^{eq}$, and by [Adl09a, Remark 5.4] it follows that $\mathrm{acl}^{eq}(Ca) \cap \mathrm{acl}^{eq}(Cb) = \mathrm{acl}^{eq}(C)$ hence $a \mathop{\smile\kern-0.75em\mid}^{aeq}_C b$. *(6)* follows from [CK12, Example 2.22], and the previous results. *(7)* is by definition. ☐

Lemma 1.2.4. *Let $\mathop{\smile\kern-0.75em\mid}$ be a relation satisfying* Symmetry, Monotonicity, Existence *and* Strong Finite Character *over C.*

$$\text{If } a \mathop{\smile\kern-0.75em\mid}^u_C b \text{ then } a \mathop{\smile\kern-0.75em\mid}_C b.$$

In particular, as $\mathop{\smile\kern-0.75em\mid}^u$ satisfies Full Existence *over models, so does $\mathop{\smile\kern-0.75em\mid}$.*

Proof. Indeed, assume $a \not\mathop{\smile\kern-0.75em\mid}_C b$ then by Strong Finite Character there is a formula $\phi(x,b) \in tp(a/Cb)$ such that if $a' \models \phi(x,b)$ then $a' \not\mathop{\smile\kern-0.75em\mid}_C b$. As $tp(a/Cb)$ is finitely satisfiable in C there is $c \in C$ such that $c \models \phi(x,b)$, so $c \not\mathop{\smile\kern-0.75em\mid}_C b$, so by Symmetry and Monotonicity $b \not\mathop{\smile\kern-0.75em\mid}_C C$ which contradicts Existence. ☐

Lemma 1.2.5. *If $\mathop{\smile\kern-0.75em\mid}$ satisfies* Invariance *and* Extension*, then $A \mathop{\smile\kern-0.75em\mid}_C B$ implies $A \mathop{\smile\kern-0.75em\mid}_C \mathrm{acl}(CB)$. If $\mathop{\smile\kern-0.75em\mid}$ satisfies* Invariance, Extension *and* Base Monotonicity*, then $\mathop{\smile\kern-0.75em\mid}$ satisfies* Closure.

Proof. Assume that $A \mathop{\smile\kern-0.75em\mid}_C B$. By Extension, let A' be such that $A' \equiv_{BC} A$ and $A' \mathop{\smile\kern-0.75em\mid}_C \mathrm{acl}(BC)$. There is an automorphism σ over BC sending A' to A hence by Invariance, $A \mathop{\smile\kern-0.75em\mid}_C \sigma(\mathrm{acl}(BC))$. Now, as sets, $\sigma(\mathrm{acl}(BC)) = \mathrm{acl}(BC)$ so $A \mathop{\smile\kern-0.75em\mid}_C \mathrm{acl}(BC)$. The last assertion is trivial, as $\mathrm{acl}(C) \subseteq \mathrm{acl}(BC)$. ☐

Remark 1.2.6. If $\mathop{\smile\kern-0.75em\mid}$ satisfies Invariance, Symmetry, Transitivity and Full Existence, then $\mathop{\smile\kern-0.75em\mid}$ satisfies Extension. Also if $\mathop{\smile\kern-0.75em\mid}$ satisfies Existence, Monotonicity and Extension then it satisfies Full Existence. Hence for relations $\mathop{\smile\kern-0.75em\mid}$ satisfying Invariance, Monotonicity, Existence, Transitivity, Symmetry, the properties Full Existence and Extension are equivalent. Unfortunately, when dealing with non-symmetrical independence relations, we need to differentiate both properties. In Chapter 7, we see an example of a relation which is not symmetric but satisfies Invariance, Monotonicity, Existence, Transitivity, Full Existence: forking independence in ACFG. We show that it also satisfies Extension by non-trivial arguments.

Remark 1.2.7. Most of the properties above are familiar to anyone who knows forking in stable or simple theories.

The property Strong Finite Character is always satisfied by forking independence relation: take the formula ϕ to be a forking formula. This property is needed to use [CR16, Proposition 5.3] and prove that under the right assumptions on T, any completion of TS is NSOP$_1$.

If $\mathscr{M}$ is a model of the ambient theory, our formulation of $\mathop{\smile\kern-0.75em\mid}$-amalgamation over $\mathscr{M}$ is what is called *The algebraically reasonable independence theorem* in [KR18], which holds for Kim-forking in any NSOP$_1$ theory (see [KR18, Theorem 2.21]). In simple theories, the forking independence relation also satisfies this property. The conclusion $A \mathop{\smile\kern-0.75em\mid}^a_{Ec} B$, $c \mathop{\smile\kern-0.75em\mid}^a_{EA} B$ and $c \mathop{\smile\kern-0.75em\mid}^a_{EB} A$ is always true in the simple case by Symmetry, Base Monotonicity and Transitivity of the forking independence relation (and Fact 1.2.3 *(5)*). In many examples, one can prove the independence theorem under weaker assumptions, for instance assuming $\mathop{\smile\kern-0.75em\mid}'$ to be $\mathop{\smile\kern-0.75em\mid}^a$, or the base set to be acl-closed. Actually, there is no known example of an NSOP$_1$ theory in which $\mathop{\smile\kern-0.75em\mid}^a$-amalgamation is *not* satisfied.

1.3 Pregeometry

This section introduces basic notions about pregeometries, as can be found in e.g. [TZ12, Appendix C]. We denote by $\mathscr{P}(S)$ the powerset of a set S.

Definition 1.3.1. A *pregeometry* (S, cl) is a set S and a *closure operator* $\mathrm{cl} : \mathscr{P}(S) \to \mathscr{P}(S)$ satisfying the following conditions, for all $A \subseteq S$ and a, b elements of S:

- (*Reflexivity*) $A \subseteq \mathrm{cl}(A)$;
- (*Finite Character*) $\mathrm{cl}(A) = \bigcup_{A_0 \subseteq B,\ A_0 \text{ finite}} \mathrm{cl}(A_0)$;
- (*Transitivity*) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;
- (*Exchange*) If $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$ then $b \in \mathrm{cl}(Aa)$.

A tuple $(b_i)_{i<\kappa}$ is *independent over* A if $b_i \notin \mathrm{cl}(A(b_j)_{j \neq i})$ for all $i < \kappa$. Similarly a set B is independent over A if for all enumeration b of B, b is independent over A. If $A \subseteq B$, and $B = \mathrm{cl}(B)$, a *basis* of B over A is a subset B' of B which is independent over A and such that $\mathrm{cl}(AB') = B$.

Fact 1.3.2. *Let (S, cl) be a pregeometry, $A \subset B \subset S$, and $B = \mathrm{cl}(B)$. Then every independent tuple in B over A can be completed into a basis of B over A, in particular B admits a basis over A. Every basis of B over A have the same cardinality, we call it the* dimension of B over A, *denoted by $\dim_{\mathrm{cl}}(B/A)$ (or $\dim_{cl}(B)$ if $A = \emptyset$).*

In any pregeometry, there is a notion of independence.

$$A \mathop{\smile\hspace{-0.9em}\mid}^{cl}_{C} B \iff \text{for all basis } A_0 \text{ of } \mathrm{cl}(A) \text{ over } C \text{ and } B_0 \text{ of } \mathrm{cl}(B) \text{ over } C,$$

$$A_0 B_0 \text{ is a basis of } \mathrm{cl}(AB) \text{ over } C$$

When there is a pregeometry in a wider context, we will say that a tuple a is $\mathop{\smile\hspace{-0.9em}\mid}^{cl}$-independent over B to precise that this is relatively to the pregeometry (S, cl).

Fact 1.3.3. *The relation $\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}$ satisfies the following properties.*

- Finite Character. *If for all finite tuple a from A we have $a \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{C} B$ then $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{C} B$.*
- Symmetry. *If $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{C} B$ then $B \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{C} A$.*
- Closure *$A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{C} B$ if and only if $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{\mathrm{cl}(C)} \mathrm{cl}(BC)$.*
- Monotonicity. *If $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{C} BD$ then $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{C} B$.*
- Base Monotonicity. *If $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{C} BD$ then $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{CD} B$.*
- Transitivity. *If $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{CB} D$ and $B \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{C} D$ then $AB \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{C} D$.*
- Existence. *For all A, C, $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{cl}}_{C} C$.*

As there are no theory lying around (yet), properties like Invariance and Full Existence doesn't make sense here.

Definition 1.3.4. A pregeometry (S, cl) is *modular* if for all $A = \mathrm{cl}(A), B = \mathrm{cl}(B)$, $\dim(AB) + \dim(A \cap B) = \dim(A) + \dim(B)$.

Fact 1.3.5. *Let (S, cl) be a pregeometry. The following are equivalent.*

(1) (S, cl) *is modular.*

(2) for all $A, B \subseteq S$ *if* $c \in \mathrm{cl}(AB)$ *then there exists* $a \in \mathrm{cl}(A)$ *and* $b \in \mathrm{cl}(B)$ *such that* $c \in \mathrm{cl}(a, b)$*.*

(3) for all A, B, C*:* $A \mathop{\smile\!\!\!\!\!\!|}^{cl}_{C} B$ *if and only if* $\mathrm{cl}(AC) \cap \mathrm{cl}(BC) = \mathrm{cl}(C)$*.*

(4) for all A, B, C *such that* $A = \mathrm{cl}(A)$*,* $B = \mathrm{cl}(B)$ *and* $C = \mathrm{cl}(C)$*, if* $C \subseteq B$ *then* $\mathrm{cl}(AB) \cap C = \mathrm{cl}(\mathrm{cl}(A \cap C), C)$*.*

Throughout, we will refer to any of these properties using “by modularity”.

Example 1.3.6 (Algebraically closed fields)**.** Let K be an algebraically closed field and cl the closure operator defined for $A \subseteq K$ by $\mathrm{cl}(A) = \overline{\mathbb{F}(A)}$ where $\overline{\mathbb{F}(A)}$ is the algebraic closure (in K) of the subfield of K generated by A and the prime field $\mathbb{F}$. Then (K, cl) defines a pregeometry. The dimension is the transcendence degree, it is not a modular pregeometry (see [Bou06b, A.V.110, §3]).

Example 1.3.7 (Vector spaces)**.** Let V be a vector space over some field k and defined the closure operator $\langle A \rangle$ to be the span of $A \subseteq V$. Then (V, cl) defines a modular pregeometry. The dimension is the dimension as a k-vector space.

In a model theoretical context, a closure operator likely to define a pregeometry in a model is the model-theoretic algebraic closure, as it always satisfies *Reflexivity*, *Finite Character* and *Transitivity*.

Definition 1.3.8. A theory T is *pregeometric* if $(\mathscr{M}, \mathrm{acl}^{\mathscr{M}}(\cdot))$ defines a pregeometry, for all model $\mathscr{M}$ of T. We denote by $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ the associated independence relation. We say that T *eliminates* $\exists^\infty$ if for all formula $\phi(x, y)$ there is an integer $n \in \mathbb{N}$ such that for all $|y|$-tuple b in any model $\mathscr{M}$ of T, $\phi(\mathscr{M}, b)$ is either infinite or of cardinality less than n. A pregeometric theory that eliminates $\exists^\infty$ is called *geometric*.

Note that if a theory is geometric, it does not mean that the algebraic closure defines a geometry, see [TZ12, Appendix C] for a definition of a geometry.

Fact 1.3.9 ([Gag05])**.** *Let* T *be a pregeometric theory and* $\mathbb{M}$ *a monster model for* T*. For all* B *small subset of* $\mathbb{M}$ *and finite tuple* x *there exists a partial type* $p_B(x)$ *such that* a *realizes* p_B *if and only if* a *is* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$*-independent over* B*. Furthermore for any type* q *in and expansion of* $\mathscr{M}$*, and* $B \subseteq D$*, if* $q \cup p_B$ *is consistent, then so is* $q \cup p_D$*. The relation* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ *satisfies* Invariance*,* Finite Character*,* Symmetry*,* Closure*,* Monotonicity*,* Base Monotonicity*,* Transitivity*,* Existence*,* Full Existence *and* Extension*.*

Proof. The first two assertions are in [Gag05], the fact that q can be choosen in an expansion of $\mathscr{M}$ follows easily by inspection of the proof. A consequence of the first part is that $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}_{C} b$ is type-definable for every basis a of $\mathrm{acl}(Ca)$ over C and b basis of $\mathrm{acl}(Cb)$ over C. As any automorphism fixes (setwise) the algebraic closure, it follows that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ satisfies Invariance. We prove that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ satisfies Extension, the rest follows from Remark 1.2.6, and Fact 1.3.3. Assume that for some finite a, $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}_{C} B$ and D is arbitrary. Let a' be a basis of $\mathrm{acl}(CBa)$ over CB. As a' realizes $tp(a'/CB) \cup p_{CB}(x)$, the type $tp(a'/CB) \cup p_{CBD}$ is consistent, let a'' be a realisation. A CB-automorphism sending a' to a'' sends a to some $\tilde{a}$ such that $\tilde{a} \subseteq \mathrm{acl}(a''CB)$. As $a'' \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}_{C} BD$, by Closure, Symmetry and Monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$, we have $\tilde{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}_{C} BD$. $\square$

Let T be a pregeometric theory with monster $\mathbb{M}$, b a tuple from $\mathbb{M}$ and $\phi(x, b)$ a formula. By $\dim(\phi(x, b))$ we mean the maximum dimension of $\mathrm{acl}(cb)$ over b, for realisations c of $\phi(x, b)$.

Fact 1.3.10. *Let T be a geometric theory and $\mathbb{M}$ a monster model for T. Then for all formula $\phi(x, y)$ there exists a formula $\theta_\phi(y)$ such that $\theta_\phi(b)$ holds if and only if there exists a realisation a of $\phi(x, b)$ which is an $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{acl}}$-independent tuple over $\mathrm{acl}_T(b)$.*

Proof. From [Gag05, Fact 2.4], for each $k \leq |x|$ there exists a formula $\theta_k(y)$ such that $\theta_k(b)$ if and only if $\dim(\phi(\mathbb{M}, b)) = k$. The formula $\theta_{|x|}(y)$ holds if and only if there is a realisation a of $\phi(x, b)$ such that $\dim(\mathrm{acl}(ab)/b) = |x|$, hence it is $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{acl}}$-independent over $\mathrm{acl}_T(b)$. ☐

Note that a reduct of a pregeometric theory is pregeometric, and the reduct of a geometric theory is also a geometric theory (see [Hil08, Fait 2.15]).

1.4 Classification Theory

Let $\mathcal{T}$ be a tree (such as $2^{<\omega}, \omega^{<\omega}, \kappa^{<\lambda}$). We denote by $\triangleleft$ the natural partial order on $\mathcal{T}$. For $\nu, \eta \in \kappa^{<\lambda}$ we denote by $\nu^\frown\eta$ the concatenation of the two.

1.4.1 Stable and simple theories

Definition 1.4.1. Let T be a complete theory, $\mathbb{M}$ a monster model of T. Let $\phi(x, y)$ be a formula.

- We say that $\phi(x, y)$ has the *order property* (or is *unstable*) if there are two indiscernible sequences $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ in $\mathbb{M}$ such that $\mathbb{M} \models \phi(a_i, b_j)$ if and only if $i < j$. A theory T is *stable* if no formula in any monster model $\mathbb{M}$ is unstable.
- We say that $\phi(x, y)$ has the *tree property* (TP) if there is a sequences $(a_\mu)_{\mu \in \omega^{<\omega}}$ in $\mathbb{M}$ and $k \in \mathbb{N}$ such that
  (1) $\{\phi(x, a_{\mu^\frown i}) \mid i < \omega\}$ is k-inconsistent, for all $\mu \in \omega^{<\omega}$;
  (2) $\{\phi(x, a_{s\restriction n}) \mid n \in \mathbb{N}\}$ is consistent, for any $s \in \omega^\omega$.

  A theory T is *simple* if no formula in any monster model $\mathbb{M}$ has the tree property.

Fact 1.4.2 ([She90, Theorem 2.13])**.** *A theory T is stable if and only if all formulas $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ are stable.*

Lemma 1.4.3. *Let $\mathscr{L}$ be any language and let T be an unstable $\mathscr{L}$-theory with monster model $\mathbb{M}$. Let $\mathscr{L}^- \subseteq \mathscr{L}$ be such that $T\restriction_{\mathscr{L}^-}$ is stable. Then there exists an $\mathscr{L}$-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ and a tuple b from $\mathbb{M}$ such that $\phi(x, b)$ is not $\mathscr{L}^-$-definable with parameters in $\mathbb{M}$.*

Proof. By Fact 1.4.2 there is an unstable $\mathscr{L}$-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$. By Ramsey and compactness (see e.g. [TZ12, Lemma 7.1.1]) we may assume that $(a_i)_{i\in\mathbb{Z}}$, $(b_i)_{i\in\mathbb{Z}}$ are two indiscernible sequences in $\mathbb{M}$ that witness the unstability of $\phi(x, y)$, i.e., $\phi(a_i, b_j)$ if and only if $i < j$. Assume towards a contradiction that $\phi(x, b_0)$ is definable by an $\mathscr{L}^-$-formula $\psi(x, c_0)$ with parameters c_0 in $\mathbb{M}$. For each $k \in \mathbb{Z}\setminus\{0\}$, as $tp(b_k/\emptyset) = tp(b_0/\emptyset)$ there is an automorphism σ_k such that $\sigma_k(b_0) = b_k$. Let $c_k = \sigma_k(c_0)$. Then $\phi(x, b_k)$ is equivalent to $\psi(x, c_k)$, and hence $\psi(a_i, c_j)$ if and only if $i < j$, a contradiction to the stability of $T\restriction_{\mathscr{L}^-}$. ☐

There is a "geometric" characterization of stable theories, which appears first in [HH84]. We give a modern presentation, see [Cas11, Theorem 12.22].

Fact 1.4.4 (Harnik-Harrington, characterisation of forking and stable theories)**.** *Let T be a complete theory, and $\mathbb{M}$ a monster model. The theory T is stable if and only if there is a ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ defined over small subsets which satisfies:* Invariance*,* Finite Character*,* Symmetry*,* Closure*,* Monotonicity*,* Base Monotonicity*,* Transitivity*,* Extension*,* Local Character *and* Stationnarity *over models. If such a relation $\mathop{\smile\!\!\!\!\!\mid}$ exists, $\mathop{\smile\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\mid}^f = \mathop{\smile\!\!\!\!\!\mid}^d$.*

Proof. One only needs to check that our set of axioms is equivalent to the set of axioms in [Cas11, Theorem 12.22]. An *independence relation* in the sense of [Cas11] satisfies SYMMETRY ([Cas11, Corollary 12.6]). We need to check first that our set of axioms implies the property *Normality*: $A \mathop{\smile\hskip-0.9em ^| \ }_C B$ implies $AC \mathop{\smile\hskip-0.9em ^| \ }_C B$, which follows from INVARIANCE, EXTENSION, MONOTONICITY and SYMMETRY (using Lemma 1.2.5). We also need to check that the set of axioms in [Cas11] implies our, the only property one needs to check is that CLOSURE follows from the set of axioms in [Cas11], which is Lemma 1.2.5. ☐

In a stable theory, $\mathop{\smile\hskip-0.9em ^| \ }^f$ coincides with the coheir independence $\mathop{\smile\hskip-0.9em ^| \ }^u$ over models (see e.g. [TZ12]).

Fact 1.4.5. *Let T be a stable theory, $\mathbb{M}$ a monster model for T, $\mathscr{M} \prec \mathbb{M}$ and a, b tuples from $\mathbb{M}$. Then $a \mathop{\smile\hskip-0.9em ^| \ }^f_{\mathscr{M}} b$ if and only if $a \mathop{\smile\hskip-0.9em ^| \ }^u_{\mathscr{M}} b$.*

There is also a classical "geometric" characterization of simple theories ([KP97]).

Fact 1.4.6 (Kim-Pillay, characterization of forking and simple theories)**.** *Let T be a complete theory, and $\mathbb{M}$ a monster model. The theory T is simple if and only if there is a ternary relation $\mathop{\smile\hskip-0.9em ^| \ }$ defined over small subsets which satisfies:* INVARIANCE, FINITE CHARACTER, SYMMETRY, CLOSURE, MONOTONICITY, BASE MONOTONICITY, TRANSITIVITY, EXTENSION, LOCAL CHARACTER *and* $\mathop{\smile\hskip-0.9em ^| \ }$-AMALGAMATION *over models. If such a relation $\mathop{\smile\hskip-0.9em ^| \ }$ exists, $\mathop{\smile\hskip-0.9em ^| \ } = \mathop{\smile\hskip-0.9em ^| \ }^f = \mathop{\smile\hskip-0.9em ^| \ }^d$.*

Proof. This follows from [Cas11, Theorem 12.21]. Indeed, the first nine axioms in our statement are equivalent to the ones of an *independence relation* in the sense of [Cas11], as we saw in the proof of Fact 1.4.4. $\mathop{\smile\hskip-0.9em ^| \ }$-AMALGAMATION over models is equivalent to the Independence Theorem over models, for any relation $\mathop{\smile\hskip-0.9em ^| \ }$ satisfying SYMMETRY, BASE MONOTONICITY, TRANSITIVITY and which is stronger than $\mathop{\smile\hskip-0.9em ^| \ }^a$, which is the case for $\mathop{\smile\hskip-0.9em ^| \ }^f$ is a simple theory by Fact 1.2.3. ☐

1.4.2 NSOP$_1$ theories and Kim-independence

Definition 1.4.7. Let T be a theory, $\mathbb{M}$ a monster for T and $\phi(x, y)$ a formula in the language of T. We say that $\phi(x, y)$ has the 1-*strong order property* (SOP$_1$) if there exists a tree of tuple $(b_\eta)_{\eta \in 2^{<\omega}}$ such that

- for all $\eta \in 2^\omega$ $\{\phi(x, b_{\eta \restriction \alpha}) \mid \alpha < \omega\}$ is consistent
- for all $\eta \in 2^{<\omega}$ if $\eta^\frown 0 \lhd \nu$ then $\{\phi(x, b_\nu), \phi(x, b_{\eta^\frown 1}\}$ is inconsistent.

If in any monster model $\mathbb{M}$ of T, no formula has SOP$_1$, then the theory is called NSOP$_1$.

Recall the definitions of Kim-dividing and Kim-forking for types.

Definition 1.4.8. (1) *Kim-dividing independence.* $a \mathop{\smile\hskip-0.9em ^| \ }^{Kd}_C b$ if and only if $tp(a/Cb)$ does not Kim-divides over C if and only if for all global C-invariant extension p of $tp(b/C)$ and sequences $(b_i)_{i<\omega}$ such that $b_i \models p \restriction Cb_{<i}$ for all $i < \omega$, there exists a' such that $a'b_i \equiv_C ab$ for all $i < \omega$;

(2) *Kim-(forking)independence.* $a \mathop{\smile\hskip-0.9em ^| \ }^K_C b$ if and only if $tp(a/Cb)$ does not Kim-fork over C if and only if for any $b' \supseteq b$ there exists $a' \equiv_{Cb} a$ such that $a' \mathop{\smile\hskip-0.9em ^| \ }^{Kd}_C b'$;

(3) A formula $\phi(x, b)$ *Kim-divides* over C if there is a global C-invariant extension p of $tp(b/C)$ and a sequence $(b_i)_{i<\omega}$ such that $b_i \models p \restriction Cb_{<i}$ for all $i < \omega$, with $\{\phi(x, b_i) \mid i < \omega\}$ inconsistent;

(4) A formula $\phi(x, b)$ *Kim-forks* over C if it implies a finite disjunction of Kim-dividing formulae.

Note that $a \mathop{\smile\hskip-0.9em ^| \ }^K_C b$ if and only if for all finite $a' \subseteq a$, no formula in $tp(a'/Cb)$ Kim-forks over C.

Remark 1.4.9. Given any b and C, a global C-invariant extension of $tp(b/C)$ need not exists. When considering Kim-independence, we will in general assume that the base set is a model, so that $tp(b/\mathscr{M})$ has a global extension finitely satisfiable in $\mathscr{M}$ hence $\mathscr{M}$-invariant. If $tp(b/C)$ has no global C invariant extension, then $a \perp^K_C b$ for all a.

Fact 1.4.10 (Kim's Lemma for Kim-dividing [KR17, Theorem 3.16])**.** *Let T be an* NSOP$_1$ *theory. Then for all formula $\phi(x,b)$, with b in a monster $\mathbb{M}$ of T and $C \subseteq \mathbb{M}$, $\phi(x,b)$ Kim-divides over C if and only if for all global C-invariant extension p of $tp(b/C)$ and a sequences $(b_i)_{i<\omega}$ such that $b_i \models p \restriction Cb_{<i}$ for all $i < \omega$, the set $\{\phi(x,b_i) \mid i < \omega\}$ inconsistent. This is actually equivalent to T being* NSOP$_1$*.*

There is also a recent "geometric" characterisation of NSOP$_1$ by Kim-independence (Definition 1.2.1), see [CR16], [KR17], [KR18].

Fact 1.4.11 (Chernikov-Kaplan-Ramsey, characterisation of Kim-independence and NSOP$_1$ theories)**.** *Let T be a complete theory, and $\mathbb{M}$ a monster model. The theory T is* NSOP$_1$ *if and only if there is a ternary relation $\perp$ which is defined when the base set is a model, which satisfies:* Invariance*,* Symmetry*,* Monotonicity*,* Existence*,* Strong Finite Character *over models, $\perp$-*amalgamation *over models and* Witnessing*. If such a relation $\perp$ exists, $\perp = \perp^K = \perp^{Kd}$.*

Proof. Only Witnessing and $\perp$-amalgamation differs from the properties in the statement of [KR17, Theorem 9.1]. It is clear that our system of axioms is stronger than the one in [KR17, Theorem 9.1], we need to show that they are equivalent. If T is NSOP$_1$, by [KR18, Theorem 2.21] $\perp^K$ satisfies the *Algebraically reasonable independence theorem*, which is exactly $\perp$-amalgamation over models. Also, $\perp^K = \perp^{Kd}$. Assume that $a \not\perp^{Kd}_{\mathscr{M}} b$, then there is a formula $\Lambda(x,b) \in tp(a/\mathscr{M}b)$ and a sequence $(b_i)_{i<\omega}$ in a global $\mathscr{M}$-invariant extension of $tp(b/\mathscr{M})$ such that $\{\Lambda(x,b_i) \mid i < \omega\}$ is inconsistent. By Kim's Lemma (Fact 1.4.10, this actually holds for all global $\mathscr{M}$-invariant extension of $tp(b/\mathscr{M})$ hence in particular for any global coheir of $tp(b/\mathscr{M})$ (which exists), hence our version of Witnessing holds. ☐

Finally, NSOP$_1$ and simple theories are linked by the following.

Fact 1.4.12 ([KR17, Propositions 8.4 and 8.8])**.** *Let T be an* NSOP$_1$ *theory and $\perp^K$ the Kim-independence. Then T is simple if and only if $\perp^K = \perp^f$ over models, if and only if $\perp^K$ satisfies* Base Monotonicity*.*

1.4.3 dp-rank, dp-minimality

We first review two equivalent definitions of dp-rank. More details about dp-rank can be found, e.g. in [Sim15]. Let $\mathbb{M}$ be a monster model of some complete $\mathscr{L}$-theory T.

Definition 1.4.13. A familly of sequences $(I_i)_{i\in S}$ is called *mutually indiscernible over B* if for all $i \in S$, the sequence I_i is indiscernible over $B(I_j)_{j\neq i}$.

Definition 1.4.14. Let $\phi(x,b)$ be an $\mathscr{L}$-formula, with parameters b from $\mathbb{M}$, and let λ be a (finite or infinite) cardinal. We say dp-rank$(\phi(x,b)) < \lambda$ if for every family $(I_i : i < \lambda)$ of mutually indiscernible sequences over b and $a \models \phi(x,b)$, there is $i < \lambda$ such that I_i is indiscernible over ab. We say that dp-rank$(\phi(x,b)) = \lambda$ if dp-rank$(\phi(x,b)) < \lambda^+$ but not dp-rank$(\phi(x,b)) < \lambda$. We say that dp-rank$(\phi(x,b)) \leq \lambda$ if dp-rank$(\phi(x,b)) < \lambda$ or dp-rank$(\phi(x,b)) = \lambda$. For a theory T we denote by dp-rank(T) the dp-rank of $(x = x)$ where $|x| = 1$. If dp-rank$(T) = 1$ we say that T is *dp-minimal*.

Note that if λ is a limit cardinal, it may happen that dp-rank$(\phi(x,b)) < \lambda$ but dp-rank$(\phi(x,b)) \geq \mu$ for all $\mu < \lambda$ (see [Sim15, Section 4.2]).

Definition 1.4.15. Let κ be some cardinal. An *ict-pattern of length κ* consists of:

- a collection of formulas $(\phi_\alpha(x; y_\alpha) : \alpha < \kappa)$, with $|x| = 1$;
- an array $(b_i^\alpha : i < \omega, \alpha < \kappa)$ of tuples; with $|b_i^\alpha| = |y_\alpha|$

such that for every $\eta : \kappa \to \omega$ there exists an element $a_\eta \in \mathbb{M}$ such that

$$\models \phi_\alpha(a_\eta; b_i^\alpha) \iff \eta(\alpha) = i.$$

We define κ_{ict} as the minimal κ such that there does not exist an ict-pattern of length κ.

Fact 1.4.16 ([Sim15, Proposition 4.22])**.** *For any cardinal κ, we have dp-rank$(T) < \kappa$ if and only if $\kappa_{ict} \leq \kappa$.*

Lemma 1.4.17. *Let $\mathscr{L} = \bigcup_{\alpha<\kappa} \mathscr{L}_\alpha$ be a language such that every atomic formula in $\mathscr{L}$ is in $\mathscr{L}_\alpha$ for some α. Let T be an $\mathscr{L}$-theory that eliminates quantifiers, and for $\alpha < \kappa$ let T_α be its reduction to $\mathscr{L}_\alpha$. Let μ_α be cardinals such that dp-rank$(T_\alpha) \leq \mu_\alpha$. Then dp-rank$(T) \leq \sum_{\alpha<\kappa} \mu_\alpha$, where $\sum$ is the cardinal sum.*

Proof. Suppose not. Let $\lambda := \sum_{\alpha<k} \mu_\alpha$. Then there is a family $(\mathcal{I}_t : t < \lambda^+)$ of mutually indiscernible sequences over $\emptyset$, $\mathcal{I}_t = (a_{t,i} : i \in I_t)$, and a singleton b, such that for all t, $\mathcal{I}_t$ is not indiscernible over b. For every $t < \lambda^+$, let $\phi_t(\bar{x}) = \phi_t(\bar{x}, b)$ be a formula over b and let $\bar{c}_{t,1}$ and $\bar{c}_{t,2}$ be two finite tuples of elements of $\mathcal{I}_t$ of length $|\bar{x}|$ such that $\phi_t(\bar{c}_{t,1})$ and $\neg\phi_t(\bar{c}_{t,2})$, i.e. witnessing the non-indiscernibility of $\mathcal{I}_t$ over b. By quantifier elimination in T, we may assume that ϕ_t is quantifier-free. Hence there must be an atomic formula $\psi_t(\bar{x}) = \psi_t(\bar{x}, b)$ such that $\psi_t(\bar{c}_{t,1})$ and $\neg\psi_t(\bar{c}_{t,2})$. By the assumption on $\mathscr{L}$, there is an $\alpha_t < \kappa$ such that $\psi_t(\bar{x}, y)$ is in $\mathscr{L}_{\alpha_t}$. Therefore, there must be an $\alpha < \kappa$ such that $|\{t < \lambda^+ : \alpha_t = \alpha\}| > \mu_\alpha$, as otherwise we get

$$\lambda^+ = \left| \bigcup_{\alpha<\kappa} \{t < \lambda^+ : \alpha_t = \alpha\} \right| \leq \sum_{\alpha<\kappa} |\{t < \lambda^+ : \alpha_t = \alpha\}| \leq \sum_{\alpha<\kappa} \mu_\alpha = \lambda,$$

a contradiction. But then $(\mathcal{I}_t : t < \lambda^+, \alpha_t = \alpha)$ is a family of more than μ_α mutually indiscernible sequences over $\emptyset$ with respect to $\mathscr{L}_\alpha$, and for all t such that $\alpha_t = \alpha$, $\mathcal{I}_t$ is not indiscernible over b with respect to $\mathscr{L}_\alpha$, a contradiction to dp-rank$(T_\alpha) \leq \mu_\alpha$. ☐

1.5 Preliminaries on fields

1.5.1 Generalities

We recall some definitions from classical field theory, as can be found e.g. in [FJ05, Chapter 2]. We assume that all fields considered are subfields of a big algebraically closed field, and we denote by $\mathbb{F}$ the prime field. For a field K we will denote by K^{alg} or $\overline{K}$ and K^s respectively the algebraic closure and the separable closure of K, i.e. the field consisting of all elements algebraic (respectively separably algebraic) over K. We denote by K^{ins} the maximal purely inseparable extension of K, if $char(K) = 0$ then $K = K^{ins}$, if $char(K) = p > 0$, K^{ins} is the field generated by $\left\{\alpha \mid \alpha^{p^{-n}} \in K,\ n \in \mathbb{N}\right\}$. We denote by L/K the fact that L is an extension of the field K. Given two fields L and K, we denote by LK the compositum of L and K. For a tuple a, $K(a)$ is the field generated by K and a. Given a prime number p and $n \in \mathbb{N}$, the field of cardinality p^n will be denoted by $\mathbb{F}_{p^n}$.

Definition 1.5.1. Let K, L be two field extensions of a field E.

(1) We say that K is *linearly disjoint* from L over E (denoted by $K \mathop{\smile\!\!\!\!\!\!\mid}^{ld}_E L$) if every finite tuple from K which is linearly independent over E is also linearly independent over L in the compositum KL.

(2) We say that K is *algebraically independent* from L over E (denoted by $K \mathop{\smile\hspace{-0.9em}\downarrow}^{alg}_{E} L$) if every finite tuple from K which is algebraically independent over E is also algebraically independent over L in the compositum KL.

(3) An extension L/K is called *separable* if $L \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{K} K^{ins}$. It is called *regular* if $L \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{K} K^{alg}$.

The definitions of $\mathop{\smile\hspace{-0.9em}\downarrow}^{ld}$ and $\mathop{\smile\hspace{-0.9em}\downarrow}^{alg}$ turn out to be symmetric, and we will sometimes say that K and L are linearly disjoint (or algebraically independent) over E. These are notions of independence only defined over fields. An easy way of extending their definition is by setting for every A, B, E subsets of some big field, $A \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{E} B$ if and only if $\mathbb{F}(AE) \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{\mathbb{F}(E)} \mathbb{F}(BE)$, and similarly with $\mathop{\smile\hspace{-0.9em}\downarrow}^{acl}$. With this extended definition, in any field F with prime field $\mathbb{F}$, the ternary relations $\mathop{\smile\hspace{-0.9em}\downarrow}^{ld}$ and $\mathop{\smile\hspace{-0.9em}\downarrow}^{alg}$ are defined over every subsets of F. Note that if K is an algebraically closed field, $\mathop{\smile\hspace{-0.9em}\downarrow}^{alg}$ defined over subsets of K is the independence relation associated with the pregeometry described in Example 1.3.6.

Fact 1.5.2. *$\mathop{\smile\hspace{-0.9em}\downarrow}^{ld}$ and $\mathop{\smile\hspace{-0.9em}\downarrow}^{alg}$satisfy* Symmetry, Finite Character, Monotonicity, Transitivity *and* Base Monotonicity. *Furthermore $\mathop{\smile\hspace{-0.9em}\downarrow}^{alg}$ satisfies* Closure: *if $K \mathop{\smile\hspace{-0.9em}\downarrow}^{alg}_{E} L$ then $K \mathop{\smile\hspace{-0.9em}\downarrow}^{alg}_{\overline{E}} \overline{L}$. We have $\mathop{\smile\hspace{-0.9em}\downarrow}^{ld} \to \mathop{\smile\hspace{-0.9em}\downarrow}^{alg}$.*

Proof. For $\mathop{\smile\hspace{-0.9em}\downarrow}^{ld}$, Symmetry is [FJ05, Lemma 2.5.1], Monotonicity, Base Monotonicity and Transitivity follow from [FJ05, Lemma 2.5.3]. Finite Character is by definition. For $\mathop{\smile\hspace{-0.9em}\downarrow}^{alg}$, it is Fact 1.3.3. The last assertion follows from the simple fact that a tuple is algebraically dependent over some field if and only if the familly of monomials of this familly is linearly independent over this field [FJ05, p. 41]. ☐

Remark 1.5.3. Note that $A \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{C} B$ implies $\mathbb{F}(AC) \cap \mathbb{F}(BC) = \mathbb{F}(C)$ whereas $A \mathop{\smile\hspace{-0.9em}\downarrow}^{alg}_{C} B$ implies $\overline{\mathbb{F}(AC)} \cap \overline{\mathbb{F}(BC)} = \overline{\mathbb{F}(C)}$.

We deduce the following classical fact:

Fact 1.5.4. *Let $E \subset K \subset L$ be three fields. Assume that L/K is separable (respectively regular). Then L/E is separable (resp. regular) if and only if K/E is separable (resp. regular).*

The relations $\mathop{\smile\hspace{-0.9em}\downarrow}^{ld}$ and $\mathop{\smile\hspace{-0.9em}\downarrow}^{alg}$ coincide when one of the extension is regular.

Fact 1.5.5 ([FJ05, Lemma 2.6.7])**.** *Let $E \subset K \cap L$ be three fields. If K/E is regular, then $K \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{E} L$ if and only if $K \mathop{\smile\hspace{-0.9em}\downarrow}^{alg}_{E} L$.*

Fact 1.5.6 ([Cha99, Lemma 3.1 (1)])**.** *Let $E \subset K \cap L$ be three fields. If K/E, L/E are regular and $K \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{E} L$ then $K^s \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{E^s} L^s$.*

Lemma 1.5.7. *Let A, B be two extensions of some field E, such that AB/E is regular and $A \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{E} B$. Then $(A^s + B^s) \cap AB = A + B$.*

Proof. First, observe that $A^sB \cap B^s = E^sB$. Indeed A/E and B/E are regular so by Fact 1.5.6, we have that $A^s \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{E^s} B^s$ hence $A^sB \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{E^sB} B^s$ and so $A^sB \cap B^s = E^sB$. Symmetrically, we have $AB^s \cap A^s = E^sA$. If $v \in AB$ is such that $v = \alpha + \beta$ for $\alpha \in A^s$ and $\beta \in B^s$, then $\alpha = v - \beta \in AB^s \cap A^s = E^sA$. Similarly $\beta \in E^sB$. Let L be a finite extension of E inside E^s such that $\alpha \in AL$ and $\beta \in BL$. We can complete $\{1\}$ to a basis $\{1, u_2, \dots, u_n\}$ of the E-vector space L. As $AB \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{E} L$, it is also a basis of the AB-vector space LAB. As $AB \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{A} LA$ and $AB \mathop{\smile\hspace{-0.9em}\downarrow}^{ld}_{B} LB$, it is also a basis of the A-vector space LA and of the B-vector space LB. Now the coordinates of $v \in AB$ in the AB-vector space LAB are $(v, 0, \dots, 0)$ as $v = v + 0u_2 + \dots + 0u_n$. Let $(a_1, \dots, a_n)$ (respectively $(b_1, \dots, b_n)$) be the coordinates of α with respect to the basis $(1, u_2, \dots, u_n)$ of the A-vector space LA (respectively of β in this basis of the B-vector space LB). As $v = \alpha + \beta$, we have, looking at the first coordinate that $v = a_1 + b_1$, so $v \in A + B$. ☐

Lemma 1.5.8. *Let K be a field and $K(X,Y)$ be a rational function field in two variables (in other words $X \mathop{\smile\hskip-0.9em\mid\hskip0.3em}^{alg}_{K} Y$ and $X, Y \notin \overline{K}$). Then*

$$XY \notin K(X) + K(Y);$$

$$X + Y \notin K(X) \cdot K(Y);$$

where $K(X) \cdot K(Y) = \{uv \mid u \in K(X), v \in K(Y)\}$.

Proof. There exists a derivative $D : K(X,Y) \to K(X,Y)$ such that $D(K(Y)) = \{0\}$ and D extends the canonical derivation on $K(X)$ (namely the *partial* derivative with respect to X, see [Mor96, Proposition 23.11]). Let $u \in K(X)$ and $v \in K(Y)$. If $XY = u + v$ then applying D we get $Y = Du \in K(X)$ a contradiction. If $X + Y = uv$ then applying D we get $1 = vDu$ hence, as $Du \in K(X)$, $v \in K(X) \cap K(Y) = K$. Now $Y = uv - X \in K(X)$ a contradiction. ☐

1.5.2 Fields and model theory

We denote by $\mathscr{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$ the language of rings. The following is [Cha99, (1.17)].

Fact 1.5.9. *Let T be any theory of fields in any language $\mathscr{L} \supseteq \mathscr{L}_{\text{ring}}$. Let $F \models T$ and $A \subseteq F$. Then $F/acl_T(A)$ is a regular extension.*

The following gives a behaviour of the Kim-independence in *any* theory of fields.

Fact 1.5.10 ([KR17, Proposition 9.28], [Cha99, Theorem 3.5])**.** *Let T be an arbitrary theory of fields, and $E \prec F \models T$. Let A, B be acl_T-closed subsets of F containing E, such that $A \mathop{\smile\hskip-0.9em\mid\hskip0.3em}^{K}_{E} B$. Then*

(1) $A \mathop{\smile\hskip-0.9em\mid\hskip0.3em}^{ld}_{E} B$;

(2) F/AB is a separable extension;

(3) $\mathrm{acl}_T(AB) \cap A^s B^s = AB$.

Lemma 1.5.11. *Let T be an arbitrary theory of fields, and $F \models T$. Let A, B, C, D be subsets of F, containing some set $k \subseteq F$, and such that $A, B \subseteq D$. Assume that A and B are acl_T-closed and $D \mathop{\smile\hskip-0.9em\mid\hskip0.3em}^{u}_{k} C$, (i.e. $tp^T(D/C)$ is finitely satisfiable in k). Then we have the following results.*

(1) $(F \cap (AC)^s + F \cap (BC)^s) \cap D = A + B$;

(2) $[(F \cap \overline{AC}) \cdot (F \cap \overline{BC})] \cap D = A \cdot B$.

Assume now that $A, B \mathop{\smile\hskip-0.9em\mid\hskip0.3em}^{u}_{A \cap B} F \cap \overline{AC} \cap \overline{BC}$. Then $F \cap \overline{AC} \cap \overline{BC} = F \cap \overline{(A \cap B)C}$.

Proof. We give the idea for *(1)*, the others are proved by a similar argument. Let $v_1 \in F \cap (AC)^s$, $v_2 \in F \cap (BC)^s$ and $u \in D$ be such that $u = v_1 + v_2$. There exist nontrivial separable polynomials $P(X, a, c)$ and $Q(X, b, c')$ with leading coefficients 1 such that v_1 is a root of $P(X, a, c)$ and v_2 is a root of $Q(X, b, c')$, a a tuple in A, b a tuple in B. The formula $\phi(z_1, z_2, z_3, c, c')$

$$\exists x \exists y \ x + y = z_1 \wedge P(x, z_2, c) = 0 \wedge Q(y, z_3, c') = 0$$

is in $tp^T(u, a, b/C)$, which is finitely satisfiable in k. Hence, there exists $d, d' \in k$ such that $\phi(z_1, z_2, z_3, d, d') \in tp^T(u, a, b/k)$ and so $u \in A + B$ as A and B are acl_T-closed. ☐

The theory ACF. Let ACF be the theory of algebraically closed fields in $\mathscr{L}_{\text{ring}}$. We recall here some basic facts about this well-known theory, as can be found e.g. in [Bou+98]. ACF is model-complete, it is the model-companion of the theory of fields in $\mathscr{L}_{\text{ring}}$. It is not complete but its completions are given by specifying the characteristic p of the field, we denote the completion obtained by ACF_p. ACF_p is *strongly minimal*, so in particular it is stable. Let $K \models \mathrm{ACF}_p$. A *Zariski-closed* subset of K^n is the set of solutions of a finite number of polynomial equations in $(X_1, \cdots, X_n)$. Those are closed subsets of a topology on all cartesian powers of K called the Zariski topology. An *affine (irreducible) variety* is a Zariski-closed set that cannot be written as the union of two proper Zariski-closed sets. Every Zariski-closed set can be decomposed into the union of finitely many affine varieties (the topology is *Noetherian*). A *quasi-affine variety* is an open subset of an affine variety, hence a set of solutions of some polynomial equations and some polynomial inequations. The theory ACF_p has quantifier eliminations in the language of rings, this means that every definable set in an algebraically closed field K is a finite union of quasi-affine varieties. A *generic* $\mathbf{x}$ of some quasi-affine variety V is a tuple in an elementary extension of K such that if $P(\mathbf{x}) = 0$ for some polynomial P with coefficients in K, then V is included in the Zariski-closed set defined by P. Informally, $\mathbf{x}$ satisfies no other equations than the one defining V. Generic points of a variety $V \subset K^n$ always exists in elementary extensions of K. We will not need much those notions except in Section 3.3. For a field K of characteristic p, the *Frobenius endomorphism* is the field endomorphism of K defined by $\mathrm{Frob} : x \mapsto x^p$.

Fact 1.5.12. *Let $K \models \mathrm{ACF}_p$. If $\xi : K \to K$ is an additive definable endomorphism, then ξ is of the form $\xi(x) = a_1 \mathrm{Frob}^{n_1}(x) + \cdots + a_k \mathrm{Frob}^{n_k}(x)$, with $n_1, \cdots, n_k \in \mathbb{Z}$.*

Proof. By [Bou+98, Chapter 4, Corollary 1.5], a definable map is given by a composition of powers of the Frobenius and rational maps, on a definable partition of K, and by [Hum98, Lemma A, VII, 20.3], additive polynomials are p-polynomials. It is easy to see that the fact follows. ☐

The theory SCF. If K is of characteristic $p > 0$, we denote by K^p the image of K by the Frobenius endomorphism. If K is separably closed and *perfect* (i.e. if K is of characteristic 0 or K is of characteristic p and $K^p = K$), K is an algebraically closed field. We assume that the characteristic of K is $p > 0$. Let $A \subseteq K$, the *p-closure of A* is the field $K^p(A)$. This defines a pregeometry on K (see for instance [Bou06a, Chapitre 5, §13]), a basis for this pregeometry is called a p-basis, and an independent set is called a p-independent set. A set A is p-independent if and only if for all finite tuple $a_1, \cdots, a_n$ from A, the set of monomials $a_1^{e_1} \cdot \cdots \cdot a_n^{e_n}$ are K^p-linearly independent, where $0 \leq n_k < p$. If K/K^p is a finite extension, it has degree p^e for some integer e, which we call the *Ershov* invariant of K (or imperfection degree). If K/K^p is infinite, we write $e = \infty$. Let $\mathscr{L}$ be the language of rings extended by n-ary relations Q_n. Let $\mathrm{SCF}_{p,e}$ be the theory of separably closed field of characteristic p and Ershov invariant e in the language $\mathscr{L}$ in which the relations Q_n represent p-independence.

Fact 1.5.13. *For all $e \leq \infty$, the theory $\mathrm{SCF}_{p,e}$ is complete, model-complete and stable. Furthermore, $\mathrm{SCF}_{p,e}$ eliminates $\exists^\infty$.*

Proof. The first part is Theorems 1 and 3 of [Woo79]. In [Del88, Proposition 61.] is proved that $\mathrm{SCF}_{p,e}$ has the NFCP, which implies elimination of $\exists^\infty$. ☐

Note that any model of $\mathrm{SCF}_{p,e}$ is existentially closed in every separable extension ([Bou+98, Chapter 9, Claim 2.2]). We have the following description of nonforking in the sense of $\mathrm{SCF}_{p,e}$ (see the remark after [Cha02, (1.2)]).

Fact 1.5.14. *Assume that A, B, C are separably closed subfields of a separably closed field F such that $C \subseteq A \cap B$. If $A \mathop{\smile\!\!\!\!\!\!\mid}^{ld}_{C} B$ and F/AB is separable, then $tp^{SCF_{p,e}}(A/B)$ does not fork over C.*

The theories ACFA_p, DCF_p. The theory ACFA_p is the model-companion of the theory of difference fields (i.e. fields with a distinguished endomorphism) of characteristic p, it was proved to be model-complete in [Mac97] and unstable but supersimple in [CH99] for any p prime or zero. DCF_p is the model-companion of differential fields of characteristic p (for $p = 0$, see [MMP96] and for $p > 0$, see [Woo73]) and is proved to be stable in [Woo76]. The theory ACFA_p eliminates $\exists^\infty$ in all characteristic, this follows easily from the definability of the σ-degree (see [CH99, Section 7]). For all p prime or 0, the theory DCF_p eliminates the quantifier $\exists^\infty$, this follows from the proof of this result in [MMP96, Theorem 2.13, p51], although it was proved in the characteristic 0 case, the proof works in all characteristics.

The theory PAC. A *pseudo algebraically closed field* is a field K which is existentially closed in every regular extension[2]. The property for a field to be pseudo algebraically closed is first order (see [FJ05]), we denote by PAC the corresponding theory. It is an incomplete theory, even when specifying the characteristic of the field (we denote the corresponding theory by PAC_p). The theory of a PAC field K is described by the isomorphism type of the field $\mathrm{acl}(\emptyset)$, the imperfection degree of K and the "first-order theory of the absolute Galois group" (in a suitable ω-sorted language, for more details, see [CDM81]). A PAC field K is *bounded* if it has finitely many algebraic extensions of degree n, for all n. It is known that a PAC field has a simple theory if and only if it is bounded (see [CP98] for the "if" and [Cha99] for the "only if"). An *ω-free* PAC field is a PAC field K which has an elementary substructure K_0 whose absolute Galois group is isomorphic to the free profinite group with countably many generators. In [FJ05, Chapter 27] is presented a language and a theory of fields for which ω-free PAC_p fields of imperfection degree 1 (if $p > 0$) are the existentially closed models: expand $\mathscr{L}_{\mathrm{ring}}$ by n-ary predicates $R_n(x_1, \cdots, x_n)$ expressing that $\exists z\ z^n + x_1 z^{n-1} + \cdots + x_n = 0$. In this expanded laguage, K is a substructure of L if and only if K is algebraically closed in L. Then the theory of ω-free PAC fields of imperfection degree 1 (if $p > 0$) is the model-companion of the theory of fields in this expanded language.

Fact 1.5.15 ([Cha02], [CR16]). *Every ω-free* PAC *field has an* NSOP_1 *theory.*

A recent result from Nick Ramsey states that a PAC field is NSOP_1 provided its Galois group has an NSOP_1 theory.

A theory of fields T in an expansion of the language of rings is *algebraically bounded* if for all formula $\phi(x, y)$ with $|x| = 1$ there are polynomials $P_1(X, Y), \cdots, P_n(X, Y)$ in $\mathbb{Z}[X, Y]$ with $|X| = 1$ and $|Y| = |y|$ such that for all $K \models T$, and b a $|y|$-tuple from K, if $\phi(K, b)$ is finite then there exists i such that $P_i(X, b)$ is finite and $\phi(K, b)$ is contained in the set of roots of $P_i(X, b)$. In particular, an algebraically bounded field eliminates the quantifier $\exists^\infty$. This notion was introduced in [Dri89], it leads to the existence of a well-behaved notion of dimension on the definable sets, in particular, any algebraically bounded field must be perfect.

Fact 1.5.16 ([CH04]). *Every perfect* PAC *field is algebraically bounded.*

The theory Psf. It is the theory of *pseudo-finite fields* (see [Ax68] or [TZ12]), fields which are PAC, perfect and 1-free (i.e. has only one extension of degree n for all n). In particular, from Fact 1.5.16 it eliminates the quantifier $\exists^\infty$. From [TZ12, Proposition B.4.13], an extension L of a pseudo-finite field K is regular if and only if K is relatively algebraically closed in L (i.e. $L \cap \overline{K} = K$), hence a Psf field K is existentially closed in every extension L in which it is relatively algebraically closed. Any non-principal ultraproduct of finite fields is a pseudo-finite field. Let $\mathscr{L}$ be the language of rings expanded by constants symbols $(c_{i,j})_{i<\omega, j<i}$, and let Psf_c be the expansion of Psf expressing that the polynomial $X^n + c_{n,n-1}X + \cdots + c_{n,0}$ is irreducible. The theory Psf_c is model-complete, see [Cha97, Section 3].

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[2] The classical definition of a pseudo algebraically bounded field is the following: K is pseudo algebraically closed if every absolutely irreducible variety defined over K has a K-rational point (see [FJ05] or [TZ12]). We do not use this definition here and prefer the equivalent in term of regular extension since it is the main property that we will use about these fields. Note that these fields were also called *regularly closed*, which would be a better name for our purpose.

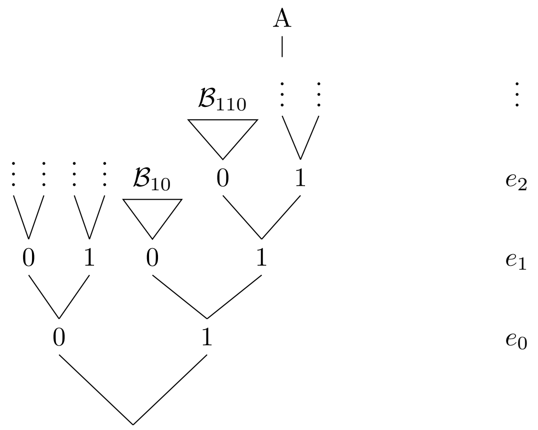

Figure 1.1: Topology on $2^{\overline{\mathbb{F}_p}}$.

1.6 The Chabauty topology on $\mathrm{Sg}(\overline{\mathbb{F}_p})$

Recall some standard facts about the topology on the Cantor space 2^ω, which can be found for instance in [Mos09] or [Kec95]. The Cantor space 2^ω is endowed with the product topology coming from the discrete topology on 2. This topology turns 2^ω into a *Polish* space; i.e. the topology admits a countable basis (*second-countable*) and admits a complete metric (*complete metrizable*). By Tychonoff's theorem, it is also *compact*, and it is *perfect*, i.e. without isolated points. It is also *totally disconnected* i.e. has no nontrivial connected subsets. Finally, a theorem of Brouwer states that 2^ω is the unique (up to homeomorphism) non-empty, totally disconnected, perfect, compact, metrizable set. Such a set is henceforth referred to as *a Cantor space.*

We fix a bijection : $e : \omega \to \overline{\mathbb{F}_p}$ and an enumeration $E = \{e_i := e(i) \mid i < \omega\}$. This enumeration induces a homeomorphism between the Cantor space 2^ω and $2^{\overline{\mathbb{F}_p}}$, hence turns the powerset $\mathscr{P}(\overline{\mathbb{F}_p})$ into a Cantor space. We give a description of the topology obtained on $\mathscr{P}(\overline{\mathbb{F}_p})$ based one the notion of *Cantor scheme*, i.e. a topology on the branchs of an infinite binary tree, as can be seen in [Kec95, Subsection I.6.A].

For each $A \subseteq \overline{\mathbb{F}_p}$, let 1_A be the function $1_A : E \to 2 := \{0,1\}$ defined by

$$1_A(b) = 1 \iff b \in A.$$

For $k < \omega$ we adopt the following notation:

$$A \restriction k := (1_A(e_0), \cdots, 1_A(e_{k-1})) \in 2^k.$$

Let $k < \omega$ and $s \in 2^k$, we define $\mathcal{B}_s = \left\{A \subseteq \overline{\mathbb{F}_p} \mid A \restriction k = s\right\}$ and for $A \subseteq \overline{\mathbb{F}_p}$, let $\mathcal{B}(A,k)$ be $\left\{B \subseteq \overline{\mathbb{F}_p} \mid B \restriction k = A \restriction k\right\} = \mathcal{B}_{A\restriction k}$. The family $(\mathcal{B}_s)_{s\in 2^k, k<\omega}$ forms a basis of the topology on $\mathscr{P}(\overline{\mathbb{F}_p})$. A convenient way of getting a picture of this topology is by representing the subsets of $\mathscr{P}(\overline{\mathbb{F}_p})$ by branchs of a binary tree in which each level of nodes represent an element of the enumeration E of $\overline{\mathbb{F}_p}$. A branch representing $A \in \mathscr{P}(\overline{\mathbb{F}_p})$ takes the value 1 at the node of level e_i if and only if $e_i \in A$. Hence, in Figure 1.1, the set A contains e_0, e_1, e_2 but not e_3. For $s \in 2^k$, the ball $\mathcal{B}_s$ contains all the sets that are represented by branchs that start with the sequence s. It is easy to see that each ball is clopen (closed and open).

The set of all subgroups of some countable group can be endowed with a topology that is compact, it is called the *Chabauty* topology. In the case of the group $(\overline{\mathbb{F}_p}, +)$, this topology has a very explicit description, in particular, it is the topology of a Cantor set. More generally the Chabauty topology of any countable group is the one of a Cantor space provided that the group is not *minimax*, see [CGP10, Proposition A].

Lemma 1.6.1. *Let* $\mathrm{Sg}(\overline{\mathbb{F}_p}) \subseteq \mathscr{P}(\overline{\mathbb{F}_p})$ *be the set of all subgroups of* $(\overline{\mathbb{F}_p}, +)$*. Then* $\mathrm{Sg}(\overline{\mathbb{F}_p})$ *is a compact subset of* $\mathscr{P}(\overline{\mathbb{F}_p})$*. Furthermore, it is a Cantor space, the topology is generated by clopens sets of the form* $\mathcal{B}(H_0, \mathbb{F}_{p^n}) = \left\{H \in \mathrm{Sg}(\overline{\mathbb{F}_p}) \mid H \cap \mathbb{F}_{p^n} = H_0\right\}$*, for some finite group* $H_0 \in \mathrm{Sg}(\overline{\mathbb{F}_p})$*.*

Proof. First, we show that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is compact. As $\mathscr{P}(\overline{\mathbb{F}_p})$ is compact, it is enough to show that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is closed. We show that its complement is open. A set $A \in \mathscr{P}(\overline{\mathbb{F}_p})$ is a not a group if and only if at least one of the following three conditions is satisfied:

- $0 \notin A$;
- $a \in A$ and $-a \notin A$;
- $a, b \in A$ and $a + b \notin A$.

The first condition is clearly open since in a metric space every singleton is closed, let $\mathscr{O}_0$ be $\mathscr{P}(\overline{\mathbb{F}_p}) \setminus \{0\}$. Let $a, b \in \overline{\mathbb{F}_p}$, let $i, j, k < \omega$ be such that $e_i = a, e_j = b$ and $e_k = a + b$. Let $S(a, b)$ be the set of all finite sequence $s \in 2^{\max(i,j,k)}$ such that $s_i = s_j = 1$ and $s_k = 0$ (see Figure 1.2). Then $\mathscr{O}_1 = \bigcup_{a,b \in \overline{\mathbb{F}_p}} \bigcup_{s \in S(a,b)} \mathcal{B}_s$ is the set of all subsets A of $\overline{\mathbb{F}_p}$ such that for some $a, b \in \overline{\mathbb{F}_p}$ we have $a, b \in A$ and $a + b \notin A$. This is clearly an open set. Similarly there is an open set $\mathscr{O}_2$ which is the set of all $A \in \mathscr{P}(\overline{\mathbb{F}_p})$ such that there exists $a \in \overline{\mathbb{F}_p}$ with $a \in A$ and $-a \notin A$. Then $\mathscr{P}(\overline{\mathbb{F}_p}) \setminus \mathrm{Sg}(\overline{\mathbb{F}_p}) = \mathscr{O}_0 \cup \mathscr{O}_1 \cup \mathscr{O}_2$ is open.

It is clear that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is again metrizable and totally disconnected. Assume that it is not perfect, and let H be an isolated point in $\mathrm{Sg}(\overline{\mathbb{F}_p})$, and $\mathcal{B}(H, k)$ a clopen containing H, for some k. Then consider the finite subgroup H_0 generated by $\{e_i \in H \mid 0 \leq i \leq k\}$. It is clear that $H_0 \in \mathcal{B}(H, k)$ since $H_0 \upharpoonright k = H \upharpoonright k$. As H_0 is finite, there exists $n \geq k$ such that for all $m \geq n$ we have $1_{H_0}(m) = 0$. If $1_H(e_n) = 0$, then $e_n \notin H$ and consider G the group generated by H_0 and e_n. If $1_H(e_n) = 1$ consider $G = H_0$. In any case we have $G \neq H$ and $G, H \in \mathcal{B}(H, k)$ hence H is not isolated. It follows that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is perfect. As it is clearly nonempty it follows that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is a Cantor space. The topology on $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is generated by $\mathcal{B}(H, k)$, as for $\mathscr{P}(\overline{\mathbb{F}_p})$. By the same agument as above, if H_0 is the subgroup generated by $\{e_i \in H \mid 0 \leq i \leq k\}$, then $H_0 \in \mathcal{B}(H, k)$ hence for some $k' \geq k$, we have $\mathcal{B}(H_0, k') \subseteq \mathcal{B}(H, k)$. Similarly, there is some $n \in \mathbb{N}$ such that $\mathcal{B}(H_0, \mathbb{F}_{p^n}) \subseteq \mathcal{B}(H_0, k')$, hence the topology is spanned by balls of the form $\mathcal{B}(H_0, \mathbb{F}_{p^n})$. ☐

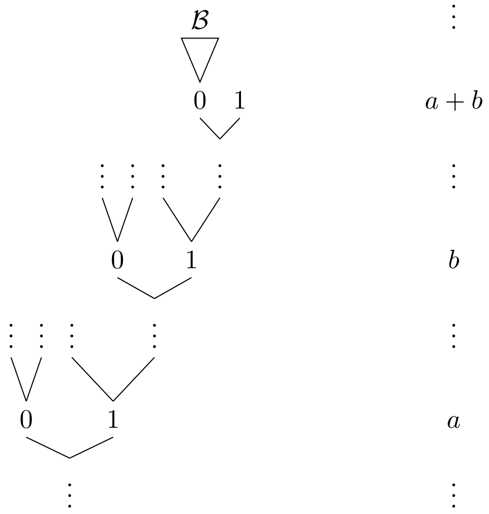

Figure 1.2: A ball that does not contain any groups.

Part A

Generic expansions

CHAPTER 2

Generic expansions by a reduct

Let T be an $\mathscr{L}$-theory. Let $\mathscr{L}_0 \subseteq \mathscr{L}$ and let T_0 be a reduct of T to the language $\mathscr{L}_0$. Let S be a new unary predicate symbol and set $\mathscr{L}_S = \mathscr{L} \cup \{S\}$. We denote by T_S the $\mathscr{L}_S$-theory of $\mathscr{L}_S$-structures $(\mathscr{M}, \mathscr{M}_0)$ where $\mathscr{M} \models T$ and $S(\mathscr{M}) = \mathscr{M}_0 \models T_0$ is a substructure of $\mathscr{M} \upharpoonright \mathscr{L}_0$. The main result of this chapter is an answer to the following question:

> Under which conditions on T and T_0 does the model-companion of the theory T_S exist?

Contents

2.1 The main result

We denote by acl_0 the algebraic closure in the sense of T_0. Assume that T_0 is pregeometric. By Section 1.3, there is an associated independence relation $\mathop{\smile\!\!\!\!\!\!|}^{0}$. It is defined over every subset of any model of T_0 and satisfies the properties FINITE CHARACTER, SYMMETRY, CLOSURE, MONOTONICITY, BASE MONOTONICITY, TRANSITIVITY. In particular, $\mathop{\smile\!\!\!\!\!\!|}^{0}$ is defined over every subset of any model of T, and we will only use it over small subsets of a monster model $\mathbb{M}$ of T. The property SYMMETRY of $\mathop{\smile\!\!\!\!\!\!|}^{0}$ will be tacitly used throughout this chapter.

Definition 2.1.1. Let t be a single variable and x, y two tuples of variables. We say that a formula $\psi(t, y)$ is *n-algebraic in t* (or just *algebraic in t*) if for all tuple b the number of realisations of $\psi(t, b)$ is at most n. In that context we say that a formula $\psi(t, x, y)$ is *strict in y* if whenever b is an $\mathop{\smile\!\!\!\!\!\!|}^{0}$-independent tuple over a, the set of realisations of $\psi(t, a, b)$ is in $\mathrm{acl}_0(a, b) \setminus \mathrm{acl}_0(a)$.

If $\psi(t, b)$ is an $\mathscr{L}_0$-algebraic formula, there exists an $\mathscr{L}_0$-formula $\tilde{\psi}(t, x)$ algebraic in t such that $\psi(\mathscr{M}, b) \subseteq \tilde{\psi}(\mathbb{M}, b)$.

Example 2.1.2. In the language of vector spaces, the formula $t = \lambda x + \mu y$ is strict in y if and only if $\mu \neq 0$.

Lemma 2.1.3. *Assume that T_0 is pregeometric. Then for u a singleton and tuples a and b, if $u \in \mathrm{acl}_0(a, b) \setminus \mathrm{acl}_0(a)$, there exists an $\mathscr{L}_0$-formula $\tau(t, x, y)$ algebraic in t and strict in y such that $u \models \tau(t, a, b)$.*

Proof. Assume that $b = b_1, \dots, b_n$. By hypothesis and using Exchange, we may assume that $b_1 \in \mathrm{acl}_0(u, a, b_2, \dots, b_n)$. Let $\tau_1(t, a, b)$ be an $\mathscr{L}_0$-formula algebraic in t isolating the type $tp^{T_0}(u/ab)$ and $\tau_2(y_1, u, a, b_2, \dots, b_n)$ algebraic in y_1 isolating $tp^{T_0}(b_1/u, a, b_2, \dots, b_n)$. Then $\tau(t, x, y) = \tau_1(t, x, y) \wedge \tau_2(y_1, t, x, y_2, \dots, y_n)$ is strict in y. Indeed assume that for some independent tuple b' over a', and singleton u' we have $\models \tau(u', a', b')$. It follows that $u' \in \mathrm{acl}_0(a'b')$ and $b'_1 \in \mathrm{acl}_0(u', a', b'_2 \dots, b'_n)$. If $u' \in \mathrm{acl}_0(a')$ then $b'_1 \in \mathrm{acl}_0(a', b'_2, \dots, b'_n)$ contradicting that b' is $\mathop{\smile\!\!\!\!\!\!|}^{0}$-independent over a', so $u' \notin \mathrm{acl}_0(a')$. ☐

Definition 2.1.4. An expansion $(\mathscr{M}, \mathscr{M}_0) \subseteq (\mathscr{N}, \mathscr{N}_0)$ is *strong* if $\mathscr{N}_0 \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathscr{M}_0} \mathscr{M}$.

Theorem 2.1.5. *Assume that the following holds:*

(H_1) *T is model complete;*

(H_2) *T_0 is model complete and for all infinite A, $\mathrm{acl}_0(A) \models T_0$;*

(H_3) *T_0 is pregeometric;*

(H_4) *for all $\mathscr{L}$-formula $\phi(x, y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for $b \in \mathscr{M} \models T$,*

$$\mathscr{M} \models \theta_\phi(b) \iff \begin{array}{l}\text{there exists } \mathscr{N} \succ \mathscr{M} \text{ and } a \in \mathscr{N} \text{ such that} \\ \phi(a, b) \text{ and } a \text{ is an } \mathop{\smile\!\!\!\!\!\!|}^{0}\text{-independent tuple over } \mathscr{M}.\end{array}$$

Then there exists a theory TS containing T_S such that

- *every model of T_S has a strong extension which is a model of TS;*
- *if $(\mathscr{M}, \mathscr{M}_0) \models TS$ and $(\mathscr{N}, \mathscr{N}_0) \models T_S$ is a strong extension of $(\mathscr{M}, \mathscr{M}_0)$ then $(\mathscr{M}, \mathscr{M}_0)$ is existentially closed in $(\mathscr{N}, \mathscr{N}_0)$.*

An axiomatization of TS *is given by adding to* T_S *the following axiom scheme: for each tuple of variables* $x = x^0x^1$*, for* $\mathscr{L}$*-formula* $\phi(x, y)$*, and* $\mathscr{L}_0$*-formulae* $(\tau_i(t, x, y))_{i<k}$ *which are algebraic in* t *and strict in* x^1*,*

$$\forall y(\theta_\phi(y) \to (\exists x\phi(x, y) \wedge x^0 \subseteq S \wedge \bigwedge_{i<k} \forall t\ (\tau_i(t, x, y) \to t \notin S))).$$

Proof. We prove the first assertion. Let $(\mathscr{M}, \mathscr{M}_0)$ be a model of T_S, $\phi(x, y)$ an $\mathscr{L}$-formula and a partition $x = x^0x^1$. Assume that for some tuple b from $\mathscr{M}$ we have $\theta_\phi(b)$. We show that the conclusion of the axiom can be satisfied in a strong extension $(\mathscr{N}, \mathscr{N}_0)$ with $\mathscr{N} \succ \mathscr{M}$. Then the result will follow by taking the union of a chain of models of T_S, which is again a model of T_S because it is an elementary chain of models of T with a predicate for models of T_0 which is inductive, by model-completeness. The fact that the union of a chain of strong extensions is again strong follows from Finite Character and Transitivity of $\mathop{\smile\hspace{-0.9em}\mid}^{0}$, and the model-completeness of T_0.

By (H_4) there exists an extension $\mathscr{N} \succ \mathscr{M}$, and a tuple $a \in \mathscr{N}$ satisfying $\phi(x, b)$ and such that a is $\mathop{\smile\hspace{-0.9em}\mid}^{0}$-independent over $\mathscr{M}$. Set $\mathscr{N}_0 = \mathrm{acl}_0(\mathscr{M}_0a^0)$. Then using Monotonicity, Base Monotonicity and Closure of $\mathop{\smile\hspace{-0.9em}\mid}^{0}$, $a^0\mathscr{M}_0 \mathop{\smile\hspace{-0.9em}\mid}^{0}_{\mathscr{M}_0} \mathscr{M}$. This means that the extension $(\mathscr{M}, \mathscr{M}_0) \subseteq (\mathscr{N}, \mathscr{N}_0)$ is strong. Now clearly $a^0 \subseteq S$. Using Base Monotonicity and Closure, it follows that $ab \mathop{\smile\hspace{-0.9em}\mid}^{0}_{a^0b} \mathscr{M}_0a^0$. Take any $\mathscr{L}_0$-formula $\tau(t, x, y)$ algebraic in t and strict in x^1, and assume that $u \in \mathscr{N}$ satisfies $\tau(t, a, b)$. As τ is strict in x^1 and a^1 is $\mathop{\smile\hspace{-0.9em}\mid}^{0}$ -independent over ba^0, we have $u \in \mathrm{acl}_0(ab) \setminus \mathrm{acl}_0(a^0b)$. If $u \in \mathscr{N}_0$ then it belongs to $\mathrm{acl}_0(ab) \cap \mathrm{acl}_0(\mathscr{M}_0a^0) \subseteq \mathrm{acl}_0(a^0b)$, a contradiction, hence $u \notin S$. It follows that $(\mathscr{N}, \mathscr{N}_0) \models \phi(a, b) \wedge a^0 \subseteq S \wedge \bigwedge_{i<k} \forall t\ (\tau_i(t, a, b) \to t \notin S)))$.

We now prove the second assertion.

Let $(\mathscr{M}, \mathscr{M}_0) \models TS$ and $(\mathscr{N}, \mathscr{N}_0) \models T_S$, a strong extension of $(\mathscr{M}, \mathscr{M}_0)$. Take finite tuples $a \in \mathscr{N}$ and $b \in \mathscr{M}$. To understand the quantifier-free $\mathscr{L}_S$-type of a over b, it is sufficient to deal with formulae of the form

$$\psi(x, b) \wedge \bigwedge_{i\in I} x_i \in S \wedge \bigwedge_{j\in J} x_j \notin S$$

with $\psi(x, y)$ an $\mathscr{L}$-formula. The reduction to formulae of this form is done by increasing the length of x (replacing $\mathscr{L}$-terms by variables), which may be greater than $|a|$. We assume that a satisfies the formula above.

Claim. There exists an $\mathop{\smile\hspace{-0.9em}\mid}^{0}$-independent tuple $a' = a^{0\prime}a^{1\prime}$ such that $\mathrm{acl}_0(\mathscr{M}a) = \mathrm{acl}_0(\mathscr{M}a')$ with

(1) $a' \mathop{\smile\hspace{-0.9em}\mid}^{0} \mathscr{M}$;

(2) $\mathrm{acl}_0(a') \cap \mathscr{N}_0 = \mathrm{acl}_0(a^{0\prime})$;

(3) $\mathscr{N}_0 \cap \mathrm{acl}_0(\mathscr{M}, a') = \mathrm{acl}_0(\mathscr{M}_0, a^{0\prime})$;

Proof of the claim. Take a tuple $a^{0\prime}$ in $\mathscr{N}_0 \cap \mathrm{acl}_0(\mathscr{M}, a)$ maximal $\mathop{\smile\hspace{-0.9em}\mid}^{0}$-independent over $\mathscr{M}_0$. We have $a^{0\prime} \mathop{\smile\hspace{-0.9em}\mid}^{0} \mathscr{M}_0$, and as the extension is strong we also have $a^{0\prime} \mathop{\smile\hspace{-0.9em}\mid}^{0} \mathscr{M}$ by Transitivity. Now take a tuple $a^{1\prime}$ in $\mathrm{acl}_0(\mathscr{M}a)$ maximal $\mathop{\smile\hspace{-0.9em}\mid}^{0}$-independent over $\mathrm{acl}_0(\mathscr{M}a^{0\prime})$. We have $a^{1\prime} \mathop{\smile\hspace{-0.9em}\mid}^{0} \mathscr{M}a^{0\prime}$ and so $a^{0\prime}a^{1\prime} \mathop{\smile\hspace{-0.9em}\mid}^{0} \mathscr{M}$. Set $a' = a^{0\prime}a^{1\prime}$ and the claim holds. □

Now as $a \subseteq \mathrm{acl}_0(\mathscr{M}, a')$ there exists a finite tuple m^1 from $\mathscr{M}$ $\mathop{\smile\hspace{-0.9em}\mid}^{0}$-independent over $\mathscr{M}_0a'$ such that $a \subseteq \mathrm{acl}_0(\mathscr{M}_0m^1a')$. Similarly there exists a finite tuple m^0 from $\mathscr{M}_0$ with $m^0 \mathop{\smile\hspace{-0.9em}\mid}^{0} m^1a'$ such that $a \subseteq \mathrm{acl}_0(m^0m^1a')$.

If $i \in I$, using *(3)*, we have $a_i \in \mathrm{acl}_0(\mathscr{M}_0a^{0\prime}) \cap \mathrm{acl}_0(m^0m^1a') = \mathrm{acl}_0(m^0a^{0\prime})$. Hence there is an $\mathscr{L}_0$-formula $\tau_i(t, a^{0\prime}, m^0)$ algebraic in t such that $a_i \models \tau_i(t, a^{0\prime}, m^0)$.

Let J_1 be the set of indices $j \in J$ such that $a_j \in \operatorname{acl}_0(a^{0\prime}, m^0, m^1)$. As $a_j \notin S$, by Lemma 2.1.3 there is an $\mathscr{L}_0$-formula $\tau_j(t, x^0, y, z)$ algebraic in t and strict in z such that $a_j \models \tau_j(t, a^{0\prime}, m^0, m^1)$.

Let $J_2 = J \setminus J_1$. Then for $j \in J_2$, we have $a_j \notin \operatorname{acl}_0(a^{0\prime}, m^0, m^1)$ so there is an $\mathscr{L}_0$-formula $\tau_j(t, x^0, x^1, y, z)$ algebraic in t and strict in x^1 such that $a_j \models \tau_j(t, a^{0\prime}, a^{1\prime}, m^0, m^1)$.

We now set $b' = bm^0m^1$ and set $\phi(a', b')$ to be the following formula

$$\begin{aligned} \exists v \psi(v, b) \quad & \wedge \quad \bigwedge_{i \in I} \tau_i(v_i, a^{0\prime}, m^0) \\ & \wedge \quad \bigwedge_{j \in J_1} \tau_j(v_j, a^{0\prime}, m^0, m^1) \\ & \wedge \quad \bigwedge_{j \in J_2} \tau_j(v_j, a^{0\prime}, a^{1\prime}, m^0, m^1) \end{aligned}$$

Step $(\star)$. By model-completeness we have that $\mathscr{N} \succ \mathscr{M}$. As a' is $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{0}$ independent over $\mathscr{M}$ it follows that $\mathscr{M} \models \theta_\phi(b')$. Using one instance of the axiom scheme, there exists $d' \in \mathscr{M}$ such that $d' \models \phi(x, b')$ with $d^{0\prime} \subseteq \mathscr{M}_0$ and for all $j \in J_2$, all the realizations of $\tau_j(t, d', m)$ are not in $\mathscr{M}_0$. Let d be the tuple whose existence is stated in $\phi(d', b')$, in particular $\mathscr{M} \models \psi(d, b)$. For $i \in I$, we have $d_i \in \operatorname{acl}_0(d^{0\prime} m^0) \subseteq \mathscr{M}_0$. For $j \in J_2$ we already saw that $d_j \notin \mathscr{M}_0$. For $j \in J_1$, as $\tau_j(t, d^{0\prime}, m^0, m^1)$ is strict in the variable of m^1 and m^1 is $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{0}$-independent over $\mathscr{M}_0$, we have that $d_j \notin \operatorname{acl}_0(d^{0\prime}, m^0)$. Recall that $m^1 \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{0} \mathscr{M}_0$, so $m^1 \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{0}_{d^{0\prime}, m^0} \mathscr{M}_0$ hence $\operatorname{acl}_0(d^{0\prime}, m^0, m^1) \cap \mathscr{M}_0 = \operatorname{acl}_0(d^{0\prime}, m^0)$, so d_j cannot belong to $\mathscr{M}_0$. We conclude that

$$(\mathscr{M}, \mathscr{M}_0) \models \psi(d, b) \wedge \bigwedge_{i \in I} d_i \in S \wedge \bigwedge_{j \in J} d_j \notin S$$

which proves that $(\mathscr{M}, \mathscr{M}_0)$ is existentially closed in $(\mathscr{N}, \mathscr{N}_0)$. □

Remark 2.1.6. Notice that if we consider $\mathscr{L}_0 = \{=\}$, the previous Theorem gives nothing more than the generic predicate (see[CP98]). The hypothesis (H_4) becomes equivalent to elimination of $\exists^\infty$ in that case. Note also that if T_0 is strongly minimal and has quantifier elimination in $\mathscr{L}_0$, the conditions (H_2) and (H_3) are satisfied.

We can forget hypothesis (H_1) to get this adapted version of Theorem 2.1.5.

Proposition 2.1.7. *Assume that the following holds.*

(H_2) *T_0 is model complete and for all A infinite, $\operatorname{acl}_0(A) \models T_0$;*

(H_3) *T_0 is pregeometric;*

(H_4) *for all $\mathscr{L}$-formula $\phi(x, y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for $b \in \mathscr{M} \models T$*

$$\mathscr{M} \models \theta_\phi(b) \iff \begin{array}{l} \textit{there exists } \mathscr{N} \succ \mathscr{M} \textit{ and } a \in \mathscr{N} \textit{ such that} \\ \phi(a, b) \textit{ and } a \textit{ is an } \mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}^{0}\textit{-independent tuple over } \mathscr{M} \end{array}$$

Then there exists a theory TS containing T_S such that

- *every model $(\mathscr{M}, \mathscr{M}_0)$ of T_S has a strong extension $(\mathscr{M}', \mathscr{M}'_0)$ which is a model of TS, such that $\mathscr{M} \prec \mathscr{M}'$;*
- *assume that $(\mathscr{M}, \mathscr{M}_0) \models TS$ and $(\mathscr{N}, \mathscr{N}_0)$ is a model of T_S which is a strong extension of $(\mathscr{M}, \mathscr{M}_0)$. If $\mathscr{M}$ is existentially closed in $\mathscr{N}$ then $(\mathscr{M}, \mathscr{M}_0)$ is existentially closed in $(\mathscr{N}, \mathscr{N}_0)$.*

An axiomatization of TS is given by adding to T_S the following axioms, for each tuple of variables $x = x^0x^1$, for $\mathscr{L}$-formula $\phi(x, y)$, and $\mathscr{L}_0$-formulae $(\tau_i(t, x, y))_{i<k}$ which are algebraic in t and strict in x^1,

$$\forall y(\theta_\phi(y) \to (\exists x\phi(x, y) \wedge x^0 \subseteq S \wedge \bigwedge_{i<k} \forall t\ (\tau_i(t, x, y) \to t \notin S))).$$

Proof. The same proof as for Theorem 2.1.5 works. In the proof of Theorem 2.1.5, the model-completeness of T was used to ensure that given any model $\mathscr{N}$ of T extending $\mathscr{M}$, then $\mathscr{M}$ is existentially closed in $\mathscr{N}$, which is now part of the second bullet. In the first bullet, the model $\mathscr{M}'$ of T extending $\mathscr{M}$ is the union of an elementary chain of extensions hence is an elementary extension of $\mathscr{M}$, this condition does not use the model-completeness of T. □

Remark 2.1.8. Assume that T, T_0 satisfies (H_1), (H_2) and (H_3). Assume that there is a class $\mathscr{C}$ of $\mathscr{L}$-formula such that for all $\mathscr{M} \models T$, for all $\mathscr{L}$-formula $\phi(x, b)$ with parameters in $\mathscr{M}$, there exists a tuple c from $\mathscr{M}$ and formulae $\vartheta_1(x, z), \cdots, \vartheta_n(x, z) \in \mathscr{C}$ such that

$$\phi(\mathscr{M}, b) = \vartheta_1(\mathscr{M}, c) \cup \cdots \cup \vartheta_n(\mathscr{M}, c).$$

Assume that condition (H_4) holds only for formulae $\vartheta(x, z) \in \mathscr{C}$. Then the conclusion of Theorem 2.1.5 applies, with the axiom-scheme restricted to formulae in $\mathscr{C}$. It is clear that the proof of the first assertion works similarly, considering only formulae in $\mathscr{C}$. For the second assertion, the proof changes at Step $(\star)$, we need to show that there exists a realisation of $\phi(x, b')$ that satisfies the right properties using the axioms. By assumption $\phi(\mathscr{M}, b') = \vartheta_1(\mathscr{M}, c) \cup \cdots \cup \vartheta_n(\mathscr{M}, c)$ for some $\vartheta_1(x, z), \cdots, \vartheta_n(x, z) \in \mathscr{C}$ and tuple c from $\mathscr{M}$. This decomposition holds also in $\mathscr{N}$ by model-completeness of T. Now as $a' \models \phi(x, b')$, there is some $i \leq n$ such that $a' \models \vartheta_i(x, c)$ hence $\mathscr{M} \models \theta_{\vartheta_i}(c)$. Using one instance of the axiom, there exists d' in $\mathscr{M}$ satisfying $\vartheta_i(x, c)$, hence also $\phi(x, b')$, and that satisfies the right properties, and the end of the proof is similar. The main example for the class $\mathscr{C}$ is the class of quasi-affine varieties in the theory ACF, see Theorem 3.3.5.

2.2 A weak converse

In this subsection, Lemmas 2.2.2 and 2.2.3 give some insight on the condition (H_4), and Proposition 2.2.4 gives a weak converse statement for the existence of TS.

In this section, we assume that T and T_0 satisfies the following conditions:

(H_1) T is model-complete;

(H_2^-) T_0 is model-complete;

(H_3) T_0 is pregeometric;

Given two tuples of variables x and y, the condition "x is $\mathop{\smile\!\!\!\!\!\!\mid}^0$-independent over $\mathrm{acl}_T(y)$" is type-definable, it is given by the set of formulae of the form

$$\forall t_1, \cdots, t_n \left(\bigwedge_{i=1}^{n} \psi_i(t_i, y) \to \bigwedge_{k=1}^{|x|} \neg\tau_k(x_k, t_1, \cdots, t_n, x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{|x|})\right)$$

for all $n \in \mathbb{N}$, $\psi_i(t, y)$ $\mathscr{L}$-formula algebraic in t and $\tau_j(t, t_1, \cdots, t_n, z)$ $\mathscr{L}_0$-formula algebraic in t with $|z| = |x| - 1$. As algebraic formulae are closed under finite disjunction and conjunction, it is clear that the previous type is equivalent to the set of all formulae of the form

$$\forall t_1, \cdots, t_n \left(\bigwedge_{i=1}^{n} \psi(t_i, y) \to \bigwedge_{k=1}^{|x|} \neg\tau(x_k, t_1, \cdots, t_n, x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{|x|})\right)$$

for all $n \in \mathbb{N}$, $\psi(t, y)$ $\mathscr{L}$-formula algebraic in t and $\tau(t, t_1, \cdots, t_n, z)$ $\mathscr{L}_0$-formula algebraic in t with $|z| = |x| - 1$. We call this type $\Sigma(x, y)$.

We work in a monster model $\mathbb{M}$ of T.

Lemma 2.2.1. *For all A, B, C acl_T-closed small sets in a monster model, then there exists $A' \equiv^T_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!\mid}^0_C B$.*

Proof. The lemma follows from Fact 1.3.9, take q to be the type of an $\mathop{\smile\!\!\!\!\!\!\mid}^0$ basis of A over C. Note that we only use hypothesis (H_3) here. ☐

Lemma 2.2.2. *Let $\phi(x, y)$ be an $\mathscr{L}$-formula, $\mathscr{M}$ an $\aleph_0$-saturated small model of T and b a $|y|$-tuple from $\mathscr{M}$. The following are equivalent:*

(1) there exists $\mathscr{N} \succ \mathscr{M}$ and some realisation a of $\phi(x, b)$ in $\mathscr{N}$ such that a is an $\mathop{\smile\!\!\!\!\!\!\mid}^0$-independent tuple over $\mathscr{M}$;

(2) there exists some realisation a of $\phi(x, b)$ in $\mathscr{M}$ such that a is $\mathop{\smile\!\!\!\!\!\!\mid}^0$-independent over $\mathrm{acl}_T(b)$.

Proof. (1) implies *(2)*. Let $\Sigma(x, b)$ be the partial type over b expressing that "x is an $\mathop{\smile\!\!\!\!\!\!\mid}^0$-independent tuple over $\mathrm{acl}_T(b)$". By *(1)*, $\Sigma(x, b)$ is finitely satisfiable in $\mathscr{M}$ hence by saturation it is realised in $\mathscr{M}$.
(2) implies *(1)*. Using Lemma 2.2.1, there exists $a' \equiv^T_b a$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathrm{acl}_T(b)} \mathscr{M}$. Using Transitivity a' is $\mathop{\smile\!\!\!\!\!\!\mid}^0$-independent over $\mathscr{M}$. For any $\mathscr{N}$ containing a', the condition *(2)* holds. ☐

Lemma 2.2.3. *Let $\phi(x, y)$ be some $\mathscr{L}$-formula. The following are equivalent.*

(1) There exists a formula $\theta_\phi(y)$ such that $\theta_\phi(b)$ holds if and only if there exists some realisation a of $\phi(x, b)$ such that a is $\mathop{\smile\!\!\!\!\!\!\mid}^0$-independent over $\mathrm{acl}_T(b)$.

(2) There exists $n \in \mathbb{N}$, an $\mathscr{L}$-formula $\psi(t, y)$ algebraic in t and an $\mathscr{L}_0$-formula $\tau(t, t_1, \ldots, t_n, z)$ algebraic in t with $|z| = |x| - 1$ such that for all b, if some realisation a of $\phi(x, b)$ is not an $\mathop{\smile\!\!\!\!\!\!\mid}^0$-independent tuple over $\mathrm{acl}_T(b)$ then there exist n realizations $c_1, \ldots, c_n$ of $\psi(t, b)$ such that for some $1 \leq k \leq |x|$, we have that a_k satisfies $\tau(t, c_1, \cdots, c_n, a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_{|x|})$.

Proof. Recall that $\Sigma(x, y)$ is the set of all formula of the form

$$\phi(x, y) \wedge \forall t_1, \cdots, t_n \left(\bigwedge_{i=1}^{n} \psi(t_i, y) \rightarrow \bigwedge_{k=1}^{|x|} \neg\tau(x_k, t_1, \cdots, t_n, x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{|x|}) \right)$$

for all $n \in \mathbb{N}$, $\psi(t, y)$ $\mathscr{L}$-formula algebraic in t and $\tau(t, t_1, \cdots, t_n, z)$ $\mathscr{L}_0$-formula algebraic in t with $|z| = |x| - 1$. Let $\Sigma(y)$ be the (consistent) partial type $\{\exists x \Gamma(x, y) \mid \Gamma(x, y) \in \Sigma\}$. By compactness, if $\theta_\phi(y)$ exists, it is equivalent to a finite fragment of $\Sigma(y)$, hence to a single formula in $\Sigma(y)$. The existence of $\theta_\phi(y)$ is equivalent to the existence of a bound $n \in \mathbb{N}$, an $\mathscr{L}$-formula $\psi(t, y)$ algebraic in t and an $\mathscr{L}_0$-formula $\tau(t, t_1, \cdots, t_n, z)$ for $|z| = |x| - 1$ such that for all b if a realizes $\phi(x, b)$, a is not $\mathop{\smile\!\!\!\!\!\!\mid}^0$-independent over $\mathrm{acl}_T(b)$ (if and) only if there are n realisations $c_1, \ldots, c_n \in \mathrm{acl}_T(b)$ of $\psi(t, b)$ such that for some $1 \leq k \leq |x|$, a_k is in $\mathrm{acl}_0(c_1, \cdots, c_n, a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_{|x|})$, witnessed by τ. ☐

Proposition 2.2.4. *Assume that there exists a theory TS such that*

- *every model of T_S has a strong extension which is a model of TS;*
- *if $(\mathscr{M}, \mathscr{M}_0) \models TS$ and $(\mathscr{N}, \mathscr{N}_0) \models T_S$ is a strong extension of $(\mathscr{M}, \mathscr{M}_0)$ then $(\mathscr{M}, \mathscr{M}_0)$ is existentially closed in $(\mathscr{N}, \mathscr{N}_0)$.*

Then the following holds:

for all $\mathscr{L}$-formula $\phi(x,y)$ and all $1 \leq k \leq |x|$, there exists an $\mathscr{L}$-formula $\theta^k_\phi(y)$ such that for all tuple b in an $\aleph_0$-saturated model $\mathscr{M}$ of T,

$$\mathscr{M} \models \theta^k_\phi(b) \iff \begin{array}{l}\text{there exists some realisation } a \text{ of } \phi(x,b) \text{ in } \mathscr{M} \text{ such that}\\ a_k \notin \operatorname{acl}_0(\operatorname{acl}_T(b), a_1, \dots, a_{k-1}, a_{k+1}, \cdots, a_{|x|}).\end{array}$$

Proof. Given a single variable t and some tuple of variables y, we denote by $\mathscr{A}_{\mathscr{L}}(t,y)$ the set of all $\mathscr{L}$-formulae without parameters that are algebraic in t with free variables (other than t) in y. Assume that the conclusion doesn't hold. Similarly to Lemma 2.2.3 there is some formula $\phi(x,y)$, some $1 \leq k \leq |x|$ and an $\aleph_0$-saturated model $\mathscr{M}$ of T such that for all $n \in \mathbb{N}$, for all $\psi(t,y) \in \mathscr{A}_{\mathscr{L}}(t,y)$ and $\tau(t,t_1,\cdots,t_n,z) \in \mathscr{A}_{\mathscr{L}_0}(t,t_1,\cdots,t_n,z)$ (with $|z| = |x|-1$) there is some $b = b(n,\psi,\tau)$ and a realisation $a = a(n,\psi,\tau)$ of $\phi(x,b)$ in $\mathscr{M}$ such that $a_k \in \operatorname{acl}_0(\operatorname{acl}_T(b), a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_{|x|})$ and for all realisations $c_1, \dots, c_n$ of $\psi(t,b)$ and all k

$$\mathscr{M} \models \neg\tau(a_k, c_1, \cdots, c_n, a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_{|x|}).$$

For convenience, we assume that $k=1$. We may assume that for all n, ψ and τ, all realisations of $\phi(x_1, a_{>1}, b)$ in $\mathscr{M}$ are in $\operatorname{acl}_0(\operatorname{acl}_T(b), a_{>1})$. Otherwise, for some n, ψ, τ as above, the formula

$$\exists x \phi(x,y) \wedge \forall t_1, \dots, t_n \left(\bigwedge_{i=1}^n \psi(t_i, y) \to \neg\tau(x_1, t_1, \cdots, t_n, x_{>1}) \right)$$

would isolate the type $\exists x\phi(x,y) \wedge$ "$x_1 \notin \operatorname{acl}_0(\operatorname{acl}_T(y), x_{>1})$" which contradicts the hypotheses. By $\aleph_0$-saturation, as $\phi(\mathscr{M}, a_{>1}, b) \subseteq \operatorname{acl}_0(\operatorname{acl}_T(b), a_{>1})$, we have that $\phi(\mathscr{M}, a_{>1}, b)$ is finite, for all (n,ψ,τ).

We define the following subset of $\mathbb{N} \times \mathscr{A}_{\mathscr{L}}(t,y) \times \left(\bigcup_{n\in\mathbb{N}} \mathscr{A}_{\mathscr{L}_0}(t,t_1,\dots,t_n,z)\right)$

$$I = \{(n,\psi,\tau) \mid n \in \mathbb{N}, \psi \in \mathscr{A}_{\mathscr{L}}(t,y), \tau \in \mathscr{A}_{\mathscr{L}_0}(t,t_1,\cdots,t_n,z)\}$$

By assumptions, $\mathscr{M}$ contains $\{a_{>1}b \mid (n,\psi,\tau) \in I\}$. We expand $\mathscr{M}$ to a model of T_S by setting $S(\mathscr{M}) = \mathscr{M}$. By hypothesis, there exists a model $(\mathscr{N}, \mathscr{N}_0)$ of TS which is a strong extension of $(\mathscr{M}, \mathscr{M})$. As T is model-complete, each realisation a_1' in $\mathscr{N}$ of $\phi(x_1, a_{>1}b)$ is still in $\operatorname{acl}_0(\operatorname{acl}_T(b), a_{>1})$, hence $\phi(\mathscr{N}, a_{>1}, b) \subset S$. Furthermore, for all $(n,\psi,\tau) \in I$, the following holds in $(\mathscr{N}, \mathscr{N}_0)$

$$\exists x_1 \phi(x_1, a_{>1}b) \wedge \forall t_1, \dots, t_n \left(\bigwedge_{i=1}^n \psi(t_i, b) \to \neg\tau(x_1, t_1, \cdots, t_n, x_{>1}) \right).$$

Let $\mathscr{U}$ be a nonprincipal ultrafilter on I and consider the ultrapower $(\mathscr{N}, \mathscr{N}_0)^{\mathscr{U}}$ of $(\mathscr{N}, \mathscr{N}_0)$, which is also a model of TS. For $\overline{a_{>1}b}$ the class of $(a_{>1}b(n,\psi,\tau))_{(n,\psi,\tau)\in I})$ in $(\mathscr{N}, \mathscr{N}_0)^{\mathscr{U}}$, every realisation of $\phi(x_1, \overline{a_{>1}b})$ in $(\mathscr{N}, \mathscr{N}_0)^{\mathscr{U}}$ is in S. On the other hand, the partial type consisting of all formulae of the form

$$\exists x_1 \phi(x_1, \overline{a_{>1}b}) \wedge \forall t_1, \dots, t_n \left(\bigwedge_{i=1}^n \psi(t_i, \bar{b}) \to \neg\tau(x_1, t_1, \cdots, t_n, \overline{a_{>1}}) \right)$$

for $(n,\psi,\tau) \in I$, is consistent. Hence there exists a realisation $\tilde{a}_1$ of $\phi(x_1, \overline{a_{>1}b})$ in $\mathscr{N}^{\mathscr{U}}$ which is not in $\operatorname{acl}_0(\operatorname{acl}_T(\bar{b})\overline{a_{>1}})$. By Lemma 2.2.1, there exists singleton $\tilde{a}_1'$ in some elementary extension $\mathscr{K}$ of $\mathscr{N}^{\mathscr{U}}$ such that $\tilde{a}_1' \equiv^T_{\operatorname{acl}_T(\bar{b})\overline{a_{>1}}} \tilde{a}_1$ and $\tilde{a}_1' \mathop{\smile\hspace{-0.9em}\mid}^{0}_{\operatorname{acl}_T(\bar{b})\overline{a_{>1}}} \mathscr{N}^{\mathscr{U}}$. Now $\tilde{a}_1' \notin \operatorname{acl}_0(\operatorname{acl}_T(\bar{b})\overline{a_{>1}})$ implies that $\tilde{a}_1' \mathop{\smile\hspace{-0.9em}\mid}^{0} \operatorname{acl}_T(\bar{b})\overline{a_{>1}}$, so by TRANSITIVITY $\tilde{a}_1' \notin \mathscr{N}^{\mathscr{U}}$. Finally observe that $(\mathscr{K}, S(\mathscr{N}^{\mathscr{U}}))$ is a strong extension of $(\mathscr{N}^{\mathscr{U}}, S(\mathscr{N}^{\mathscr{U}}))$, hence $(\mathscr{N}^{\mathscr{U}}, S(\mathscr{N}^{\mathscr{U}}))$ is existentially closed in $(\mathscr{K}, S(\mathscr{N}^{\mathscr{U}}))$, but

$$(\mathscr{K}, S(\mathscr{N}^{\mathscr{U}})) \models \exists x_1 \phi(x_1, \overline{a_{>1}b}) \wedge x_1 \notin S$$

a contradiction. $\square$

Remark 2.2.5. A consequence of Proposition 2.2.4 is that if TS exists, then T eliminates $\exists^\infty$. A question one might ask is wether it is a sufficient condition for the existence of the theory TS. The answer is no, the theory ACF_0 eliminates $\exists^\infty$ but the model companion of the theory of algebraically closed fields of characteristic 0 with a predicate for an additive subgroup is not first order axiomatisable, see Proposition 3.2.7. On the other hand, the existence of TS under the reduction of the hypothesis (H_4) to formulae $\phi(x,y)$ with $|x| = 2$ would be a good improvement, as it would be much easier to check.

2.3 Suitable triple

In Sections 2.1 and 2.2, we have listed minimal hypotheses in order to have (weakly) necessary and sufficient conditions for the existence of a generic theory TS. We now consider a stronger assumption on T_0 which encompass the conditions of Sections 2.1 and 2.2: the *modularity* of the pregeometry in T_0. This hypothesis make obsolete the notion of *strong* extension. As a consequence, the theory TS becomes the *model-companion* of the theory T_S.

Definition 2.3.1. We say that a triple $(T, T_0, \mathscr{L}_0)$ is *suitable* if it satisfies the following

(H_1) T is model complete;

(H_2) T_0 is model complete and for all infinite A, $\mathrm{acl}_0(A) \models T_0$;

(H_3^+) acl_0 defines a modular pregeometry;

(H_4) for all $\mathscr{L}$-formula $\phi(x,y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for $b \in \mathscr{M} \models T$

$$\mathscr{M} \models \theta_\phi(b) \iff \begin{array}{l}\text{there exists } \mathscr{N} \succ \mathscr{M} \text{ and } a \in \mathscr{N} \text{ such that} \\ \phi(a,b) \text{ and } a \text{ is } \mathop{\smile\mkern-10mu|}^{0}\text{-independent over } \mathscr{M}.\end{array}$$

Remark 2.3.2. Let $(T, T_0, \mathscr{L}_0)$ be a suitable triple. By Fact 1.2.3, in T, the relation $\mathop{\smile\mkern-10mu|}^{a}$ defined by $A \mathop{\smile\mkern-10mu|}^{a}_{C} B$ if and only if $\mathrm{acl}_T(AC) \cap \mathrm{acl}_T(BC) = \mathrm{acl}_T(C)$ satisfies FULL EXISTENCE, so for all A, B, C subsets of $\mathbb{M}$ there exists $A' \equiv^T_C A$ such that $\mathrm{acl}_T(A'C) \cap \mathrm{acl}_T(BC) = \mathrm{acl}_T(C)$. As acl_0 is modular, it follows that $\mathrm{acl}_T(A'C) \mathop{\smile\mkern-10mu|}^{0}_{\mathrm{acl}_T(C)} \mathrm{acl}_T(BC)$, this gives another proof of Lemma 2.2.1 in that context.

From Section 2.1, we immediately get the following.

Proposition 2.3.3. *Let $(T, T_0, \mathscr{L}_0)$ be a suitable triple. Then TS exists and is the model-companion of the theory T_S.*

Lemma 2.3.4. *Let $(\mathscr{M}, \mathscr{M}_0)$ and $(\mathscr{N}, \mathscr{N}_0)$ are two models of T_S, such that $\mathscr{M}_0 \mathop{\smile\mkern-10mu|}^{0}_{\mathscr{N}_0} \mathscr{N}$ and $\mathscr{N}_0 \mathop{\smile\mkern-10mu|}^{0}_{\mathscr{M}_0} \mathscr{M}$. Then, there exists a model $(\mathscr{K}, \mathscr{K}_0)$ of TS extending both $(\mathscr{M}, \mathscr{M}_0)$ and $(\mathscr{N}, \mathscr{N}_0)$. If furthermore $(\mathscr{M}, \mathscr{M}_0)$ and $(\mathscr{N}, \mathscr{N}_0)$ are models of TS, then $(\mathscr{K}, \mathscr{K}_0)$ is an elementary extension of both $(\mathscr{M}, \mathscr{M}_0)$ and $(\mathscr{N}, \mathscr{N}_0)$.*

Proof. Let $\mathscr{K}'$ be a model of T extending $\mathscr{M}$ and $\mathscr{N}$. Now set $\mathscr{K}_0' = \mathrm{acl}_0(\mathscr{M}_0, \mathscr{N}_0)$. Clearly $(\mathscr{K}', \mathscr{K}_0')$ is a model of T_S. By hypothesis we have $\mathscr{K}_0' \mathop{\smile\mkern-10mu|}^{0}_{\mathscr{M}_0} \mathscr{M}$ and $\mathscr{K}_0' \mathop{\smile\mkern-10mu|}^{0}_{\mathscr{N}_0} \mathscr{N}$. Using Theorem 2.1.5, there is a model $(\mathscr{K}, \mathscr{K}_0)$ of TS extending $(\mathscr{K}', \mathscr{K}_0')$, $(\mathscr{M}, \mathscr{M}_0)$ and $(\mathscr{N}, \mathscr{N}_0)$. We conclude by model-completeness. ☐

Proposition 2.3.5. *Let $(T, T_0, \mathscr{L}_0)$ be an adapted triple.*

(1) *Let* $(\mathscr{M},\mathscr{M}_0)$ *and* $(\mathscr{N},\mathscr{N}_0)$ *be two models of* TS *and* A *be a common subset of* $\mathscr{M}$ *and* $\mathscr{N}$. *Then we have*

$$(\mathscr{M},\mathscr{M}_0) \equiv^{TS}_A (\mathscr{N},\mathscr{N}_0) \quad\iff\quad \begin{array}{l} \textit{there exists } f : \mathrm{acl}_T(A) \to \mathrm{acl}_T(A) \\ T\textit{-elementary bijection over } A, \\ \textit{such that } f(\mathscr{M}_0 \cap \mathrm{acl}_T(A)) = \mathscr{N}_0 \cap \mathrm{acl}_T(A). \end{array}$$

(2) *For any* a, b, A *in a monster model of* TS

$$a \equiv^{TS}_A b \quad\iff\quad \begin{array}{l} \textit{there exists } f : \mathrm{acl}_T(Aa) \to \mathrm{acl}_T(Ab) \\ \textit{a } T\textit{-elementary bijection over } A \textit{ with } f(a) = b, \\ \textit{such that } f(S(\mathrm{acl}_T(Aa))) = S(\mathrm{acl}_T(Ab)).. \end{array}$$

We call such a function a T-elementary $\mathscr{L}_S$-isomorphism between $(\mathrm{acl}_T(Aa), S(\mathrm{acl}_T(Aa))$ and $(\mathrm{acl}_T(Ab), S(\mathrm{acl}_T(Ab))$.

(3) *The completions of* TS *are given by the* T*-elementary* $\mathscr{L}_S$*-isomorphism types of*

$$(\mathrm{acl}_T(\emptyset), S(\mathrm{acl}_T(\emptyset))).$$

(4) *For all* A, $\mathrm{acl}_{TS}(A) = \mathrm{acl}_T(A)$.

Proof. *(1)* The left to right implication is standard. From right to left. Note that, under hypotheses, we may assume that $A = \mathrm{acl}_T(A)$ is a subset of both $\mathscr{M}$ and $\mathscr{N}$ and that $\mathscr{M}_0 \cap A = \mathscr{N}_0 \cap A$. By Lemma 2.2.1, there exists $\mathscr{M}' \equiv^T_A \mathscr{M}$ such that $\mathscr{M}' \mathop{\smile\hskip-0.9em ^| \ }^0_A \mathscr{N}$. There is an $\mathscr{L}$-isomorphism g between $\mathscr{M}'$ and $\mathscr{M}$ that fixes A, so we may define $\mathscr{M}_0' = g^{-1}(\mathscr{M}_0)$ and turn $(\mathscr{M}', \mathscr{M}_0')$ into a model of TS. By MONOTONICITY and BASE MONOTONICITY we have $\mathscr{M}_0' \mathop{\smile\hskip-0.9em ^| \ }^0_{\mathscr{N}_0} \mathscr{N}$. Similarly we have $\mathscr{N}_0 \mathop{\smile\hskip-0.9em ^| \ }^0_{\mathscr{M}_0'} \mathscr{M}'$ hence by Lemma 2.3.4 there exists a model $(\mathscr{K}, \mathscr{K}_0)$ of TS that is an elementary extension of both $(\mathscr{M}', \mathscr{M}_0')$ and $(\mathscr{N}, \mathscr{N}_0)$, hence $(\mathscr{M}', \mathscr{M}_0') \equiv^{TS}_A (\mathscr{K}, \mathscr{K}_0) \equiv^{TS}_A (\mathscr{N}, \mathscr{N}_0)$.

(2) This is similar to *(1)*.

(3) This is an obvious application of *(1)*.

(4) We only need to show that $\mathrm{acl}_{TS}(A) \subseteq \mathrm{acl}_T(A)$. Assume that $b \notin \mathrm{acl}_T(A)$. Let $(\mathscr{M}, \mathscr{M}_0)$ be a model of TS containing b. There exists a model $\mathscr{N}$ of T and a T-isomorphism $f : \mathscr{N} \to \mathscr{M}$ over A such that $\mathscr{N} \mathop{\smile\hskip-0.9em ^| \ }^0_{\mathrm{acl}_T(A)} \mathscr{M}$. Consider $\mathscr{N}_0 = f^{-1}(\mathscr{M}_0)$, then $(\mathscr{N}, \mathscr{N}_0)$ and $(\mathscr{M}, \mathscr{M}_0)$ are $\mathscr{L}_S$-isomorphic. Now set $b' = f^{-1}(b)$, we have $b' \equiv^{TS}_A b$ and $b \neq b'$ because $b \mathop{\smile\hskip-0.9em ^| \ }^0_{\mathrm{acl}_T(A)} b'$ and $b \notin \mathrm{acl}_T(A)$. Since $\mathscr{N} \mathop{\smile\hskip-0.9em ^| \ }^0_{\mathrm{acl}_T(A)} \mathscr{M}$, we may do as in *(1)* and find a model of TS extending both $\mathscr{M}$ and $\mathscr{N}$ in which the condition *(3)* is satisfied. Similarly we can produce as many conjugates of b over A as we want inside some bigger model so $b \notin \mathrm{acl}_{TS}(A)$. ☐

Proposition 2.3.6. *Let* $\mathbb{M}$ *be a monster model of* T. *Let* $\mathscr{M} \prec \mathbb{M}$ *and* $\mathscr{M}_0 \subseteq \mathscr{M}$ *such that* $(\mathscr{M}, \mathscr{M}_0)$ *is a model of* TS. *Let* $B \subset \mathscr{M}$, *and* X *a small subset of* $\mathbb{M}$. *Let* $S_{XB} \subseteq \mathrm{acl}_T(XB) \subset \mathbb{M}$ *be some* acl_0*-closed set containing* $S(\mathrm{acl}_T(B))$ *and such that:*

(1) $S_{XB} \cap \mathscr{M} = S(\mathrm{acl}_T(B))$

(2) $\mathrm{acl}_T(XB) \cap \mathscr{M} = \mathrm{acl}_T(B)$.

Then the type (over B*) associated to the* T*-elementary* $\mathscr{L}_S$*-isomorphism type of* $(\mathrm{acl}_T(XB), S_{XB})$ *is consistent in* $Th(\mathscr{M}, \mathscr{M}_0)$.

Proof. Let $\mathbb{M}_0' = \mathrm{acl}_0(\mathscr{M}_0, S_{XB})$. We have that $(\mathbb{M}, \mathbb{M}_0')$ is a model of T_S and an extension of $(\mathscr{M}, \mathscr{M}_0)$. Indeed, $\mathbb{M}_0' \cap \mathscr{M} = \mathrm{acl}_0(\mathscr{M}_0, S_{XB}) \cap \mathscr{M} = \mathrm{acl}_0(\mathscr{M}_0, S_{XB} \cap \mathscr{M})$ by modularity. By hypothesis *(1)*, $S_{XB} \cap \mathscr{M} = S(\mathrm{acl}_T(B) \subseteq \mathscr{M}_0$ hence $\mathbb{M}_0' \cap \mathscr{M} = \mathscr{M}_0$. By Theorem 2.1.5 there exists a model $(\mathscr{N}, \mathscr{N}_0)$ of TS extending $(\mathbb{M}, \mathbb{M}_0')$ which is an elementary extension of $(\mathscr{M}, \mathscr{M}_0)$. Now

$$\begin{aligned}
\mathrm{acl}_T(XB) \cap \mathscr{N}_0 &= \mathrm{acl}_T(XB) \cap \mathbb{M}_0 \\
&= \mathrm{acl}_T(XB) \cap \mathrm{acl}_0(\mathscr{M}_0, S_{XB}) \\
&= \mathrm{acl}_0(S_{XB}, \mathrm{acl}_T(XB) \cap \mathscr{M}_0) && \text{by modularity} \\
&= \mathrm{acl}_0(S_{XB}, \mathrm{acl}_T(B) \cap \mathscr{M}_0) && \text{by } (2) \\
&= \mathrm{acl}_0(S_{XB}, S(\mathrm{acl}_T(B))) \\
&= S_{XB}.
\end{aligned}$$

It follows that in $(\mathscr{N}, \mathscr{N}_0)$, $tp^{TS}(X/B)$ is given by the T-elementary $\mathscr{L}_S$-isomorphism type of $(\mathrm{acl}_T(XB), S_{XB})$. ☐

2.4 Iterating the construction

Let T be an $\mathscr{L}$-theory, $\mathscr{L}_1, \cdots, \mathscr{L}_n$ be sublanguages of $\mathscr{L}$ and let $T_i = T \restriction \mathscr{L}_i$. Let $S_1, \cdots, S_n$ be new unary predicate and let $\mathscr{L}_{S_1 \dots S_n}$ be the language $\mathscr{L} \cup \{S_1, \cdots, S_n\}$. Let $T_{S_1 \dots S_n}$ be the $\mathscr{L}_{S_1 \dots S_n}$-theory which models are models $\mathscr{M}$ of T in which $\mathscr{M}_i := S_i(\mathscr{M})$ is an $\mathscr{L}_i$-substructure of $\mathscr{M}$ and a model of T_i. The following give a condition for the existence of a model companion for $T_{S_1 \dots S_n}$.

Proposition 2.4.1. *Assume inductively that $(TS_1 \dots S_i, T_{i+1}, \mathscr{L}_{i+1})$ is a suitable triple for $i = 0, \cdots, n-1$, and let $TS_1 \dots S_{i+1}$ be the model companion of the theory $TS_1, \dots, S_{i S_{i+1}}$ of models of $TS_1, \dots, S_i$ with a predicate S_{i+1} for an $\mathscr{L}_{i+1}$ submodel of T_{i+1}. Then $TS_1 \cdots S_n$ is the model-companion of the theory $T_{S_1 \dots S_n}$.*

Proof. We show the following:

(1) every model $(\mathscr{M}, \mathscr{M}_1, \dots \mathscr{M}_n)$ of $T_{S_1 \dots S_n}$ can be extended to a model $(\mathscr{N}, \mathscr{N}_1, \cdots, \mathscr{N}_n)$ of $TS_1 \dots S_n$;

(2) every model $(\mathscr{N}, \mathscr{N}_1, \dots, \mathscr{N}_n)$ of $TS_1 \dots S_n$ is existentially closed in an extension $(\mathscr{M}, \mathscr{M}_1, \dots, \mathscr{M}_n)$ model of $T_{S_1, \dots S_n}$.

(1) Start by extending $(\mathscr{M}, \mathscr{M}_1)$ to a model $(\mathscr{N}^1, \mathscr{N}_1^1)$ of TS_1. Then $(\mathscr{N}^1, \mathscr{N}_1^1, \mathscr{M}_2)$ is a model of TS_{1S_2} so can be extended to a model $(\mathscr{N}^2, \mathscr{N}_1^2, \mathscr{N}_2^2)$ of TS_1S_2. The structure $(\mathscr{N}^2, \mathscr{N}_1^2, \mathscr{N}_2^2)$ is also an extension of $(\mathscr{M}, \mathscr{M}_1, \mathscr{M}_2)$. We iterate this process to end with a model $(\mathscr{N}^n, \mathscr{N}_1^n, \cdots, \mathscr{N}_n^n)$ of $TS_1 \cdots S_n$ extending $(\mathscr{M}, \mathscr{M}_1, \cdots, \mathscr{M}_n)$.
(2) Let $(\mathscr{N}, \mathscr{N}_1, \cdots, \mathscr{N}_n)$ be a model of $TS_1 \cdots S_n$ and $(\mathscr{M}, \mathscr{M}_1, \cdots, \mathscr{M}_n)$ be a model of $T_{S_1 \dots S_n}$ extending it. By (1) there exists a model $(\mathscr{M}', \mathscr{M}_1', \cdots, \mathscr{M}_n')$ of $TS_1 \cdots S_n$ extending $(\mathscr{M}, \mathscr{M}_1, \cdots, \mathscr{M}_n)$. As $(\mathscr{N}, \mathscr{N}_1, \cdots, \mathscr{N}_n)$ is a model of $TS_1 \cdots S_n$ it is existentially closed in any model of $TS_1 \cdots S_{n-1 S_n}$ extending it, in particular, it is existentially closed in $(\mathscr{M}', \mathscr{M}_1', \cdots, \mathscr{M}_n')$ and hence also in $(\mathscr{M}, \mathscr{M}_1, \dots, \mathscr{M}_n)$. ☐

In a model of $TS_1 \cdots S_n$, the relations between the S_i are very generic. For example, it is not possible that $S_i \subseteq S_j$ for some i, j, since one can always extend the predicate S_i by a new element which is not in S_j. In a sense, those generic predicates are invisible from one another. A way to impose relations between the S_i, is by considering, for instance, a slightly stronger version of the generic expansion by a reduct –analogously to the generic predicate in [CP98]. Consider a suitable triple $(T, T_0, \mathscr{L}_0)$ and P a 0-definable predicate in T such that in any model

$\mathscr{M}$ of T, P is a model of T_0 which is a substructure of $\mathscr{M}$. One may do the construction of the generic expansion by a substructure S inside P. In that case, assume that $T_i = T_j$ for all $i, j \leq n$. One may construct TS_1 then add a generic substructure S_2 inside S_1 and iterate. This would be the model companion of the theory $T_{S_1 \dots S_n} \cup \{S_1 \supseteq S_2 \supseteq \dots \supseteq S_n\}$. One may also consider the case in which T_i is not the theory of a substructure but of a structure 0-definable in T.

CHAPTER 3

Examples of generic expansion by a reduct

In this chapter, we apply the results of Chapter 2 to construct new examples of generic expansions.

Contents

3.1 Generic vector subspaces over a finite field

Let $\mathbb{F}_q$ be a finite field. In this section, we let $\mathscr{L}_0 = \left\{(\lambda_\alpha)_{\alpha\in\mathbb{F}_q}, +, 0\right\}$, and $\mathscr{L}$ a language containing $\mathscr{L}_0$. We let T be a complete $\mathscr{L}$-theory which contains the $\mathscr{L}_0$-theory T_0 of infinite-dimensional $\mathbb{F}_q$-vector spaces. For A a subset of a model of T, the set $\mathrm{acl}_0(A)$ is the vector space spanned by A, and we denote it by $\langle A\rangle$. Let $\mathscr{L}_V = \mathscr{L}\cup\{V\}$, with V a unary predicate and T_V the $\mathscr{L}_V$-theory whose models are the models of T in which V is an infinite vector subspace.

Definability and notations. For $\alpha = \alpha_1, \dots, \alpha_n \in \mathbb{F}_q$ and any n-tuple x of variables let $\lambda_\alpha(x)$ be the term

$$\lambda_{\alpha_1}(x_1) + \cdots + \lambda_{\alpha_n}(x_n).$$

Let z be a tuple of variables of length $s = q^n - 1$ and $z' = z_0 z$ a tuple of length $s+1 = q^n$. Let $\psi(t)$ be any $\mathscr{L}_V$-formula, t a single variable. We fix an enumeration $\alpha^1, \dots, \alpha^s$ of $(\mathbb{F}_q)^n \setminus (0, \dots, 0)$. We denote by

$z = \langle x\rangle_0$	the formula	$\bigwedge_{i=1,\dots,s} z_i = \lambda_{\alpha^i}(x)$
$z' = \langle x\rangle$	the formula	$z_0 = 0 \wedge z_1, \cdots, z_s = \langle x\rangle_0$
$t \in \langle x\rangle$	the formula	$\forall z' \left(z' = \langle x\rangle \to \bigvee_{i=0}^{s} t = z_i\right)$
$t \in \langle xy\rangle \setminus \langle y\rangle$	the formula	$t \in \langle xy\rangle \wedge \neg t \in \langle y\rangle$
$\langle x\rangle \cap \psi = \langle y\rangle$	the formula	$\forall t\, (t \in \langle x\rangle \wedge \psi(t) \leftrightarrow t \in \langle y\rangle)\,.$

The formulae above have the obvious meaning, for instance, for any a, b in a model of T, if $\mathscr{M} \models b = \langle a\rangle_0$ then b is an enumeration of all non-trivial $\mathbb{F}_q$-linear combinations of a.

The following is [CP98, Lemma 2.3]:

Fact 3.1.1. *Assume that T is a theory that eliminates the quantifier $\exists^\infty$. Then for any formula $\phi(x, y)$ there is a formula $\theta_\phi(y)$ such that in any $\aleph_0$-saturated model $\mathscr{M}$ of T the set $\theta_\phi(\mathscr{M})$ consists of tuples b from $\mathscr{M}$ such that there exists a realisation a of $\phi(x, b)$ with $a_i \notin \mathrm{acl}_T(b)$ for all i.*

Theorem 3.1.2. *If T is model complete and eliminates the quantifier $\exists^\infty$, then $(T, T_0, \mathscr{L}_0)$ is a suitable triple. It follows that the theory T_V admits a model companion, which we denote by TV.*

Proof. We have to show that the triple $(T, T_0, \mathscr{L}_0)$ is suitable, the existence of the model-companion then follows from Proposition 2.3.3. We check the conditions of Definition 2.3.1:

(H_1) T is model complete;

(H_2) T_0 model complete and for all infinite A, $\langle A\rangle \models T_0$;

(H_3^+) $\langle\cdot\rangle$ defines a modular pregeometry;

(H_4) for all $\mathscr{L}$-formula $\phi(x, y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for $b \in \mathscr{M} \models T$

$$\mathscr{M} \models \theta_\phi(b) \iff \text{there exists a saturated } \mathscr{N} \succ \mathscr{M} \text{ and } a \in \mathscr{N} \text{ such that } \phi(a, b) \text{ and } a \text{ is } \mathop{\smile\mkern-13mu\vert}^{0}\text{-independent over } \mathscr{M}.$$

Condition (H_1) holds by hypothesis. Conditions (H_2) and (H_3^+) are also clear, these are basic properties of the theory of infinite dimensional vector spaces. As A is infinite, $\langle A\rangle$ is an infinite dimensional $\mathbb{F}_q$-vector space.

We prove condition (H_4). Let $\phi(x,y)$ be an $\mathscr{L}$-formula. For some tuple of variables z of suitable length, let $\tilde{\phi}(z,y)$ be the following formula

$$\exists x \; z = \langle x\rangle_0 \wedge \phi(x,y).$$

Now apply Fact 3.1.1 with $\tilde{\phi}(z,y)$. We get a formula $\theta_{\tilde{\phi}}(y)$ such that for any $\aleph_0$-saturated model $\mathscr{N}$ of T and $b \in \mathscr{N}$ we have that $\mathscr{N} \models \theta_{\tilde{\phi}}(b)$ if and only if there exist tuples a and c in $\mathscr{N}$ such that $\phi(a,b)$ holds, $c = \langle a\rangle_0$ and for all i, $c_i \notin \operatorname{acl}_T(b)$. Equivalently $\mathscr{N} \models \theta_{\tilde{\phi}}(b)$ if and only if there exists a tuple a from $\mathscr{N}$ such that a is $\mathbb{F}_q$-linearly independent over $\operatorname{acl}_T(b)$ and $\mathscr{N} \models \phi(a,b)$. By Lemma 2.2.2, this condition is equivalent to (H_4), hence the triple $(T, T_0, \mathscr{L}_0)$ is suitable. $\square$

Lemma 3.1.3. *Let $\psi(x,y)$ be an $\mathscr{L}_V$-formula. Assume that in a saturated model $(\mathscr{M}, V)$ of T_V the following holds for some tuple b from $\mathscr{M}$, for all $\mathscr{L}$-formula $\phi(x,y)$:*

$$\theta_\phi(b) \to \exists x \phi(x,b) \wedge \psi(x,b).$$

Then for all $\phi(x,y)$, if $\mathscr{M} \models \theta_\phi(b)$ then there exists a realisation a of $\phi(x,b)\wedge\psi(x,b)$ such that a is linearly independent over $\operatorname{acl}_T(b)$.

Proof. Let $\Sigma(x,y)$ be the partial type expressing "x is linearly independent over $\operatorname{acl}_T(y)$" (see Section 2.2). We claim that $\{\phi(x,b)\wedge\psi(x,b)\}\cup\Sigma(x,b)$ is consistent. Indeed, let $\Lambda(x,b)$ be a finite conjunction of formulae in $\Sigma(x,b)$. As $\theta_\phi(b)$ holds, by Lemma 2.2.2 there exists a realisation a of $\phi(x,b)$ which is $\mathbb{F}_q$-linearly independent over $\operatorname{acl}_T(b)$, hence in particular a satisfies $\phi(x,b)\wedge\Lambda(x,b)$, hence $\mathscr{M} \models \theta_{\phi\wedge\Lambda}(b)$. By hypothesis, the formula $\phi(x,b)\wedge\Lambda(x,b)\wedge\psi(x,b)$ is consistent, hence we conclude by compactness. $\square$

Proposition 3.1.4 (Axioms for TV)**.** *The theory TV is axiomatised by adding to T_V the following $\mathscr{L}_V$-sentences, for all tuples of variable $y_V \subset y$, $x_V \subset x$ and $\mathscr{L}$-formula $\phi(x,y)$*

$$\forall y(\langle y\rangle \cap V = \langle y_V\rangle \wedge \theta_\phi(y)) \to (\exists x \phi(x,y) \wedge \langle xy\rangle \cap V = \langle x_V y_V\rangle). \tag{A_1}$$

Equivalently, the theory TV is axiomatised by adding to T_V the following $\mathscr{L}_V$-sentences, for all tuples of variable $y^1 \subseteq y$, $x_V \subset x$ and $\mathscr{L}$-formula $\phi(x,y)$

$$\forall y(\langle y^1\rangle \cap V = \{0\} \wedge \theta_\phi(y)) \to (\exists x \phi(x,y) \wedge \langle xy^1\rangle \cap V = \langle x_V\rangle). \tag{A_2}$$

Proof. It is clear that the system of axioms (A_1) is equivalent to the one given in Theorem 2.1.5. It is also clear that the system of axioms (A_1) implies the system of axioms (A_2). We show that the two systems are equivalent. Assume that the system (A_2) is satisfied in an $\aleph_0$ saturated model $(\mathscr{M}, V)$ of T_V. Let $\phi(x,y)$ be given, and subtuples y_V of y and x_V of x. We show that $(\mathscr{M}, V)$ satisfies the axiom of the form (A_1) given by $y_V \subset y$, $x_V \subset x$ and $\phi(x,y)$. Assume that for some tuple b from $\mathscr{M}$, the formula $\langle b\rangle \cap V = \langle b_V\rangle \wedge \theta_\phi(b)$ holds. Let b^1 be a subtuple of b which is a basis of $\langle b\rangle$ over $\langle b_V\rangle$. We have $\langle b^1\rangle \cap V = \{0\}$ hence using an instance of an axiom (A_2), there exists a realisation a of $\phi(x,b)$ such that $\langle ab^1\rangle \cap V = \langle a_V\rangle$. Since $b_V \subseteq V$, it follows from BASE MONOTONICITY that $\langle ab\rangle \cap V = \langle a_V b_V\rangle$. $\square$

Lemma 3.1.5. *Assume that T is model complete and eliminates the quantifier $\exists^\infty$. Then TV eliminates the quantifier $\exists^\infty$, so $(TV, T_0, \mathscr{L}_0)$ is also a suitable triple.*

Proof. Assume that $|x| = 1$. From the description of types (see Proposition 2.3.5), types in TS are obtained by adding to the types in T the description of V on the algebraic closure. By compactness, every $\mathscr{L}_V$-formula $\phi(x,y)$ is equivalent to a disjunction of formulae of the form

$$\exists z \psi(x,z,y) \wedge \langle xz\rangle \cap V = \langle z_V\rangle$$

where $\psi(x,z,y)$ is an $\mathscr{L}$-formula (not necessarily quantifier-free) and z_V a subtuple of variables of z[1]. In order to prove elimination of $\exists^\infty$, by the pigeonhole principle , we may assume that $\phi(x,y)$ is equivalent to such a formula. Now let u, v be two tuples of variables such that $|u|+|v| \leq |z|+1$, and let $u_V \subset u$, $v_V \subset v$ be two subtuples. Let $\Gamma^{uv}_{u_V v_V}(u, yv)$ be the following $\mathscr{L}$-formula

$$\exists xz\psi(x,z,y) \wedge \langle xz\rangle = \langle uv\rangle \wedge \langle z_v\rangle = \langle u_V v_V\rangle \wedge x \in \langle uv\rangle \setminus \langle v\rangle.$$

Let $\Lambda(y)$ be the formula

$$\bigvee_{|uv|\leq|z|+1, u_V\subseteq u, v_V\subseteq v, |u|\geq 1} \exists v(\langle v\rangle \cap V = \langle v_V\rangle \wedge \theta_{\Gamma^{uv}_{u_V v_V}}(yv)).$$

Claim: For all tuple b from a saturated model $(\mathscr{M}, V)$ of TV, $(\mathscr{M}, V) \models \Lambda(b)$ if and only if there exists $a \in \mathscr{M}$ such that $(\mathscr{M}, V) \models \phi(a,b)$ and $a \notin \mathrm{acl}_T(b)$.

From left to right. If $\Lambda(b)$ holds for some b, there exists a formula $\Gamma = \Gamma^{uv}_{u_V v_V}$ and some tuple e from $\mathscr{M}$ and a subtuple e_V of e such that $V \cap \langle e\rangle = \langle e_V\rangle$ and $\mathscr{M} \models \theta_\Gamma(be)$. Using one instance of the axioms (A_1) (Proposition 3.1.4) and Lemma 3.1.3, there exists a realisation d of $\Gamma(u, be)$ such that $\langle dbe\rangle \cap V = \langle d_V b_V e_V\rangle$, for d_V the subtuple associated to the variables u_V and such that d is linearly independent over $\mathrm{acl}_T(be)$. Using that d is linearly independent over $\langle de\rangle$, we obtain that $\langle de\rangle \cap V = \langle d_V e_V\rangle$. As $(\mathscr{M}, V) \models \Gamma(d, be)$, there exists a and a tuple c from $\mathscr{M}$ such that

- $\mathscr{M} \models \psi(a,c,b)$
- $\langle ac\rangle = \langle de\rangle$
- $\langle c_V\rangle = \langle d_V e_V\rangle$
- $a \in \langle de\rangle \setminus \langle e\rangle$.

Now as $\langle de\rangle \cap V = \langle d_V e_V\rangle$ we have $\langle ac\rangle \cap V = \langle c_V\rangle$ so $(\mathscr{M}, V) \models \phi(a,b)$. Now as d is linearly independent over $\mathrm{acl}_T(be)$ and $a \in \langle de\rangle \setminus \langle e\rangle$ we have $a \notin \mathrm{acl}_T(be)$ so $a \notin \mathrm{acl}_T(b)$.

From right to left. Assume that $(\mathscr{M}, V) \models \phi(a,b)$ and $a \notin \mathrm{acl}_T(b)$. Let c be such that $c \models \psi(a,z,b)$ and $\langle ac\rangle \cap V = \langle c_V\rangle$. Let e_V be a basis of $\mathrm{acl}_T(b) \cap V \cap \langle ac\rangle$, and complete it in a basis e of $\mathrm{acl}_T(b) \cap \langle ac\rangle$. Let d_V be a basis of a complement of $\langle e_V\rangle$ inside $\langle ac\rangle \cap V$ and complete it in a basis d of a complement of $\langle ed_V\rangle$ inside $\langle ac\rangle$. As $a \in \langle de\rangle \setminus \mathrm{acl}_T(b)$ we have $a \in \langle de\rangle \setminus \langle e\rangle$. It is clear that $(\mathscr{M}, V) \models \Gamma^{uv}_{u_V v_V}(d, be)$ for the appropriate choice of subtuple of variables $u_V \subseteq u$ and $v_V \subseteq v$. Furthermore, as d is linearly independent over $\mathrm{acl}_T(b) = \mathrm{acl}_T(be)$, we have $\theta_\Gamma(be)$, and so $\Lambda(b)$ holds. ☐

Corollary 3.1.6. *Assume that T is model-complete and eliminates $\exists^\infty$. Let $T_{V_1\ldots V_n}$ be the theory whose models are models of T in which V_i is a predicate for a vector subspace over $\mathbb{F}_q$. Then $T_{V_1\ldots V_n}$ admits a model companion $TV_1\ldots V_n$.*

Proof. This is an immediate consequence of Lemma 3.1.5 and Proposition 2.4.1. ☐

Example 3.1.7 (Generic vector subspace of a vector space)**.** Consider the theory T of infinite $\mathbb{F}_q$-vector spaces in the language $\mathscr{L} = \{(\lambda_\alpha)_{\alpha\in\mathbb{F}_q}, +, 0\}$. Applying Corollary 3.1.6 the theory $T_{V_1\ldots V_n}$ admits a model companion $TV_1\ldots V_n$. It is easy to check that TV_1 is the theory of belles paires (see [Poi83]) of the theory T, hence as T is NFCP, TV_1 is stable. One can easily show that TV_1 has U-rank 2, and one expects that $TV_1\ldots V_n$ has U-rank $n+1$. This is a particular case of Proposition 3.4.1.

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[1] Actually we might assume that every realisation of z in ψ is algebraic over the realisations of x, y in ψ, but we don't need this fact here. Also, we may replace the condition $\langle xz\rangle \cap V = \langle z_V\rangle$ by $\langle z\rangle \cap V = \langle z_V\rangle$, but we assume that the formula gives a description of V on $\langle xz\rangle$ in order to simplify the proof.

3.2 Fields with generic additive subgroups

Let $p > 0$ be a prime number. Let $\mathscr{L} = \{+, -, \cdot, 0, 1, \dots\}$ and T an $\mathscr{L}$-theory of an infinite field of characteristic p. Let $\mathbb{F}_{q_1}, \cdots, \mathbb{F}_{q_n}$ be finite subfields in any model of T. Consider the theory T' obtained by adding to the language a constant symbol for each element of $\mathbb{F}_{q_1} \cup \cdots \cup \mathbb{F}_{q_n}$. Then T and T' have the same models. It follows that for each i we may consider that the theory of infinite $\mathbb{F}_{q_i}$-vector space in the language $\mathscr{L}_i = \left\{+, 0, (\lambda_\alpha)_{\alpha \in \mathbb{F}_{q_i}}\right\}$ is a reduct of T.

Proposition 3.2.1. *Let $\mathscr{L} \supseteq \mathscr{L}_{\text{ring}}$ and T an $\mathscr{L}$-theory of an infinite field of characteristic p. Let $\mathbb{F}_{q_1}, \cdots, \mathbb{F}_{q_n}$ be finite subfields in any model of T. Assume that*

(1) T is model-complete;

(2) T eliminates $\exists^\infty$.

Let $T_{V_1 \dots V_n}$ be the theory whose models are models of T in which each V_i is a predicate for an $\mathbb{F}_{q_i}$-vector subspace. By Corollary 3.1.6 the theory $T_{V_1 \dots V_n}$ admits a model-companion.

An additive subgroup of a field of characteristic p is an $\mathbb{F}_p$-vector space, hence Proposition 3.2.1 translates as follows.

Proposition 3.2.2. *Let $\mathscr{L} \supseteq \mathscr{L}_{\text{ring}}$ and T an $\mathscr{L}$-theory of an infinite field of characteristic p. Assume that*

(1) T is model-complete;

(2) T eliminates $\exists^\infty$.

Let $T_{G_1 \dots G_n}$ be the theory whose models are models of T in which each G_i is a predicate for an additive subgroup. By Corollary 3.1.6 the theory $T_{G_1 \dots G_n}$ admits a model-companion, which we denote by $TG_1 \dots G_n$.

Example 3.2.3. The hypotheses of Propositions 3.2.1 and 3.2.2 are satisfied by the following theories by Subsection 1.5.2:

- ACF_p, $\mathrm{SCF}_{p,e}$ for e finite or infinite, Psf_c,
- ACFA_p, DCF_p.

Example 3.2.4 ($\mathrm{ACFV}_1 \cdots \mathrm{V}_n$ and ACFG)**.** Let $\mathbb{F}_{q_1}, \cdots, \mathbb{F}_{q_n}$ be any finite fields of characteristic p. We denote by $\mathrm{ACFV}_1 \cdots \mathrm{V}_n$ and ACFG respectively the theories $\mathrm{ACF}_p\mathrm{V}_1 \cdots \mathrm{V}_n$ and ACF_pG. Chapters 5, 6 and 7 are dedicated to a detailed study of the theory ACFG, which is NSOP_1 and not simple (see also Example 4.4.3).

Recall from Subsection 1.5.2 that a pseudo-algebraically closed field is a field K which is existentially closed in every regular extension. The theory PAC is incomplete but eliminates $\exists^\infty$ if the field is perfect (Fact 1.5.16).

Proposition 3.2.5. *Let PAC_{pG} be the theory whose models are perfect PAC_p-fields in $\mathscr{L}_{\text{ring}}$ with a predicate G for an additive subgroup. Then there exists a theory PAC_pG such that*

(1) every model (F, G') of PAC_{pG} extends to a model (K, G) of PAC_pG such that K is a regular extension of F;

(2) every model (K, G) of PAC_pG is existentially closed in every extension (F, G') such that F is a regular extension of K.

Let T be a theory of perfect PAC_p-fields in a language containing $\mathscr{L}_{\mathrm{ring}}$ such that T is model-complete, and $T_{G_1\cdots G_n}$ be the theory whose models are models of T with predicates G_i for additive subgroups. Then $T_{G_1\cdots G_n}$ admits a model-companion, $TG_1\cdots G_n$.

Proof. Perfect PAC_p-fields in $\mathscr{L}_{\mathrm{ring}}$ satisfies (H_4), the proof of this in Theorem 3.1.2 does not use the model-completeness of the theory T, so the first statement follows from Proposition 2.1.7. The second statement is Corollary 3.1.6. ☐

Remark 3.2.6. Note that the *perfect* assumption is only here to ensure that the fields eliminate the quantifier $\exists^\infty$. It should be true that all PAC fields eliminate the quantifier $\exists^\infty$ although we did not find any reference in the literature.

However, in the characteristic 0 case the model-companion does not exist.

Proposition 3.2.7. *Let T be the theory of a field of characteristic 0 in a language $\mathscr{L}$ containing $\mathscr{L}_{\mathrm{ring}}$, such that T is inductive. Let $\mathscr{L}_G = \mathscr{L} \cup \{G\}$ and let T_G be the $\mathscr{L}_G$-theory of models of T in which G is a predicate for an additive subgroup of the field. Let (K, G) be an existentially closed model of T_G. Then*

$$S_K(G) := \{a \in K \mid aG \subseteq G\} = \mathbb{Z}.$$

In particular, the theory T_G does not admit a model-companion.

Proof. The right to left inclusion is trivial. Assume that $a \in K\setminus\mathbb{Z}$, let L be a proper elementary extension of K and $t \in L \setminus K$. Then $(L, G + \mathbb{Z}\frac{t}{a})$ is an $\mathscr{L}_G$-extension of (K, G). Furthermore, as $a \notin \mathbb{Z}$, we have $t \notin G + \mathbb{Z}\frac{t}{a}$. Then $\frac{t}{a} \in G + \mathbb{Z}\frac{t}{a}$ and $a\frac{t}{a} \notin G + \mathbb{Z}\frac{t}{a}$. As (K, G) is existentially closed in $(L, G + \mathbb{Z}\frac{t}{a})$, we have that

$$(K, G) \models \exists x(x \in G \wedge ax \notin G)$$

hence $a \notin S_K(G)$. The class of existentially closed models of T_G is not axiomatisable as the definable infinite set $S_L(G)$ is of fixed cardinality. As T_G is inductive, this is equivalent to saying that T_G does not admit a model-companion. ☐

Remark 3.2.8. Let T be the theory of a field of characteristic 0 in a language $\mathscr{L}$ containing $\mathscr{L}_{\mathrm{ring}}$, such that T is inductive. Let $\mathscr{L}_D = \mathscr{L} \cup \{D\}$ and let T_D be the $\mathscr{L}_D$-theory of models of T in which D is a predicate for a *divisible* additive subgroup of the field. Let (K, D) be an existentially closed model of T_D. A similar argument yields that $\{a \in K \mid aD = D\} = \mathbb{Q}$, so T_D does not admits a model-companion either.

Remark 3.2.9. Let $K = \mathbb{C}$ (or $\mathbb{R}$). Using Remark 3.2.8 and Lemmas 2.2.2 and 2.2.3, one deduces that there exist $k, l \in \mathbb{N}$ and a constructible set if $K = \mathbb{C}$ (or a semialgebraic set if $K = \mathbb{R}$) $V \subseteq K^k \times K^l$ such that for all polynomials $P(X, Y) \in K[X, Y]$ with $|X| = 1$, $|Y| = l$ and for all $n \in \mathbb{N}$ and all $q_1, \dots q_n, s_1, \dots, s_k \in \mathbb{Q}$ there exists $b \in K^l$ such that for all $a \in K^k$, if $(a, b) \in V$ then

(1) a is not $\mathbb{Q}$-linearly independent over $\overline{\mathbb{Q}(b)} \cap K$;

(2) $\sum_{i=1}^k s_i a_i \notin q_1 R + \cdots + q_n R$ for R the set of roots of $P(X, b)$ in K.

3.3 Algebraically closed fields with a generic multiplicative subgroup

We are now interested in using Theorem 2.1.5 to prove that the theory of algebraically closed fields of fixed arbitrary characteristic with a predicate for a multiplicative subgroup admits a model companion. Consider $\mathscr{L}_{\text{field}} = \{+, -, \cdot, ^{-1}, 0, 1\}$ and $\mathscr{L}_0 = \{\cdot, ^{-1}, 1\} \subseteq \mathscr{L}_{\text{field}}$.

The pure multiplicative group of any field is an $\aleph_1$-categorical abelian group, its model theory is described in [Mac71], see also [Che76, Chapter VI].

Fix p a prime or 0. Consider the theory ACF_p. The theory $\mathrm{ACF}_p \restriction \mathscr{L}_0$ is complete and we will identify it with the theory of the multiplicative group of an algebraically closed field of characteristic p, denoted by T_p. The theory T_p is axiomatised by adding to the theory of abelian groups the following sets of axiom:

- *If* $p > 0$: $\{\forall x\ \exists^{=n} y\ \ y^n = x \mid n \in \mathbb{N} \setminus p\mathbb{N}\} \cup \left\{\forall x \exists^{=1} y\ y^p = x\right\}$
- *If* $p = 0$: $\{\forall x\ \exists^{=n} y\ \ y^n = x \mid n \in \mathbb{N} \setminus \{0\}\}$.

Proposition 3.3.1. *The theory T_p has quantifier elimination in the language $\mathscr{L}_0$. It is strongly minimal hence $\aleph_1$-categorical. Furthermore for any subset A of a model M of T_p, the algebraic closure is given by*

$$\mathrm{acl}_p(A) := \{u \in M, u^n \in \langle A\rangle \textit{ for some } n \in \mathbb{N} \setminus \{0\}\}$$

where $\langle A\rangle$ is the group spanned by A. Every algebraically closed set is a model of T_p. Furthermore acl_p defines a pregeometry which is modular and the associated independence relation in T_p is given by

$$A \mathop{\smile\!\!\!\!\!\!\;\mid}^{p}_{C} B :\iff \mathrm{acl}_p(AC) \cap \mathrm{acl}_p(BC) = \mathrm{acl}_p(C).$$

See Subsection 1.5.2 for basics about affine varieties and generics of a variety.

Lemma 3.3.2. *Let $K \models \mathrm{ACF}$, $V \subset K^n$ an affine (irreducible) variety, $\mathcal{O} \subset K^n$ a Zariski open set. The following are equivalent:*

(1) for all $k_1, \dots, k_n \in \mathbb{N}$, $c \in K$ the quasi affine variety $V \cap \mathcal{O}$ is not included in the zero set of $x_1^{k_1} \cdot \dots \cdot x_n^{k_n} = c$

(2) for all $k_1, \dots, k_n \in \mathbb{N}$, $c \in K$ the variety V is not included in the zero set of $x_1^{k_1} \cdot \dots \cdot x_n^{k_n} = c$

(3) there exist $L \succ K$ and a tuple a which is multiplicatively independent over K and with $a \in (V \cap \mathcal{O})(L)$

Proof. *(1)* implies *(2)* is trivial. We show that *(2)* implies *(3)*. Assume that *(3)* does not hold. Take a generic a over K of the variety V in some $L \succ K$. We have $a \in \mathcal{O}$. Then there exists $k_1, \dots, k_n \in \mathbb{N}$ such that $a_1^{k_1} \cdot \dots \cdot a_n^{k_n} = c$ for some $c \in K$. By genericity of a, it follows that V is included in the zero set of $x_1^{k_1} \cdot \dots \cdot x_n^{k_n} = c$, hence *(2)* does not hold. *(3)* implies *(1)* follows easily from the fact that V and $\mathcal{O}$ are definable over K. $\square$

The following fact was first observed in the proof of Theorem 1.2 in [BGH13], it is also Corollary 3.12 in [Tra17].

Fact 3.3.3. *Let p be a prime number or 0. Let $\phi(x, y)$ an $\mathscr{L}_{\text{field}}$-formula such that for all tuple b in a model of ACF_p, $\phi(x, b)$ defines an affine variety. Then there exists an $\mathscr{L}_{\text{field}}$-formula $\theta_\phi(y)$ such that for any model K of ACF_p and tuple b from K, we have $K \models \theta_\phi(b)$ if and only if for all $k_1, \dots, k_n \in \mathbb{N}$, $c \in K$, the set $\phi(K, b)$ is not included in the zero set of $x_1^{k_1} \cdot \dots \cdot x_n^{k_n} = c$.*

By Subsection 1.5.2, every definable set in ACF_p can be written as a finite union of quasi-affine varieties. Furthermore, it is standard that given any $\mathscr{L}_{\text{ring}}$-formula $\vartheta(x, z)$, the set of c such that $\vartheta(x, c)$ is a quasi-affine variety is a definable set ([Tra17, Lemma 3.10]). Let $\mathscr{C}$ be the class of formulae $\vartheta(x, z)$ such that for all $K \models \mathrm{ACF}_p$ and c tuple from K, the set $\vartheta(K, c)$ is a quasi-affine variety.

Lemma 3.3.4. *Let p be a prime number or* 0. *For any $\vartheta(x,z) \in \mathscr{C}$ there exists an $\mathscr{L}_{\text{field}}$-formula $\theta_\vartheta(z)$ such that for any model K of* ACF_p *and tuple c from K, we have $K \models \theta_\vartheta(c)$ if and only if there exists a such that $\models \vartheta(a,c)$ and a is $\mathop{\smile\!\!\!\!\!\!|}^{p}$-independent over K.*

Proof. Let $K \models \mathrm{ACF}_p$ and $\vartheta(x,z) \in \mathscr{C}$. Using [Joh16, Theorem 10.2.1], there exists a formula $\tilde{\vartheta}(x,z)$ such that for all tuple c from K, the set $\tilde{\vartheta}(K,c)$ is the Zariski closure of $\vartheta(K,c)$. Now by Fact 3.3.3, there exists a formula $\theta(z)$ such that $K \models \theta(c)$ if and only if $\tilde{\vartheta}(K,c)$ is not included in the zero set of $x_1^{k_1} \cdot \dots \cdot x_n^{k_n} = d$, for all $d \in K$, $k_1, \cdots, k_n \in \mathbb{N}$. By Lemma 3.3.2, $K \models \theta(c)$ if and only if there exist $L \succ K$ and a tuple a which is multiplicatively independent over K and with $a \models \vartheta(x,c)$. ☐

If $G^\times$ is a symbol for a unary predicate, we denote by $\mathrm{ACF}_{G^\times}$ the theory in the language $\mathscr{L}_{\text{ring}} \cup \{G^\times\}$ whose models are algebraically closed fields of characteristic p in which the predicate $G^\times$ consists of a multiplicative subgroup.

Theorem 3.3.5. *The theory* $\mathrm{ACF}_{G^\times}$ *admits a model companion, which we denote by* $\mathrm{ACFG}^\times$.

Proof. We check the conditions of Definition 2.3.1

(H_1) ACF_p is model complete;

(H_2) T_p is model-complete and for all infinite A, $\mathrm{acl}_p(A) \models T_p$;

(H_3^+) acl_p defines a modular pregeometry;

(H_4) for all $\mathscr{L}_{\text{field}}$-formula $\phi(x,y)$ there exists an $\mathscr{L}_{\text{field}}$-formula $\theta_\phi(y)$ such that for $b \in K \models \mathrm{ACF}_p$

$$\mathscr{M} \models \theta_\phi(b) \iff \begin{array}{l}\text{there exists } L \succ K \text{ and } a \in L \text{ such that}\\ \phi(a,b) \text{ and } a \text{ is } \mathop{\smile\!\!\!\!\!\!|}^{p}\text{-independent over } K.\end{array}$$

ACF_p is model complete by quantifier elimination. Conditions (H_2) and (H_3) follow from Proposition 3.3.1. We don't have condition (H_4) for all formulae, but only for the formulae in $\mathscr{C}$ (Lemma 3.3.4), which is sufficient for the existence of the model-companion by Remark 2.1.8. ☐

3.4 Pairs of geometric structures

Let T be an $\mathscr{L}$-theory. Let $\mathscr{L}_S$ be the expansion of $\mathscr{L}$ by a unary predicate S. A *pair* of models of T is an $\mathscr{L}_S$-structure $(\mathscr{M}, \mathscr{M}_0)$, where $\mathscr{M} \models T$ and $S(\mathscr{M}) = \mathscr{M}_0$ is a substructure of $\mathscr{M}$ model of T. We call T_S the theory of the *pairs of models of T*. This is consistent with the notations in Chapter 2.

Proposition 3.4.1. *Let T be a model-complete geometric theory (see Section 1.3) in a language $\mathscr{L}$. Assume that every acl_T-closed set is a model of T. Then there exists an $\mathscr{L}_S$-theory TS containing T_S such that:*

(1) every model $(\mathscr{N}, \mathscr{N}_0)$ of T_S has a strong extension which is a model of TS;

(2) every model of TS is existentially closed in every strong extension model of T_S.

Furthermore, TS satisfies the conclusions of Proposition 2.3.5.

Proof. We check that $T, T_0, \mathscr{L}_0$ satisfies the hypotheses of Theorem 2.1.5. (H_1), (H_2) and (H_3) are clear, and (H_4) is Fact 1.3.10. ☐

We call this theory the *weak model companion of the pairs of models of T*. If the pregeometry is modular, it is the model-companion.

Example 3.4.2. The theory of pairs of any strongly minimal theory with quantifier elimination admits a weak model companion. For instance, the weak model companion of the theory of pairs of algebraically closed fields is the theory of proper pairs of algebraically closed fields and coincides with the theory of belle paires of algebraically closed fields (see [Del12], [Poi83]). The theory RCF also satisfies the hypotheses of Proposition 3.4.1, hence the theory of pairs of real closed fields admits a weak model-companion. Connections with lovely pairs of geometric structures ([BV10]) could be made, although we did not investigate.

CHAPTER 4

Preservation of NSOP_1

The aim of this chapter is to establish when the construction presented in Chapter 2 preserves NSOP_1. More precisely, given some suitable triple $(T, T_0, \mathscr{L}_0)$ such that T is NSOP_1, we establish a condition on the triple $(T, T_0, \mathscr{L}_0)$ so that TS is NSOP_1. This condition (see (A) in Theorem 4.2.1) expresses how the pregeometry given by acl_0 is controled by the Kim-independence in T, and how the latter interacts with $\mathop{\smile\!\!\!\!\!\!\mid}^0$.

Contents

4.1 Independence relations in T and TS

We set up the context for this section, Section 4.2 and Section 4.3. Let $(T, T_0, \mathscr{L}_0)$ be a suitable triple (see Definition 2.3.1 and Corollary 2.3.3). We work in a monster model $(\mathbb{M}, \mathbb{M}_0)$ of TS such that $\mathbb{M}$ is a monster model of T. In particular we fix some completion of TS. Also, $\mathbb{M}$ is a monster model for T (see Section 1.1). All small sets $A, B, C, \dots$ or models $\mathscr{M}, \mathscr{N}$ of T, or models $(\mathscr{M}, \mathscr{M}_0), (\mathscr{N}, \mathscr{N}_0)$ of TS are seen as subsets of $\mathbb{M}$, respectively elementary substructures of $\mathbb{M}$ or elementary substructures of $(\mathbb{M}, \mathbb{M}_0)$. For instance we have $S(\mathscr{M}) = \mathscr{M} \cap S(\mathbb{M}) = \mathscr{M} \cap \mathbb{M}_0 = \mathscr{M}_0$. We will start with a ternary relation ($\mathop{\smile\kern-0.9em\mid}^{T}$) defined over subsets of $\mathbb{M}$ and construct from it a ternary relation ($\mathop{\smile\kern-0.9em\mid}^{w}$) taking into account the predicate $S(\mathbb{M}) = \mathbb{M}_0$.

We denote by $\overline{A}$ the set $\mathrm{acl}_T(A)$ which, as we saw, equals $\mathrm{acl}_{TS}(A)$.

Assumption. There exists a ternary relation $\mathop{\smile\kern-0.9em\mid}^{T}$ defined over subsets of $\mathbb{M}$, such that $\mathop{\smile\kern-0.9em\mid}^{T} \to \mathop{\smile\kern-0.9em\mid}^{a}$, where $A \mathop{\smile\kern-0.9em\mid}^{a}_{C} B \iff \overline{AC} \cap \overline{BC} = \overline{C}$.

In particular, if $A \mathop{\smile\kern-0.9em\mid}^{T}_{C} B$ then $\mathrm{acl}_T(AC) \mathop{\smile\kern-0.9em\mid}^{0}_{\mathrm{acl}_T(C)} \mathrm{acl}_T(BC)$, by modularity.

Definition 4.1.1. We call *weak independence* the relation $\mathop{\smile\kern-0.9em\mid}^{w}$ defined by

$$A \mathop{\smile\kern-0.9em\mid}^{w}_{C} B \iff A \mathop{\smile\kern-0.9em\mid}^{T}_{C} B \text{ and } S(\mathrm{acl}_0(\overline{AC}, \overline{BC})) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC})).$$

We call *strong independence* the relation $\mathop{\smile\kern-0.9em\mid}^{st}$ defined by

$$A \mathop{\smile\kern-0.9em\mid}^{st}_{C} B \iff A \mathop{\smile\kern-0.9em\mid}^{T}_{C} B \text{ and } S(\overline{ABC}) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC})).$$

Obviously $\mathop{\smile\kern-0.9em\mid}^{st} \to \mathop{\smile\kern-0.9em\mid}^{w}$.

We will show that if $\mathop{\smile\kern-0.9em\mid}^{T}$ satisfies most of the properties listed in Section 1.2 relatively to the theory T, then so does $\mathop{\smile\kern-0.9em\mid}^{w}$ relatively to the theory TS. The property SYMMETRY of $\mathop{\smile\kern-0.9em\mid}^{0}$, $\mathop{\smile\kern-0.9em\mid}^{T}$ and $\mathop{\smile\kern-0.9em\mid}^{w}$ will be tacitly used throughout this chapter.

Lemma 4.1.2. *If $\mathop{\smile\kern-0.9em\mid}^{T}$ satisfies* INVARIANCE, CLOSURE, SYMMETRY, EXISTENCE *and* MONOTONICITY, *then so does $\mathop{\smile\kern-0.9em\mid}^{w}$.*

Proof. INVARIANCE is clear because $S(\mathrm{acl}_0(\overline{AC}, \overline{BC})) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC}))$ is an $\mathscr{L}_S$-invariant condition. CLOSURE, SYMMETRY and EXISTENCE are trivial.

For MONOTONICITY, let A, B, C, D such that $A \mathop{\smile\kern-0.9em\mid}^{w}_{C} BD$. By hypothese, $A \mathop{\smile\kern-0.9em\mid}^{T}_{C} B$. Now

$$\begin{aligned} S(\mathrm{acl}_0(\overline{AC}, \overline{BC})) &= S(\mathrm{acl}_0(\overline{AC}, \overline{BCD})) \cap \mathrm{acl}_0(\overline{AC}, \overline{BC}) \\ &= \mathrm{acl}_0(S(\overline{AC}), S(\overline{BCD})) \cap \mathrm{acl}_0(\overline{AC}, \overline{BC}). \end{aligned}$$

Since $S(\overline{AC}) \subseteq \mathrm{acl}_0(\overline{AC}, \overline{BC})$, we have by modularity

$$\mathrm{acl}_0(S(\overline{AC}), S(\overline{BCD})) \cap \mathrm{acl}_0(\overline{AC}, \overline{BC}) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BCD}) \cap \mathrm{acl}_0(\overline{AC}, \overline{BC})).$$

Using that $\mathop{\smile\kern-0.9em\mid}^{T} \to \mathop{\smile\kern-0.9em\mid}^{a}$, it follows from the hypotheses that $\overline{AC} \mathop{\smile\kern-0.9em\mid}^{0}_{\overline{C}} \overline{BCD}$ hence by BASE MONOTONICITY of $\mathop{\smile\kern-0.9em\mid}^{0}$ we have $\overline{BCD} \cap \mathrm{acl}_0(\overline{AB}, \overline{BC}) = \overline{BC}$ hence

$$S(\overline{BCD}) \cap \mathrm{acl}_0(\overline{AC}, \overline{BC}) = S(\overline{BC}).$$

It follows that $S(\mathrm{acl}_0(\overline{AC}, \overline{BCD})) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC}))$ and so $A \mathop{\smile\kern-0.9em\mid}^{w}_{C} B$. ☐

Lemma 4.1.3. *If $\mathop{\smile\kern-0.8em\vert}^T$ satisfies* FULL EXISTENCE, *then $\mathop{\smile\kern-0.8em\vert}^{st}$ and $\mathop{\smile\kern-0.8em\vert}^w$ satisfy* FULL EXISTENCE.

Proof. We show that $\mathop{\smile\kern-0.8em\vert}^{st}$ satisfies FULL EXISTENCE. Let A, B, C be contained in some model $(\mathscr{M}, \mathscr{M}_0)$ of TS. By FULL EXISTENCE for $\mathop{\smile\kern-0.8em\vert}^T$, there exists $A' \equiv_C^T A$ with $A' \mathop{\smile\kern-0.8em\vert}^T_C \mathscr{M}$, in particular $\overline{A'C} \cap \overline{BC} = \overline{C}$. Using FULL EXISTENCE of $\mathop{\smile\kern-0.8em\vert}^a$ we may assume that $\overline{A'BC} \cap \mathscr{M} = \overline{BC}$. Let $f : \overline{A'C} \to \overline{AC}$ be a T-elementary isomorphism over C and $S_{A'C} := f^{-1}(S(\overline{AC}))$. Let $S_{A'BC} = \operatorname{acl}_0(S_{A'C}, S(\overline{BC}))$. It is easy to see that

- $S_{A'BC} \cap \mathscr{M} = S_{A'BC} \cap \overline{BC} = S(\overline{BC})$
- $S_{A'BC} \cap \overline{A'C} = S_{A'C}$

Using $\overline{A'BC} \cap \mathscr{M} = \overline{BC}$ and the first item, the type over BC defined by the pair $(\overline{A'BC}, S_{A'BC})$ is consistent (see Proposition 2.3.6). We may assume that $A' \subseteq \mathbb{M}$ realizes this type. From the second item, we have that $A' \equiv_C^{TS} A$, and it is clear that $S(\overline{A'BC})$ is equal to $\operatorname{acl}_0(S(\overline{A'C}), S(\overline{BC}))$ so $A' \mathop{\smile\kern-0.8em\vert}^{st}_C B$. We conclude that FULL EXISTENCE is satisfied by $\mathop{\smile\kern-0.8em\vert}^{st}$. As $\mathop{\smile\kern-0.8em\vert}^{st} \to \mathop{\smile\kern-0.8em\vert}^w$, FULL EXISTENCE is also satisfied by $\mathop{\smile\kern-0.8em\vert}^w$. ☐

Lemma 4.1.4. *If $\mathop{\smile\kern-0.8em\vert}^T$ satisfies* STRONG FINITE CHARACTER *over algebraically closed sets, then the relation $\mathop{\smile\kern-0.8em\vert}^w$ satisfies* STRONG FINITE CHARACTER *over algebraically closed sets.*

Proof. Assume that $a \mathop{\not\smile\kern-0.8em\vert}^w_C b$ and $C = \overline{C}$. If $a \mathop{\not\smile\kern-0.8em\vert}^T_C b$, we have a formula witnessing STRONG FINITE CHARACTER over C by hypothesis. Otherwise, assume that $a \mathop{\smile\kern-0.8em\vert}^T_C b$, set $A = \overline{Ca}$, $B = \overline{Cb}$ and assume that there exists $s \in S(\operatorname{acl}_0(A, B)) \setminus \operatorname{acl}_0(S(A), S(B))$. Let $u \in A \setminus S(A)$ and $v \in B \setminus S(B)$ be such that $s \in \operatorname{acl}_0(u, v)$. There exists $\mathscr{L}_S$-formulae $\psi_u(y, a, c)$ and $\psi_v(z, b, c)$ isolating respectively $tp^{TS}(u/Ca)$ and $tp^{TS}(v/Cb)$ for some tuple c in C. There is also an $\mathscr{L}_0$-formula $\phi(t, y, z)$ algebraic in t, strict in y and strict in z, such that $s \models \phi(t, u, v)$.

Claim. $v \notin \operatorname{acl}_0(S(B), C)$.

Proof of the claim. Assuming otherwise, by modularity there exists singletons $s_b \in S(B)$ and $c \in C$ such that $v \in \operatorname{acl}_0(s_b, c)$ and so $s \in \operatorname{acl}_0(s_b, c, u)$. As $cu \subseteq A$, by modularity there exists a singleton $u' \in A$ such that $s \in \operatorname{acl}_0(s_b, u')$ and by Exchange $u' \in \operatorname{acl}_0(s_b, s) \cap A \subseteq S(A)$, this contradicts the hypothesis on s. ☐

In particular for any other realisation v' of $\psi_v(z, b, c)$ we have $v' \notin \operatorname{acl}_0(S(B), C)$. Now let $\Lambda(x, b, c)$ be the following formula

$$\exists y \exists z \exists t \psi_u(y, x, c) \wedge \psi_v(z, b, c) \wedge \phi(t, y, z) \wedge t \in S.$$

We have that $\Lambda(x, b, c) \in tp^{TS}(a/bC)$. Assume that $a' \models \Lambda(x, b, c)$. If $a' \mathop{\not\smile\kern-0.8em\vert}^T_C b$ then we are done, so we may assume that $a' \mathop{\smile\kern-0.8em\vert}^T_C b$, in particular $\overline{Ca'} \cap B = C$ as C is algebraically closed. There exists $u' \in \overline{Ca'}$ and $v' \in B \setminus \operatorname{acl}_0(S(B), C)$ such that there is $s' \in \operatorname{acl}_0(u', v') \cap S$. In particular $v' \in \operatorname{acl}_0(s', u')$ as $\phi(t, y, z)$ is strict in z. Now assume that $s' \in \operatorname{acl}_0(S(B), S(\overline{Ca'}))$, then $v' \in \operatorname{acl}_0(\overline{Ca'}, S(B))$ and also $v' \in B$. By modularity,

$$\operatorname{acl}_0(S(B), \overline{Ca'}) \cap B = \operatorname{acl}_0(S(B), \overline{Ca'} \cap B) = \operatorname{acl}_0(S(B), C)$$

so $v' \in \operatorname{acl}_0(S(B), C)$, a contradiction. We conclude that

$$s' \in S(\operatorname{acl}_0(\overline{Ca'}, B)) \setminus \operatorname{acl}_0(S(\overline{Ca'}), S(B))$$

so $a' \mathop{\not\smile\kern-0.8em\vert}^w_C B$. ☐

Theorem 4.1.5. *Assume that* $\mathop{\smile\hspace{-0.9em}\mid}^{T}$ *satisfies the hypotheses of Lemmas 4.1.2. Assume that for some subset* E *of* $\mathbb{M}$*, the following two properties hold:*

(A_1) $\mathop{\smile\hspace{-0.9em}\mid}'$*-amalgamation over* E *for some* $\mathop{\smile\hspace{-0.9em}\mid}' \to \mathop{\smile\hspace{-0.9em}\mid}^{a}$, $\mathop{\smile\hspace{-0.9em}\mid}'$ *satisfying* Monotonicity, Symmetry *and* Closure*;*

(A_2) *For all* A, B, C *algebraically closed containing* E*, if* $C \mathop{\smile\hspace{-0.9em}\mid}^{T}_{E} A, B$ *and* $A \mathop{\smile\hspace{-0.9em}\mid}'_{E} B$ *then*

$$(\overline{AC}, \overline{BC}) \underset{A,B}{\mathop{\smile\hspace{-0.9em}\mid}^{0}} \overline{AB}.$$

Then $\mathop{\smile\hspace{-0.9em}\mid}^{w}$ *satisfies* $\mathop{\smile\hspace{-0.9em}\mid}'$*-amalgamation over* E*.*

Proof. Let c_1, c_2, A, B be in a $(\mathscr{M}, \mathscr{M}_0) \prec (\mathbb{M}, \mathbb{M}_0)$ such that

- $c_1 \equiv^{TS}_{E} c_2$
- $A \mathop{\smile\hspace{-0.9em}\mid}'_{E} B$
- $c_1 \mathop{\smile\hspace{-0.9em}\mid}^{w}_{E} A$ and $c_2 \mathop{\smile\hspace{-0.9em}\mid}^{w}_{E} B$

As $\mathop{\smile\hspace{-0.9em}\mid}'$ satisfies Symmetry, Closure and Monotonicity, we have that $A \mathop{\smile\hspace{-0.9em}\mid}'_{E} B \iff \overline{AE} \mathop{\smile\hspace{-0.9em}\mid}'_{E} \overline{BE}$, hence we may assume that A, B are algebraically closed and contain E. By hypothesis there is a T-elementary $\mathscr{L}_S$-isomorphism $h : \overline{Ec_1} \to \overline{Ec_2}$ over E sending c_1 to c_2. Let C_1 be an enumeration of $\overline{Ec_1}$ and let C_2 be the enumeration $h(C_1)$. We have $C_1 \equiv^{T}_{E} C_2$.

We have $C_1 \mathop{\smile\hspace{-0.9em}\mid}^{T}_{E} A$, $C_2 \mathop{\smile\hspace{-0.9em}\mid}^{T}_{E} B$ and $C_1 \equiv^{T}_{E} C_2$. By (A_1), there exists C such that $C \equiv^{T}_{A} C_1$, $C \equiv^{T}_{B} C_2$ with $C \mathop{\smile\hspace{-0.9em}\mid}^{T}_{E} AB$, $A \mathop{\smile\hspace{-0.9em}\mid}^{a}_{C} B$, $C \mathop{\smile\hspace{-0.9em}\mid}^{a}_{B} A$ and $C \mathop{\smile\hspace{-0.9em}\mid}^{a}_{A} B$. We may assume that $\overline{ABC} \cap \mathscr{M} = \overline{AB}$ using Full Existence of $\mathop{\smile\hspace{-0.9em}\mid}^{a}$. There exists two T-elementary bijections $f : \overline{AC} \to \overline{AC_1}$ over A and $g : \overline{BC} \to \overline{BC_2}$ over B such that $g \restriction C = h \circ (f \restriction C)$.

We define $S_{AC} = f^{-1}(S(\overline{AC_1})) \subseteq \overline{AC}$ and $S_{BC} = g^{-1}(S(\overline{BC_2})) \subseteq \overline{BC}$, and set $S_{ABC} = \operatorname{acl}_0(S_{AB}, S_{AC}, S_{BC})$, with $S_{AB} = S(\overline{AB})$. The following is easy to check, it uses that $A \mathop{\smile\hspace{-0.9em}\mid}^{a}_{C} B$, $C \mathop{\smile\hspace{-0.9em}\mid}^{a}_{B} A$ and $C \mathop{\smile\hspace{-0.9em}\mid}^{a}_{A} B$:

- $S_{AB} \cap S_{AC} = S_{AB} \cap A = S_{AC} \cap A = S(A) =: S_A$;
- $S_{AB} \cap S_{BC} = S_{AB} \cap B = S_{BC} \cap B = S(B) =: S_B$;
- $S_{AC} \cap S_{BC} = S_{AC} \cap C = S_{BC} \cap C = f^{-1}(S(C_1)) = g^{-1}(S(C_2)) =: S_C$.

Furthermore, with $S^{-}_{AB} = S_{AB} \cap \operatorname{acl}_0(A, B)$, $S^{-}_{AC} = S_{AC} \cap \operatorname{acl}_0(A, C)$ and $S^{-}_{BC} = S_{BC} \cap \operatorname{acl}_0(B, C)$, it follows from $c_1 \mathop{\smile\hspace{-0.9em}\mid}^{w}_{E} A$ and $c_2 \mathop{\smile\hspace{-0.9em}\mid}^{w}_{E} B$ that

1. $S^{-}_{AC} = \operatorname{acl}_0(S_A, S_C)$;
2. $S^{-}_{BC} = \operatorname{acl}_0(S_B, S_C)$.

Claim. We have the following

- $S_{ABC} \cap \overline{AB} = S_{AB}$;
- $S_{ABC} \cap \overline{AC} = S_{AC}$;
- $S_{ABC} \cap \overline{BC} = S_{BC}$.

Proof of the claim. As $A \mathop{\smile\hspace{-0.9em}\mid}^{a}_{C} B$, $C \mathop{\smile\hspace{-0.9em}\mid}^{a}_{B} A$ and $C \mathop{\smile\hspace{-0.9em}\mid}^{a}_{A} B$, we have that $\overline{AC} \mathop{\smile\hspace{-0.9em}\mid}^{0}_{C} \overline{BC}$, $\overline{BC} \mathop{\smile\hspace{-0.9em}\mid}^{0}_{B} \overline{AB}$ and $\overline{AC} \mathop{\smile\hspace{-0.9em}\mid}^{0}_{A} \overline{AB}$. By hypothesis ($A_2$) and Transitivity of $\mathop{\smile\hspace{-0.9em}\mid}^{0}$ we have the following:

- $(\overline{AC}, \overline{BC}) \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{A,B} \overline{AB}$;
- $(\overline{AB}, \overline{BC}) \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{A,C} \overline{AC}$;
- $(\overline{AC}, \overline{AB}) \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{B,C} \overline{BC}$.

In order to prove the first item of the claim, by modularity, it suffices to show that $\mathrm{acl}_0(S_{AC}, S_{BC}) \cap \overline{AB} \subseteq S_{AB}$. We will in fact show that

$$\mathrm{acl}_0(S_{AC}, S_{BC}) \cap \overline{AB} = S^{-}_{AB}.$$

We have that $(\overline{AB}, \overline{BC}) \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{A,C} \overline{AC}$. Since $S^{-}_{AC} = S_{AC} \cap \mathrm{acl}_0(A, C)$ and $S_{BC} \subseteq \overline{BC}$ we deduce $S_{AC} \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{S^{-}_{AC}} \overline{AB}, S_{BC}$. Now since $S^{-}_{AC} = \mathrm{acl}_0(S_A, S_C)$ we can use Base Monotonicity of $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}$ and the fact that $S_C \subseteq S_{BC}$ to get

$$S_{AC} \mathop{\smile\hspace{-0.9em}\mid}\limits^{0}_{S_A, S_B, S_{BC}} \overline{AB}.$$

On the other hand, $\overline{BC} \cap \overline{AB} = B$ so $S_{BC} \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{S_B} \overline{AB}$. Using Base Monotonicity of $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}$ we also have that $S_{BC} \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{S_A, S_B} \overline{AB}$ so using Transitivity of $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}$ it follows that $(S_{AC}, S_{BC}) \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{S_A, S_B} \overline{AB}$.

For the second item, it is sufficient to prove that $\mathrm{acl}_0(S_{AB}, S_{BC}) \cap \overline{AC} \subseteq S_{AC}$. We do similarly as before paying attention to the fact that S_{AB} and S_{AC} do not play a symmetric role. We get first that $S_{BC} \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{S^{-}_{BC}} (\overline{AC}, S_{AB})$ using $(\overline{AC}, \overline{AB}) \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{B,C} \overline{BC}$. Now $S^{-}_{BC} = \mathrm{acl}_0(S_B, S_C)$, so we deduce $S_{BC} \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{S_C, S_B} (\overline{AC}, S_{AB})$ and by Base Monotonicity of $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}$ and the fact that $S_B, S_A \subseteq S_{AB}$ we deduce

$$S_{BC} \mathop{\smile\hspace{-0.9em}\mid}\limits^{0}_{S_C, S_A, S_{AB}} \overline{AC}.$$

Now by Base Monotonicity of $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}$, we have $S_{AB} \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{S_A, S_C} \overline{AC}$. We conclude using Transitivity of $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}$ that $(S_{AB}, S_{BC}) \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{S_A, S_C} \overline{AC}$. The proof of the last assertion is similar. $\square$

We know that $\overline{ABC} \cap \mathscr{M} = \overline{AB}$. Moreover, it follows from the first point of the claim that $S_{ABC} \cap \mathscr{M} = S_{ABC} \cap \overline{AB} = S_{AB}$. Consequently, by Proposition 2.3.6, the type in the sense of the theory TS defined by the pair $(\overline{ABC}, S_{ABC})$ is consistent, so we may consider that it is realised in $(\mathbb{M}, \mathbb{M}_0)$, by say C. It follows that $C = \overline{Ec}$ with c such that $c \equiv^{TS}_{A} c_1$ and $c \equiv^{TS}_{B} c_2$. What remains to show is that $C \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{w}_{E} A, B$. We already have that $C \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{T}_{E} A, B$ so we will prove that

$$S(\mathrm{acl}_0(C, \overline{AB})) = \mathrm{acl}_0(S(C), S(\overline{AB})).$$

By modularity, it suffices to show that $\mathrm{acl}_0(S_{AC}, S_{BC}) \cap \mathrm{acl}_0(C, \overline{AB}) \subseteq \mathrm{acl}_0(S_C, S_{AB})$. We in fact prove that $(S_{AC}, S_{BC}) \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{S_A, S_B, S_C} (\overline{AB}, C)$. As before, using $(\overline{AB}, \overline{BC}) \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{A,C} \overline{AC}$ we have that $S_{AC} \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}_{S^{-}_{AC}} (\overline{AB}, \overline{BC})$, so as $S^{-}_{AC} = \mathrm{acl}_0(S_A, S_C)$ we have

$$S_{AC} \mathop{\smile\hspace{-0.9em}\mid}\limits^{0}_{S_A, S_C} (\overline{AB}, S_{BC}, C).$$

Using Base Monotonicity of $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{0}$, we have

$$S_{AC} \mathop{\smile\hspace{-0.9em}\mid}\limits^{0}_{S_A, S_B, S_C, S_{BC}} (\overline{AB}, C).$$

On the other hand, from $(\overline{AC},\overline{AB}) \mathop{\smile\hspace{-0.9em}\mid}^0_{B,C} \overline{BC}$ and Monotonicity of $\mathop{\smile\hspace{-0.9em}\mid}^0$, we have that $\overline{BC} \mathop{\smile\hspace{-0.9em}\mid}^0_{B,C} (\overline{AB},C)$. It follows that $S_{BC} \cap \mathrm{acl}_0(\overline{AB},C) \subseteq S^-_{BC} = \mathrm{acl}_0(S_B,S_C)$ so $S_{BC} \mathop{\smile\hspace{-0.9em}\mid}^0_{S_B,S_C} (\overline{AB},C)$. Using Base Monotonicity of $\mathop{\smile\hspace{-0.9em}\mid}^0$ we have

$$S_{BC} \mathop{\smile\hspace{-0.9em}\mid}^0_{S_B,S_A,S_C} (\overline{AB},C).$$

Now using Transitivity of $\mathop{\smile\hspace{-0.9em}\mid}^0$, we get $(S_{AC},S_{BC}) \mathop{\smile\hspace{-0.9em}\mid}^0_{S_A,S_B,S_C} (\overline{AB},C)$. ☐

Lemma 4.1.6. *Assume that $a \not\mathop{\smile\hspace{-0.9em}\mid}^w_C b$ and $a \mathop{\smile\hspace{-0.9em}\mid}^T_C b$ with $C = \overline{C}$. Then there is a formula $\Lambda(x,b,c) \in tp(a/Cb)$ such that for all sequence $(b_i)_{i<\omega}$ such that*

(1) $b_i \equiv^{TS}_C b$ for all $i < \omega$,

(2) $b_i \mathop{\smile\hspace{-0.9em}\mid}^a_C b_j$ and $S(\mathrm{acl}_0(\overline{Cb_i},\overline{Cb_j})) = \mathrm{acl}_0(S(\overline{Cb_i}),S(\overline{Cb_j}))$ for all $i,j < \omega$,

the partial type $\{\Lambda(x,b_i,c) \mid i < \omega\}$ is inconsistent.

Proof. Let $A = \overline{Ca}$, $B = \overline{Cb}$. As $a \not\mathop{\smile\hspace{-0.9em}\mid}^w_C b$ there exists $s \in S(\mathrm{acl}_0(A,B)) \setminus \mathrm{acl}_0(S(A),S(B))$. As we saw in the proof of Lemma 4.1.4, there exist $u \in A \setminus S(A)$, $v \in B \setminus S(B)$ and $\mathscr{L}_S(C)$-formulae $\psi_u(y,a)$ algebraic in y and $\psi_v(z,b)$ algebraic in z, satisfied respectively by u and v. There is also an $\mathscr{L}_0$-formula $\phi(t,y,z)$ algebraic in t, strict in y and strict in z, such that $s \models \phi(t,u,v)$. Again, as $v \notin \mathrm{acl}_0(S(B),C)$ and $\psi_v(z,b)$ isolates the type $tp^{TS}(v/Cb)$, every v' satisfying $\psi_v(z,b)$ will satisfy $v' \notin \mathrm{acl}_0(S(B),C)$. Let $\Lambda(x,b,c) \in tp^{TS}(a/Cb)$ be the following formula, for a tuple c from C

$$\exists y \exists z \exists t \psi_u(y,x) \wedge \psi_v(z,b) \wedge \phi(t,y,z) \wedge t \in S.$$

As we saw in the proof of Lemma 4.1.4, it witnesses Strong Finite Character over C. Note that if $b' \equiv^{TS}_C b$, then no realization of $\psi_v(y,b')$ is in $\mathrm{acl}_0(S(\overline{Cb'}),C)$.

Now let $(b_i)_{i<\omega}$ be as in the hypothesis. By contradiction, assume that $\{\Lambda(x,b_i,c) \mid i < \omega\}$ is consistent, and realised by some a'. Assume that $\psi_u(t,a')$ does not have more than k distinct realisations. As

$$\bigwedge_{i<k+1} \Lambda(a',b_i,c)$$

is consistent, there is $u' \in \overline{Ca'}$ and $i < j < k+1$ such that v_i, v_j are two realisations of $\psi_v(z,b_i)$ and $\psi_v(z,b_j)$ respectively –we assume $i = 1, j = 2$ for convenience– and such that there exist $s_1 \in \mathrm{acl}_0(u',v_1) \cap S$ and $s_2 \in \mathrm{acl}_0(u',v_2) \cap S$. As $v_2 \notin \mathrm{acl}_0(S(\overline{Cb_2}),C)$ it follows that $v_2 \notin \mathrm{acl}_0(u')$, hence $u' \in \mathrm{acl}_0(s_2,v_2)$ so $s_1 \in \mathrm{acl}_0(s_2,v_1,v_2)$. By modularity, it means that there is some $w \in \mathrm{acl}_0(v_1,v_2)$ such that $s_1 \in \mathrm{acl}_0(s_2,w)$. We have that $w \in \mathrm{acl}_0(s_1,s_2)$, so $w \in \mathrm{acl}_0(v_1,v_2) \cap S$. As $S(\mathrm{acl}_0(\overline{Cb_1}),\mathrm{acl}_0(\overline{Cb_2})) = \mathrm{acl}_0(S(\mathrm{acl}_0(\overline{Cb_1}),S(\mathrm{acl}_0(\overline{Cb_2})))$ there is some $s_1^b \in S(\overline{Cb_1})$ and $s_2^b \in S(\overline{Cb_2})$ such that $w \in \mathrm{acl}_0(s_1^b,s_2^b)$. Now, as $v_1 \notin C$, it follows that $v_1 \notin \mathrm{acl}_0(v_2)$ hence $v_1 \in \mathrm{acl}_0(w,v_2)$, and so $v_1 \in \mathrm{acl}_0(s_1^b,s_2^b,v_2)$. So there is $v_2' \in \mathrm{acl}_0(s_2^b,v_2) \subseteq \overline{Cb_2}$ such that $v_1 \in \mathrm{acl}_0(s_1^b,v_2')$. It follows that $v_2' \in \mathrm{acl}_0(s_1^b,v_1)$ so $v_2' \in \overline{Cb_1} \cap \overline{Cb_2} = C$, hence $v_2' \in C$. Now $v_1 \in \mathrm{acl}_0(S(\overline{Cb_1}),C)$ and this is a contradiction. ☐

Lemma 4.1.7. *Assume that $\mathop{\smile\hspace{-0.9em}\mid}^T$ satisfies the hypothesis of Lemma 4.1.2 and 4.1.4. If $\mathop{\smile\hspace{-0.9em}\mid}^T$ satisfies Witnessing, then so does $\mathop{\smile\hspace{-0.9em}\mid}^w$.*

Proof. Assume that $a \not\mathop{\smile\hspace{-0.9em}\mid}^w_{\mathscr{M}} b$, and let $\Lambda(x,b,m)$ be as in Lemma 4.1.6 and set $p(x) = tp^{TS}(a/\mathscr{M}b)$, $p_{\mathscr{L}} = p \upharpoonright \mathscr{L} = tp^T(a/\mathscr{M}b)$. Let $q(x)$ be a global extension of $tp^{TS}(b/\mathscr{M})$ finitely satisfiable in $\mathscr{M}$, $q_{\mathscr{L}} = q \upharpoonright \mathscr{L}$. It is clear that $q_{\mathscr{L}}$ is finitely satisfiable in $\mathscr{M}$. Let $(b_i)_{i<\omega}$ be a sequence in $\mathbb{M}$ such that $b_i \models q \upharpoonright \mathscr{M}b_{<i}$ for all $i < \omega$. Observe that for $j < i$ we have $tp^{TS}(b_i/\mathscr{M}b_j)$ is finitely satisfiable in $\mathscr{M}$. By hypothesis, $\mathop{\smile\hspace{-0.9em}\mid}^w$ satisfies in particular Symmetry, Monotonicity, Existence, and Strong Finite Character over models, hence by Lemma 1.2.4, $b_i \mathop{\smile\hspace{-0.9em}\mid}^w_{\mathscr{M}} b_j$.

In particular $b_i \mathop{\smile\hspace{-0.9em}\mid}^{a}_{\mathscr{M}} b_j$ and $S(\mathrm{acl}_0(\overline{\mathscr{M} b_i}, \overline{\mathscr{M} b_j})) = \mathrm{acl}_0(S(\overline{\mathscr{M} b_i}), S(\overline{\mathscr{M} b_j}))$ for all $i, j < \omega$. If $\{\Lambda(x, b_i, m) \mid i < \omega\}$ is inconsistent, we conclude. If $\{\Lambda(x, b_i, m) \mid i < \omega\}$ is consistent, by Lemma 4.1.6 we have $a \mathop{\not\smile\hspace{-0.9em}\mid}^{T}_{\mathscr{M}} b$. Now also $b_i \models q_{\mathscr{L}} \restriction \mathscr{M} b_{<i}$, hence as $\mathop{\smile\hspace{-0.9em}\mid}^{T}$ satisfies WITNESSING, we conclude. ☐

Lemma 4.1.8. *Assume that* $\mathop{\smile\hspace{-0.9em}\mid}^{T}$ *satisfies* BASE MONOTONICITY. *The following are equivalent.*

(1) $\mathop{\smile\hspace{-0.9em}\mid}^{w}$ *satisfies* BASE MONOTONICITY*;*

(2) For all algebraically closed sets A, B, C, D *such that* A, B, D *contain* C *and* $A \mathop{\smile\hspace{-0.9em}\mid}^{T}_{C} BD$*, the following holds*

$$\mathrm{acl}_0(A, \overline{BD}) \cup \overline{AD} = \mathrm{acl}_0(\overline{AD}, \overline{BD}).$$

In particular if acl_0 *is trivial or if* $\mathrm{acl}_0 = \mathrm{acl}_T$ *then* $\mathop{\smile\hspace{-0.9em}\mid}^{w}$ *satisfies* BASE MONOTONICITY.

Proof. Assume that there exist A, B, C, D that does not satisfy *(2)*. Let $w \in \mathrm{acl}_0(\overline{AD}, \overline{BD}) \setminus (\mathrm{acl}_0(A, \overline{BD}) \cup \overline{AD})$, and $S_0 := S(\mathrm{acl}_T(\emptyset))$. We define $S_{ABD} = \mathrm{acl}_0(S_0, w)$. The type (over $\emptyset$) defined by the pair $(\overline{ABD}, S_{ABD})$ is consistent. As $S_{ABD} \cap \mathrm{acl}_0(A, \overline{BD}) = S_{ABD} \cap A = S_{ABD} \cap \overline{BD} = S_0$ and $A \mathop{\smile\hspace{-0.9em}\mid}^{T}_{C} BD$ we have that $A \mathop{\smile\hspace{-0.9em}\mid}^{w}_{C} BD$. Now $w \in S_{ABD} \cap \mathrm{acl}_0(\overline{AD}, \overline{BD})$ whereas $S_{ABD} \cap \overline{AD} = S_{ABD} \cap \overline{BD} = S_0$, hence

$$S_0 = \mathrm{acl}_0(S_{ABD} \cap \overline{AD}, S_{ABD} \cap \overline{BD}) \subsetneq S_{ABD} \cap \mathrm{acl}_0(\overline{AD}, \overline{BD}).$$

It follows that $A \mathop{\not\smile\hspace{-0.9em}\mid}^{w}_{D} B$, so $\mathop{\smile\hspace{-0.9em}\mid}^{w}$ doesn't satisfies BASE MONOTONICITY.

Conversely if $\mathop{\smile\hspace{-0.9em}\mid}^{w}$ doesn't satisfies BASE MONOTONICITY, it means that there exist A, B, C, D such that $A \mathop{\smile\hspace{-0.9em}\mid}^{w}_{C} BD$ and $A \mathop{\not\smile\hspace{-0.9em}\mid}^{w}_{CD} B$. We may assume that A, B, D are algebraically closed and contains C. As $\mathop{\smile\hspace{-0.9em}\mid}^{T}$ satisfies BASE MONOTONICITY we have that

$$S(\mathrm{acl}_0(\overline{AD}, \overline{BD})) \supsetneq \mathrm{acl}_0(S(\overline{AD}), S(\overline{BD})).$$

Let w be in $S(\mathrm{acl}_0(\overline{AD}, \overline{BD})) \setminus \mathrm{acl}_0(S(\overline{AD}), S(\overline{BD}))$. As $w \in S$ we have that $w \notin \overline{AD}$ and $w \notin \overline{BD}$. It remains to show that $w \notin \mathrm{acl}_0(A, \overline{BD})$. Assume that $w \in \mathrm{acl}_0(A, \overline{BD})$. As $w \in S$ we have that $w \in S(\mathrm{acl}_0(A, \overline{BD}))$. From $A \mathop{\smile\hspace{-0.9em}\mid}^{w}_{C} BD$ we have that $S(\mathrm{acl}_0(A, \overline{BD})) = \mathrm{acl}_0(S(A), S(\overline{BD}))$ so $w \in \mathrm{acl}_0(S(A), S(\overline{BD}))$ which contradicts that $w \notin \mathrm{acl}_0(S(\overline{AD}), S(\overline{BD}))$. So it follows that $w \in \mathrm{acl}_0(\overline{AD}, \overline{BD}) \setminus (\mathrm{acl}_0(A, \overline{BD}) \cup \overline{AD})$. ☐

4.2 Preservation of NSOP_1

In this section, we use the results of the previous section to prove that if T is NSOP_1 and T satisfies an additional hypothesis then TS is also NSOP_1. This additional hypothesis (namely (A) below) translates how $\mathop{\smile\hspace{-0.9em}\mid}^{0}$ in the reduct T_0 is controlled by $\mathop{\smile\hspace{-0.9em}\mid}^{T}$ in T. We work in the same context as the previous section, with small sets and small models in a monster model for TS, when $(T, \mathscr{L}_0, T_0)$ is a suitable triple.

Theorem 4.2.1. *Assume that* $(T, \mathscr{L}_0, T_0)$ *is a suitable triple. Assume that* T *is* NSOP_1 *and that* $\mathop{\smile\hspace{-0.9em}\mid}^{T}$ *is the Kim-independence relation in* T*. If*

(A) all $\mathscr{M} \models T$ *and* A, B, C *algebraically closed containing* $\mathscr{M}$*, if* $C \mathop{\smile\hspace{-0.9em}\mid}^{T}_{\mathscr{M}} A, B$ *and* $A \mathop{\smile\hspace{-0.9em}\mid}^{T}_{\mathscr{M}} B$ *then*

$$(\overline{AC}, \overline{BC}) \mathop{\smile\hspace{-0.9em}\mid}^{0}_{A,B} \overline{AB}.$$

Then TS *is* NSOP_1 *and the Kim-independence relation in* TS *is given by* $\mathop{\smile\hspace{-0.9em}\mid}^{w}$*, i.e. the relation*

$$A \mathop{\smile\hspace{-0.9em}\mid}^{T}_{\mathscr{M}} B \text{ and } S(\mathrm{acl}_0(\overline{A\mathscr{M}}, \overline{B\mathscr{M}})) = \mathrm{acl}_0(S(\overline{A\mathscr{M}}), S(\overline{B\mathscr{M}})).$$

Proof. From [KR17], if T is NSOP$_1$ the Kim-independence $\mathop{\smile\hspace{-0.9em}\vert}^{T}$ satisfies Invariance, Symmetry, Monotonicity, Existence and Strong Finite Character all over models. Furthermore, by [KR18, Theorem 2.21], it also satisfies $\mathop{\smile\hspace{-0.9em}\vert}^{T}$-amalgamation over models. By Lemmas 4.1.2, 4.1.4 and Theorem 4.1.5, all these properties are also satisfied over models by $\mathop{\smile\hspace{-0.9em}\vert}^{w}$ (relatively to the theory TS). By Proposition 5.3 in [CR16], TS is NSOP$_1$. As $\mathop{\smile\hspace{-0.9em}\vert}^{T}$ satisfies Witnessing, so does $\mathop{\smile\hspace{-0.9em}\vert}^{w}$ by Lemma 4.1.7. Using [KR17, Theorem 9.1] (and [KR17, Remark 9.2]), it follows that $\mathop{\smile\hspace{-0.9em}\vert}^{w}$ and Kim-independence in TS coincide over models. ☐

The results of the previous section give more than the previous Theorem. Indeed, most of the nice features that may happen in T for $\mathop{\smile\hspace{-0.9em}\vert}^{T}$ are preserved when expanding T to TS. For instance, if $\mathop{\smile\hspace{-0.9em}\vert}^{T}$ is defined over every small base set, so is $\mathop{\smile\hspace{-0.9em}\vert}^{w}$. If the independence theorem in T is satisfied by $\mathop{\smile\hspace{-0.9em}\vert}^{T}$ not only over models but over a wider class of small sets then the same holds in TS for $\mathop{\smile\hspace{-0.9em}\vert}^{w}$. We summarize these features in the next result.

Theorem 4.2.2. *Assume that $(T, \mathscr{L}_0, T_0)$ is a suitable triple. Assume that there is a ternary relation $\mathop{\smile\hspace{-0.9em}\vert}^{T}$ over small sets of a monster model of T that satisfies*

- Invariance*;*
- Symmetry*;*
- Closure*;*
- Monotonicity*;*
- Existence*;*
- Full Existence*;*
- Strong Finite Character *over E for $E = \overline{E}$;*
- $\mathop{\smile\hspace{-0.9em}\vert}'$-amalgamation *over E for $E = \overline{E}$, where $\mathop{\smile\hspace{-0.9em}\vert}'$ is such that $\mathop{\smile\hspace{-0.9em}\vert}^{T} \to \mathop{\smile\hspace{-0.9em}\vert}' \to \mathop{\smile\hspace{-0.9em}\vert}^{a}$ and $\mathop{\smile\hspace{-0.9em}\vert}'$ satisfies* Monotonicity, Symmetry *and* Closure*;*

(A) *For $E = \overline{E}$ and A, B, C algebraically closed containing E, if $C \mathop{\smile\hspace{-0.9em}\vert}^{T}_{E} A, B$ and $A \mathop{\smile\hspace{-0.9em}\vert}^{T}_{E} B$ then $\overline{AC} \mathop{\smile\hspace{-0.9em}\vert}^{0}_{C} \overline{BC}$ and*

$$(\overline{AC}, \overline{BC}) \mathop{\smile\hspace{-0.9em}\vert}^{0}_{A,B} \overline{AB};$$

- Witnessing.

(In particular T is NSOP$_1$*, and $\mathop{\smile\hspace{-0.9em}\vert}^{T}$ coincide with Kim-independence over models of T, by [CR16, Proposition 5.3] and [KR17, Theorem 9.1]).*

Then any completion of TS is NSOP$_1$ *and $\mathop{\smile\hspace{-0.9em}\vert}^{w}$ and the Kim-forking independence relation in TS coincide over models. Furthermore $\mathop{\smile\hspace{-0.9em}\vert}^{w}$ satisfies all these properties, relatively to the theory TS.*

Finally, using [KR17, Proposition 8.8] we give a condition on $(T, T_0, \mathscr{L}_0)$ that characterizes the simplicity of TS, assuming that T satisfies the hypotheses of Theorem 4.2.2.

Corollary 4.2.3. *Let $(T, \mathscr{L}_0, T_0)$ be a suitable triple satisfying all the assumptions of Theorem 4.2.2. The following are equivalent.*

(1) *Any completion of TS is not simple*

(2) *T is not simple or there exist algebraically closed sets A, B, C, D such that A, B, D contain C and $A \mathop{\smile\!\!\!\!\!\!\mid}^T_C BD$, and such that*

$$\mathrm{acl}_0(A, \overline{BD}) \cup \overline{AD} \neq \mathrm{acl}_0(\overline{AD}, \overline{BD}).$$

In particular if acl_0 is trivial or if $\mathrm{acl}_0 = \mathrm{acl}_T$ the theory TS is simple if and only if T is simple. If TS is simple, $\mathop{\smile\!\!\!\!\!\!\mid}^w$ is forking independence over models.

Proof. From Theorem 4.2.2, we know that the relation $\mathop{\smile\!\!\!\!\!\!\mid}^w$ is Kim-independence over models. By [KR17, Proposition 8.8], TS is simple if and only if $\mathop{\smile\!\!\!\!\!\!\mid}^w$ satisfies Base Monotonicity. The equivalence follows from Lemma 4.1.8. The fact that Kim-independence and forking independence coincide is [KR17, Proposition 8.4]. ☐

Corollary 4.2.4. *Assume that T is a complete $\mathscr{L}$-theory and $\mathscr{L}_1, \ldots, \mathscr{L}_n$ are sublanguages of $\mathscr{L}$. Let $T_1 = T \restriction \mathscr{L}_1, \ldots, T_n = T \restriction \mathscr{L}_n$ such that $(TS_1 \ldots S_i, T_{i+1}, \mathscr{L}_{i+1})$ is a suitable triple for each $i = 0, \cdots, n-1$. By Proposition 2.4.1, let $TS_1 \ldots S_n$ be the model companion of the theory of models of T with a predicate S_i for an $\mathscr{L}_i$ substructure.*

(1) *Assume that T is NSOP_1, with Kim-independence $\mathop{\smile\!\!\!\!\!\!\mid}^T$ in T and that for all i we have (for A, B, C algebraically closed containing $\mathscr{M} \models T$)*

$$\text{if } C \mathop{\smile\!\!\!\!\!\!\mid}^T_{\mathscr{M}} A, B \text{ and } A \mathop{\smile\!\!\!\!\!\!\mid}^T_{\mathscr{M}} B \text{ then } (\overline{AC}, \overline{BC}) \mathop{\smile\!\!\!\!\!\!\mid}^i_{A,B} \overline{AB}.$$

Then $TS_1 \ldots S_n$ is NSOP_1 and Kim-independence in TS is given by

$$A \mathop{\smile\!\!\!\!\!\!\mid}^T_{\mathscr{M}} B \text{ and for all } i \leq n \ S_i(\mathrm{acl}_i(\overline{A\mathscr{M}}, \overline{B\mathscr{M}})) = \mathrm{acl}_i(S_i(\overline{A\mathscr{M}}), S_i(\overline{B\mathscr{M}}))$$

(for acl_i, $\mathop{\smile\!\!\!\!\!\!\mid}^i$ the algebraic closure and independence in the sense of the pregeometric theory T_i).

(2) *If there exists $\mathop{\smile\!\!\!\!\!\!\mid}^T$ that satisfies the hypotheses of Theorem 4.2.2 (relatively to each theory T_i), then $TS_1 \ldots S_n$ is NSOP_1 and the relation*

$$A \mathop{\smile\!\!\!\!\!\!\mid}^T_C B \text{ and for all } i \leq n \ S_i(\mathrm{acl}_i(\overline{AC}, \overline{BC})) = \mathrm{acl}_i(S_i(\overline{AC}), S_i(\overline{BC}))$$

agrees with Kim-independence over models. Furthermore this relation satisfies all the properties listed in Theorem 4.2.2.

4.3 Mock stability and stability

We keep the same hypotheses on T and $\mathop{\smile\!\!\!\!\!\!\mid}^T$ as in the previous section. Mock stability is a notion introduced in [Adl08a] by Adler.

Definition 4.3.1. A theory T is *mock stable* if there is a relation satisfying Invariance, Finite Character, Closure, Symmetry, Monotonicity, Base Monotonicity, Transitivity, Full Existence, Stationnarity over models.

Remark 4.3.2. In the original definition of mock stability ([Adl08a]), Adler asks for slightly different properties but as in the proof of Fact 1.4.4, it is easy to see that our set of properties is equivalent to the one in [Adl08a].

Lemma 4.3.3. *Assume that $\mathop{\smile\!\!\!\!\!\!\mid}^T$ satisfies Invariance, Finite Character, Symmetry, Closure, Monotonicity, Base Monotonicity, Transitivity, Full Existence then so does $\mathop{\smile\!\!\!\!\!\!\mid}^{st}$. Furthermore, for any $E = \overline{E}$, if $\mathop{\smile\!\!\!\!\!\!\mid}^T$ satisfies Stationnarity over $E = \overline{E}$, so does $\mathop{\smile\!\!\!\!\!\!\mid}^{st}$. In particular if T is mock stable, so is TS.*

Proof. INVARIANCE, FINITE CHARACTER, SYMMETRY, CLOSURE are trivial. FULL EXISTENCE is Lemma 4.1.3. It remains to show MONOTONICITY, BASE MONOTONICITY, TRANSITIVITY and STATIONNARITY over algebraically closed sets.

MONOTONICITY. Assume that $A \mathop{\smile\hskip-0.9em\vert}\nolimits^{st}_{C} BD$. We only need to check that $S(\overline{ABC}) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC})$. We have

$$\begin{aligned} S(\overline{ABC}) &= \mathrm{acl}_0(S(\overline{AC}), S(\overline{BCD})) \cap \overline{ABC} \\ &= \mathrm{acl}_0(S(\overline{AC}), S(\overline{BCD}) \cap \overline{ABC} && \text{by modularity)} \\ &= \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC})) && \text{as } \overline{BCD} \cap \overline{ABC} = \overline{BC}\ (\mathop{\smile\hskip-0.9em\vert}\nolimits^{T} \to \mathop{\smile\hskip-0.9em\vert}\nolimits^{a}). \end{aligned}$$

BASE MONOTONICITY. If $A \mathop{\smile\hskip-0.9em\vert}\nolimits^{st}_{C} BD$ then by BASE MONOTONICITY of $\mathop{\smile\hskip-0.9em\vert}\nolimits^{T}$ we have $A \mathop{\smile\hskip-0.9em\vert}\nolimits^{T}_{CD} B$. As $S(\overline{ABCD}) = \mathrm{acl}_0(S(\overline{CA}), S(\overline{CBD})$, in particular $S(\overline{ABCD}) \subseteq \mathrm{acl}_0(S(\overline{ACD}), S(\overline{BCD}) \subseteq S(\overline{ABCD})$, so $A \mathop{\smile\hskip-0.9em\vert}\nolimits^{st}_{CD} B$.

TRANSITIVITY. Assume that $A \mathop{\smile\hskip-0.9em\vert}\nolimits^{st}_{CB} D$ and $B \mathop{\smile\hskip-0.9em\vert}\nolimits^{st}_{C} D$. By CLOSURE, we may assume that $A = \overline{ABC}, B = \overline{BC}, D = \overline{CD}$. By MONOTONICITY, it is sufficient to show that $A \mathop{\smile\hskip-0.9em\vert}\nolimits^{st}_{C} D$. We have $A \mathop{\smile\hskip-0.9em\vert}\nolimits^{T}_{C} D$ by TRANSITIVITY of $\mathop{\smile\hskip-0.9em\vert}\nolimits^{T}$. We show that $S(\overline{AD}) = \mathrm{acl}_0(S(A), S(D)$. By $A \mathop{\smile\hskip-0.9em\vert}\nolimits^{st}_{B} D$ we have $S(\overline{AD}) = \mathrm{acl}_0(S(A), S(\overline{BD}))$. By $B \mathop{\smile\hskip-0.9em\vert}\nolimits^{st}_{C} D$, $S(\overline{BD}) = \mathrm{acl}_0(S(B), S(D))$ hence $S(\overline{AD}) = \mathrm{acl}_0(S(A), S(B), S(D)) = \mathrm{acl}_0(S(A), S(D)$.

STATIONNARITY. Assume that $c_1 \mathop{\smile\hskip-0.9em\vert}\nolimits^{st}_{E} A$ and $c_2 \mathop{\smile\hskip-0.9em\vert}\nolimits^{st}_{E} A$ and $c_1 \equiv^{TS}_{E} c_2$. We may assume that A is algebraically closed and contains E. There is a T-elementary S-preserving map $f : \overline{Ec_1} \to \overline{Ec_2}$ over E. By STATIONNARITY over E, we can extend f to $\tilde{f} : \overline{Ac_1} \to \overline{Ac_2}$ T-elementary over A. But as $S(\overline{Ac_1})) = \mathrm{acl}_0(S(\overline{Ec_1}), S(A))$ and $S(\overline{Ac_2}) = \mathrm{acl}_0(S(\overline{Ec_2}), S(A))$, $\tilde{f}$ preserves S, so $c_1 \equiv^{TS}_{B} c_2$. ☐

Proposition 4.3.4. *If T is stable and $\mathrm{acl}_0 = \mathrm{acl}_T$, then the theory TS is stable.*

Proof. By Corollary 4.2.3, TS is simple and $\mathop{\smile\hskip-0.9em\vert}\nolimits^{w}$ is the forking independence, in particular it satisfies LOCAL CHARACTER. As $\mathrm{acl}_T = \mathrm{acl}_0$ it follows that $\mathop{\smile\hskip-0.9em\vert}\nolimits^{st} = \mathop{\smile\hskip-0.9em\vert}\nolimits^{w}$, hence as $\mathop{\smile\hskip-0.9em\vert}\nolimits^{T}$ is stationnary over models, so is $\mathop{\smile\hskip-0.9em\vert}\nolimits^{w}$ by Lemma 4.3.3. Hence TS is stable by Fact 1.4.4. Note that the fact that forking independence is stationnary over models gives directly the stability. ☐

Remark 4.3.5. Assume that T is stable and that acl_0 is trivial, then TS is not necessary stable. From Corollary 4.2.3, TS is simple and $\mathop{\smile\hskip-0.9em\vert}\nolimits^{w}$ is forking independence. As acl_0 is trivial, we have $\mathop{\smile\hskip-0.9em\vert}\nolimits^{w} = \mathop{\smile\hskip-0.9em\vert}\nolimits^{T}$, (with $\mathop{\smile\hskip-0.9em\vert}\nolimits^{T}$ forking independence in T) which is not likely to be stationnary. The easiest example of a reduct T_0 for which acl_0 is trivial is the particular case of $\mathscr{L}_0 = \{=\}$. Then TS is the theory of the generic predicate on T (see Remark 2.1.6 and [CP98]), which does not preserve stability. Indeed [CP98, (2.10) Proposition, Errata] gives a sufficient condition on T so that TS have the independence property (hence is unstable): there exists a model $\mathscr{M}$ of T and two elements a and b such that $b \mathop{\smile\hskip-0.9em\vert}\nolimits^{u}_{\mathscr{M}} a$ and $\overline{\mathscr{M}ab} \neq \overline{\mathscr{M}a} \cup \overline{\mathscr{M}b}$. It follows that adding a generic predicate to an algebraically closed field result in a simple unstable theory (take a and b two generics independent over $\mathscr{M}$).

Example 4.3.6. We saw in Example 3.1.7 that the generic theory $TV_1 \cdots V_n$ of infinite $\mathbb{F}_q$-vector spaces with predicates for $\mathbb{F}_q$-vector subspaces $V_1, \cdots, V_n$ is stable for $n = 1$ as it is the theory of a belle paire of infinite $\mathbb{F}_q$-vector space. Proposition 4.3.4 gives us inductively that $TV_1 \cdots V_n$ is stable for all $n \in \mathbb{N}$.

Example 4.3.7. Assume that T is a model-complete geometric theory such that every acl_T-closed set is a model of T (Proposition 3.4.1). If T is stable, then the weak model-companion of the pairs of models of T is stable.

4.4 NSOP_1 expansions of fields

4.4.1 Fields with generic additive subgroups

In this section, we give some condition under which the theory obtained in Proposition 3.2.1 is NSOP_1. In this section, for A in some field, we denote by acl_T the model-theoretic algebraic closure, A^s the separable closure and $\overline{A}$ the field theoretic algebraic closure.

Theorem 4.4.1. *Let T be a model-complete theory of an NSOP_1 field that eliminates $\exists^\infty$ and let $\mathbb{F}_{q_1}, \cdots, \mathbb{F}_{q_n}$ be subfields. Assume that T satisfies the following assumption for all acl_T-closed A, B and $E \models T$ contained in A and B:*

$$\text{if } A \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{T}_{E} B \text{ then } \mathrm{acl}_T(AB) \subseteq \overline{AB}.$$

Then $TV_1 \dots V_n$ is NSOP_1 and Kim-independence in $TV_1 \dots V_n$ is given by

$$A \mathop{\smile\hspace{-0.9em}\mid}\limits^{w}_{E} B \iff A \mathop{\smile\hspace{-0.9em}\mid}\limits^{T}_{E} B \text{ and for all } i \leq n \ V_i(A+B) = V_i(A) + V_i(B)$$

(for A, B, C acl_T-closed, A, B containing E, $E \models T$).

Proof. We prove that $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{T}$ satisfies the conditions of Corollary 4.2.4. Let $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{i}$ the independence in the sense of $\mathbb{F}_{q_i}$-vector space, we want to show that for all $i = 1, \dots, n$,

(A) for all model E of T and A, B, C algebraically closed containing E, if $C \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{T}_{E} A, B$ and $A \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{T}_{E} B$ then
$$(\mathrm{acl}_T(AC), \mathrm{acl}_T(BC)) \mathop{\smile\hspace{-0.9em}\mid}\limits^{i}_{A,B} \mathrm{acl}_T(AB).$$

Let $F \models T$, let $E \prec F$ and A, B, C in F containing E, with $C \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{T}_{E} A, B$ and $A \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{T}_{E} B$. For all $i = 1, \cdots, n$, the condition $(\mathrm{acl}_T(AC), \mathrm{acl}_T(BC)) \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{i}_{A,B} \mathrm{acl}_T(AB)$ is equivalent to

$$(\mathrm{acl}_T(AC) + \mathrm{acl}_T(BC)) \cap \mathrm{acl}_T(AB) = A + B.$$

From Fact 1.5.10 *(2)*, F/AB, F/BC and F/AC are separable extension. By our assumptions on T and A, B and C we have that $\mathrm{acl}_T(AB) \subseteq (AB)^s$, $\mathrm{acl}_T(AC) \subseteq (AC)^s$ and $\mathrm{acl}_T(BC) \subseteq (BC)^s$, so
$$(\mathrm{acl}_T(AC) + \mathrm{acl}_T(BC)) \cap \mathrm{acl}_T(AB) \subseteq ((AC)^s + (BC)^s) \cap (AB)^s.$$

Claim. $((AC)^s + (BC)^s) \cap (AB)^s = A^s + B^s$

Proof of the claim. First, observe that as fields, E^s is an elementary substructure of F^s. Indeed, by model completeness of $Th(E^s)$ (which is $\mathrm{SCF}_{p,e}$ for some $e \leq \infty$, see Subsection 1.5.2) we have to check that they have the same imperfection degree (which is clear as $F \succ E$) and that F^s/E^s is separable (the later follows from the fact that F/E is a regular extension). Now by Fact 1.5.10 *(1)* we have $C \mathop{\smile\hspace{-0.9em}\mid}\nolimits^{ld}_{E} AB$. As E is a model, C/E and AB/E are regular extensions[1], by Fact 1.5.6 we have that
$$C^s \mathop{\smile\hspace{-0.9em}\mid}\limits^{ld}_{E^s} (AB)^s. \quad (*)$$

Moreover F^s/ABC is separable, (as so are F^s/F and F/ABC, the latter using Fact 1.5.10 *(2)*) and so is $C^s(AB)^s/ABC$. It follows that the following extension is separable

$$F^s/C^s(AB)^s. \quad (**)$$

From $(*)$ and $(**)$, using Fact 1.5.14 we have that $tp_{\mathrm{SCF}}(C^s/(AB)^s)$ does not fork over E^s. By stability, as E^s is an elementary submodel of the ambiant model F^s of $\mathrm{SCF}_{p,e}$, $tp_{\mathrm{SCF}}(C^s/(AB)^s)$

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[1] In fact here we only use that $E = \mathrm{acl}_T(E)$, and Fact 1.5.9.

is a coheir of $tp_{\mathrm{SCF}}(C^s/E^s)$ (Fact 1.4.5). From Lemma 1.5.11, it follows that $((AC)^s+(BC)^s)\cap(AB)^s = A^s+B^s$.

By the claim $(\mathrm{acl}_T(AC)+\mathrm{acl}_T(BC))\cap\mathrm{acl}_T(AB)\subseteq(A^s+B^s)\cap\mathrm{acl}_T(AB)$. Now by Fact 1.5.10 *(3)*, we have that $A^sB^s\cap\mathrm{acl}_T(AB)=AB$ so $(A^s+B^s)\cap\mathrm{acl}_T(AB)\subseteq(A^s+B^s)\cap AB$. Finally, by Lemma 1.5.7, as AB/E is regular and $A\perp^{ld}_E B$, we have $(A^s+B^s)\cap AB=A+B$. □

Proposition 4.4.2. *Let T be a theory of fields satisfying the same hypotheses as Theorem 4.4.1. Then $TV_1\cdots V_n$ is not simple.*

Proof. To prove that $TV_1\cdots V_n$ is not simple, it is sufficient to prove that TV is not simple. Let $E\prec F$ be models of T and a,b,d elements of F be such that $a\perp^T_E b,d$ and $b\perp^T_E d$. We show that

$$ad+b\in[\mathrm{acl}_T(Ead)+\mathrm{acl}_T(Ebd)]\setminus[(\mathrm{acl}_T(Ea)+\mathrm{acl}_T(Ebd))\cup\mathrm{acl}_T(Ead)],$$

then TV is not simple, byy Corollary 4.2.3. Since $b\notin\mathrm{acl}_T(Ead)$, it is clear that $ad+b\notin\mathrm{acl}_T(Ead)$. Assume that $ad+b\in\mathrm{acl}_T(Ea)+\mathrm{acl}_T(Ebd)$. Then $ad\in\mathrm{acl}_T(Ea)+\mathrm{acl}_T(Ebd)$, let $u\in\mathrm{acl}_T(Ea)$ and $v\in\mathrm{acl}_T(Ebd)$ be such that $ad=u+v$. From Fact 1.5.10, we have that $\mathrm{acl}_T(Ea)\perp^{ld}_E\mathrm{acl}_T(Ebd)$, hence $\mathrm{acl}_T(Ea)(d)\perp^{ld}_{E(d)}\mathrm{acl}_T(Ebd)$ so $\mathrm{acl}_T(Ea)(d)\cap\mathrm{acl}_T(Ebd)=E(d)$. Similarly, $\mathrm{acl}_T(Ebd)(a)\cap\mathrm{acl}_T(Ea)=E(a)$. It follows that

$$u=ad-v\in\mathrm{acl}_T(Ebd)(a)\cap\mathrm{acl}_T(Ea)=E(a)$$
$$v=ad-u\in\mathrm{acl}_T(Ea)(d)\cap\mathrm{acl}_T(Ebd)=E(d)$$

hence $ad\in E(a)+E(b)$, which contradicts Lemma 1.5.8. □

Example 4.4.3 (The theories ACFV$_1\dots$V$_n$ and ACFG)**.** Let ACFV$_1\dots$V$_n$ and ACFG be the theories as in Example 3.2.4. By Theorem 4.4.1 and Proposition 4.4.2 those theories are NSOP$_1$ not simple. In ACFV$_1\dots$V$_n$, Kim-independence agrees with the relation

$$A\perp^w_C B\iff A\perp^{\mathrm{ACF}}_C B \text{ and for all } i\le n,\ V_i(\overline{AC}+\overline{BC})=V_i(\overline{AC})+V_i(\overline{BC}).$$

Furthermore, $\perp^w$ satisfies

- Strong Finite Character over algebraically closed sets. For algebraically closed E, if $a\not\perp^w_E b$, then there is a formula $\phi(x,b,e)\in tp^{ACFV_1\dots V_n}(a/bE)$ such that for all a', if $a'\models\phi(x,b,e)$ then $a'\not\perp^w_E b$.
- $\perp^a$-amalgamation over algebraically closed sets. For algebraically closed set E if there exists tuples c_1,c_2 and sets A,B such that
  - $c_1\equiv^{ACFV_1\dots V_n}_E c_2$
  - $\overline{AE}\cap\overline{BE}=E$
  - $c_1\perp^w_E A$ and $c_2\perp^w_E B$

  then there exists $c\perp^w_E A,B$ such that $c\equiv^{ACFV_1\dots V_n}_A c_1$, $c\equiv^{ACFV_1\dots V_n}_B c_2$, $A\perp^a_{Ec}B$, $c\perp^a_{EA}B$ and $c\perp^a_{EB}A$.

This is Theorem 4.2.2, knowing that $\perp^{\mathrm{ACF}}$ is stationary over algebraically closed sets hence satisfies the independence theorem over algebraically closed sets without any assumption on the parameters.

Example 4.4.4. Perfect ω-free PAC$_p$ fields are NSOP$_1$ (see Subsection 1.5.2), furthermore, as they are algebraically bounded, the condition on the algebraic closure in Theorem 4.4.1 is satisfied. If T is a theory of a perfect ω-free PAC$_p$-field in an expansion of the language $\mathscr{L}_{\mathrm{ring}}$ such that T is model-complete, then $TG_1\cdots G_n$ (Proposition 3.2.5) is NSOP$_1$. This holds of course for any NSOP$_1$ perfect PAC$_p$ field.

4.4.2 Algebraically closed fields with a generic multiplicative subgroup

Let $\mathrm{ACFG}^\times$ be the theory obtained in Theorem 3.3.5. We denote by $A \cdot B$ the product set $\{a \cdot b \mid a \in A,\ b \in B\}$.

Theorem 4.4.5. *Any completion of* $\mathrm{ACFG}^\times$ *is* NSOP_1 *and not simple. Furthermore, Kim-independence coincide over models with the relation*

$$A \mathop{\smile\!\!\!\!\!\!|}\limits^{w}_{C} B \iff A \mathop{\smile\!\!\!\!\!\!|}\limits^{\mathrm{ACF}}_{C} B \text{ and } G^\times(\overline{AC} \cdot \overline{BC}) = G^\times(\overline{AC}) \cdot G^\times(\overline{BC}).$$

Furthermore, $\mathop{\smile\!\!\!\!\!\!|}^{w}$ *satisfies*

- STRONG FINITE CHARACTER OVER ALGEBRAICALLY CLOSED SETS. *For algebraically closed* E*, if* $a \not\mathop{\smile\!\!\!\!\!\!|}^{w}_{E} b$*, then there is a formula* $\phi(x, b, e) \in tp^{\mathrm{ACFG}^\times}(a/bE)$ *such that for all* a'*, if* $a' \models \phi(x, b, e)$ *then* $a' \not\mathop{\smile\!\!\!\!\!\!|}^{w}_{E} b$*.*
- $\mathop{\smile\!\!\!\!\!\!|}^{a}$-AMALGAMATION OVER ALGEBRAICALLY CLOSED SETS. *For algebraically closed set* E *if there exists tuples* c_1, c_2 *and sets* A, B *such that*
  - $c_1 \equiv^{\mathrm{ACFG}^\times}_{E} c_2$
  - $\overline{AE} \cap \overline{BE} = E$
  - $c_1 \mathop{\smile\!\!\!\!\!\!|}^{w}_{E} A$ *and* $c_2 \mathop{\smile\!\!\!\!\!\!|}^{w}_{E} B$

  then there exists $c \mathop{\smile\!\!\!\!\!\!|}^{w}_{E} A, B$ *such that* $c \equiv^{\mathrm{ACFG}^\times}_{A} c_1$*,* $c \equiv^{\mathrm{ACFG}^\times}_{B} c_2$*,* $A \mathop{\smile\!\!\!\!\!\!|}^{a}_{Ec} B$*,* $c \mathop{\smile\!\!\!\!\!\!|}^{a}_{EA} B$ *and* $c \mathop{\smile\!\!\!\!\!\!|}^{a}_{EB} A$*.*

Proof. Using Theorem 4.2.1, it is enough to show that for E algebraically closed and A, B, C algebraically closed containing E, if $C \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{E} A, B$ and $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{E} B$ then

$$\overline{AC} \cdot \overline{BC} \cap \overline{AB} = A \cdot B.$$

This easily follows from the fact that $tp^{\mathrm{ACF}}(C/AB))$ is finitely satisfiable in E, as in the proof of Theorem 4.4.1. The rest is Theorem 4.2.2, knowing that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}$ is stationnary over algebraically closed sets, similarly to Example 4.4.3. To prove that $\mathrm{ACFG}^\times$ is not simple, we use Corollary 4.2.3, as in the proof of Proposition 4.4.2. Let E be a model of ACF_p and a, b, d in an extension be such that $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{E} b, d$ and $b \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{E} d$. We claim that

$$(a+d)b \in \left[\overline{Ead} \cdot \overline{Ebd}\right] \setminus \left[(\overline{Ea} \cdot \overline{Ebd}) \cup \overline{Ead}\right].$$

Since $b \notin \overline{Ead}$, it is clear that $(a+d)b \notin \overline{Ead}$. Assume that $(a+d)b \in \overline{Ea} \cdot \overline{Ebd}$. Then $a + d \in \overline{Ea} \cdot \overline{Ebd}$, let $u \in \overline{Ea}$ and $v \in \overline{Ebd}$ be such that $a + d = uv$. We have that $\overline{Ea} \mathop{\smile\!\!\!\!\!\!|}^{ld}_{E} \overline{Ebd}$, hence $\overline{Ea}(d) \mathop{\smile\!\!\!\!\!\!|}^{ld}_{E(d)} \overline{Ebd}$ so $\overline{Ea}(d) \cap \overline{Ebd} = E(d)$. Similarly, $\overline{Ebd}(a) \cap \overline{Ea} = E(a)$. It follows that

$$\begin{aligned} u &= (a+d)v^{-1} \in \overline{Ebd}(a) \cap \overline{Ea} = E(a) \text{ and} \\ v &= (a+d)u^{-1} \in \overline{Ea}(d) \cap \overline{Ebd} = E(d) \end{aligned}$$

hence $a + d \in E(a) \cdot E(d)$, which contradicts Lemma 1.5.8. □

UN CADET, *chantonnant d'un air détaché.*

To lo lo!...

DE GUICHE, *s'arrêtant et le regardant.*

Qu'avez-vous, vous?... Vous êtes tout rouge!

LE CADET

Moi?... Mais rien. C'est le sang. On va se battre : il bouge!

CHAPTER 5

A study of ACFG

Let $p > 0$ be a fixed prime number. Unless stated otherwise, every field we consider has characteristic p. Let $\mathscr{L}_{\text{ring}}$ be the language of rings and $\mathscr{L}_G = \mathscr{L}_{\text{ring}} \cup \{G\}$ for G a unary predicate. Let ACF_G be the $\mathscr{L}_G$-theory whose models are algebraically closed fields of characteristic p in which G is a predicate for an additive subgroup. Let ACFG be the model companion of ACF_G, see Examples 3.2.4 and 4.4.3. In this chapter, we give a basic study of the theory ACFG. First, we give a precise description of the Kim-independence. Then we investigate some algebraic properties of any models. Finally, we construct inductively a model inside $\overline{\mathbb{F}_p}$ and prove that such models are numerous in $\overline{\mathbb{F}_p}$.

Contents

5.1 Preliminaries, axioms and types

The following is Proposition 3.1.4.

Proposition 5.1.1 (Axiomatisation of ACFG)**.** *The theory* ACFG *is axiomatised by adding to* ACF_G *the following* $\mathscr{L}_G$*-sentences, for all tuples of variables* $x' \subset x$*,* $y' \subset y$ *and* $\mathscr{L}_{\mathrm{ring}}$*-formula* $\phi(x, y)$

$$\forall y(\langle y'\rangle \cap G = \{0\} \wedge \theta_\phi(y)) \to (\exists x\phi(x, y) \wedge \langle xy'\rangle \cap G = \langle x'\rangle),$$

where $\theta_\phi(y)$ *such that* $K \models \theta_\phi(b)$ *if and only if in an elementary extension of* K*, there exists a tuple of realisations of* $\phi(x, b)$ *which is* $\mathbb{F}_p$*-linearly independent over* K *(see Theorem 3.1.2).*

By Proposition 2.3.5 we have the following, for $(K, G) \models$ ACFG sufficiently saturated, and a, b, C in K

(1) $\mathrm{acl}_{\mathrm{ACFG}}(C) = \mathrm{acl}_{\mathrm{ACF}}(C) =: \overline{C}$;

(2) $a \equiv_C b$ if and only if there exists an $\mathscr{L}_G$-isomorphism $\sigma : \overline{Ca} \to \overline{Cb}$ over C such that $\sigma(a) = b$;

(3) the completions of ACFG are given by the $\mathscr{L}_G$-isomorphism type of $(\overline{\mathbb{F}_p}, G(\overline{\mathbb{F}_p}))$.

Let x be a tuple from a field extension of K and H be an additive subgroup of the field $\overline{Cx}$. If

$$\overline{Cx} \cap K = \overline{C} \text{ and } H \cap \overline{C} = G(\overline{C})$$

then, by Proposition 2.3.6, the type associated to the $\mathscr{L}_G$-isomorphism class of the pair $(\overline{Cx}, H)$ is consistent in (K, G), i.e. there exists a tuple a from K such that there is a $\mathscr{L}_G$-isomorphism over C

$$f : (\overline{Ca}, G(\overline{Ca})) \to (\overline{Cx}, H)$$

with $f(a) = x$.

Example 5.1.2 (Empty types)**.** Let (K, G) be a κ-saturated model of ACFG, $C \subseteq K$ such that $|C| < \kappa$ and x a finite tuple algebraically independent over K. By previously, the type associated to the pair $(\overline{Cx}, G(\overline{C}))$ is consistent. Hence there is some tuple a from K, algebraically independent over C such that $G(\overline{Ca}) = G(\overline{C})$. This type is unique if $G(\overline{C}) \subseteq C$: let a and a' realise this type, meaning that $G(\overline{Ca}) = G(\overline{Ca'}) = G(\overline{C})$. Then $a \equiv_C a'$. Indeed if σ is a field isomorphism over C between $\overline{Ca}$ and $\overline{Ca'}$, then it fixes $G(\overline{C})$ so it is an $\mathscr{L}_G$-isomorphism. The type is unique in particular if C is algebraically closed. This uniqueness is a special case of the stationarity of the strong independence (cf. Lemma 4.3.3).

5.2 Independence relations in (K, G)

We work in a monster model (K, G) of ACFG.

Definition 5.2.1 (Weak and strong independence)**.** Let A, B, C be subsets of K. Let $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}$ be the forking independence in the sense of ACF. Recall the *weak independence relation*:

$$A \mathop{\smile\!\!\!\!\!\!|}\limits_C^{w} B \text{ if and only if } A \mathop{\smile\!\!\!\!\!\!|}\limits_C^{\mathrm{ACF}} B \text{ and } G(\overline{AC} + \overline{BC}) = G(\overline{AC}) + G(\overline{BC}),$$

and the *strong independence relation*:

$$A \mathop{\smile\!\!\!\!\!\!|}\limits_C^{st} B \text{ if and only if } A \mathop{\smile\!\!\!\!\!\!|}\limits_C^{\mathrm{ACF}} B \text{ and } G(\overline{ABC}) = G(\overline{AC}) + G(\overline{BC}).$$

Theorem 5.2.2. *The relation* $\mathop{\smile\hspace{-0.9em}\mid}^{w}$ *satisfies* Invariance, Closure, Symmetry, Full Existence, Monotonicity, Existence, Local Character, Transitivity, Strong Finite Character *over algebraically closed sets,* $\mathop{\smile\hspace{-0.9em}\mid}^{a}$-amalgamation *over algebraically closed sets.*

Proof. Apart from Transitivity and Local Character, all properties has been proven in Theorem 4.2.2 and Example 4.4.3.

Transitivity. Assume that $A \mathop{\smile\hspace{-0.9em}\mid}^{w}_{CB} D$ and $B \mathop{\smile\hspace{-0.9em}\mid}^{w}_{C} D$. We may assume that $A = \overline{ABC}, B = \overline{CB}$ and $D = \overline{CD}$. By Monotonicity, it is sufficient to show that $A \mathop{\smile\hspace{-0.9em}\mid}^{w}_{C} D$. We clearly have $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{ACF}}_{C} D$ by Transitivity of $\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{ACF}}$. We show that $G(A+D) = G(A) + G(D)$. By $A \mathop{\smile\hspace{-0.9em}\mid}^{w}_{B} D$ we have $G(A + \overline{BD}) = G(A) + G(\overline{BD})$. It follows that $G(A+D)$ is included in $(A+D) \cap (G(A) + G(\overline{BD}))$, which, by modularity, is equal to

$$G(A) + (A+D) \cap G(\overline{BD}) = G(A) + G(A \cap \overline{BD} + D).$$

As $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{ACF}}_{B} D$, $A \cap \overline{BD} = B$. By $B \mathop{\smile\hspace{-0.9em}\mid}^{w}_{C} D$, $G(B+D) = G(B) + G(D)$ hence

$$G(A+D) = G(A) + G(B) + G(D) = G(A) + G(D).$$

Local Character. We start with a claim.

Claim. Let A, B be subsets of (K, G) with B subgroup of $(K, +)$, then there exists $C \subseteq B$ with $|C| \leq |A|$ such that

$$G(A+B) = G(A+C) + G(B).$$

Proof of the claim. For each $a \in A$ define $C(a)$ to be the set of those $b \in B$ such that $a + b \in G$. Take $c(a) \in C(a)$ for each a such that $C(a)$ is nonempty, and set

$$C = \{c(a) \mid a \in A \text{ and } C(a) \neq \emptyset\}.$$

Now if $g \in G(A+B)$ then $g = a + b$ with $a \in A$, $b \in B$. We have $C(a)$ nonempty so we can write for $c = c(a)$

$$g = (a+c) + (b-c).$$

It follows that $b - c \in G(B)$ hence $g \in G(A+C) + G(B)$. The reverse inclusion is trivial. □

Let a be a finite tuple and B an algebraically closed set. We construct two sequences $(A_i)_{i<\omega}$ and $(D_i)_{i<\omega}$ such that the following holds for all $n < \omega$:

(1) $A_n \subseteq A_{n+1} \subseteq \overline{Ba}$ and $D_n \subseteq A_n$

(2) $G(A_n + B) \subseteq G(A_{n+1}) + G(B)$

(3) $A_n \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{ACF}}_{D_n} B$

(4) $|A_n| \leq \aleph_0$

Using Local Character for $\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{ACF}}$ there exists a countable set $D_0 \subseteq B$ such that $a \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{ACF}}_{D_0} B$. We define $A_0 = \overline{aD_0}$. Assume that D_n and A_n has been constructed and that $|A_n| \leq \aleph_0$. By the claim there exists $C \subseteq B$ with $|C| \leq \aleph_0$ such that $G(A_n + B) = G(A_n + C) + G(B)$. Using Local Character[1] of $\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{ACF}}$ on the set A_nC there exists $D_{n+1} \subseteq B$ with $|D_{n+1}| \leq \aleph_0$ such

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[1] Here we use a stronger version of Local Character which holds in any simple (countable) theory (see [Cas11, Proposition 5.5]): for all countable set A and arbitrary set B there exists $B_0 \subseteq B$ with $|B_0| \leq \aleph_0$ with $A \mathop{\smile\hspace{-0.9em}\mid}_{B_0} B$.

that $A_n C \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_{D_{n+1}} B$. We set $A_{n+1} = \overline{A_n C D_{n+1}}$. Note that $A_n + C \subseteq A_{n+1}$ so $G(A_n + B) \subseteq G(A_{n+1}) + G(B)$.

Now set $A_\omega = \bigcup_{i<\omega} A_i$ and $D_\omega = \bigcup_{i<\omega} D_i$. We have $|A_\omega| \leq \aleph_0$ and $|D_\omega| \leq \aleph_0$. We claim that

$$A_\omega \mathop{\smile\!\!\!\!\!\!\mid}^{w}_{D_\omega} B.$$

If u is a finite tuple from A_ω, then $u \subseteq A_n$ for some n, so as $A_n \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_{D_n} B$ we have $u \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_{D_n} B$. Now as $D_\omega \subseteq B$, we use BASE MONOTONICITY of $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}$ to conclude that $u \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_{D_\omega} B$. As this holds for every finite tuple u from A_ω, we conclude that

$$A_\omega \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_{D_\omega} B.$$

It remains to show that $G(A_\omega + B) = G(A_\omega) + G(B)$. If $g \in G(A_\omega + B)$ then there is some n such that $g \in A_n + B$ and so

$$g \in G(A_n + B) \subseteq G(A_{n+1}) + G(B) \subseteq G(A_\omega) + G(B).$$

The reverse inclusion being trivial, we conclude that $G(A_\omega + B) = G(A_\omega) + G(B)$, so $A_\omega \mathop{\smile\!\!\!\!\!\!\mid}^{w}_{D_\omega} B$. As $a \subseteq A_\omega$ we conclude by MONOTONICITY of $\mathop{\smile\!\!\!\!\!\!\mid}^{w}$. ☐

Proposition 5.2.3. *Assume that $C = \overline{C}$. If $a \mathop{\smile\!\!\!\!\!\!\mid}^{w}_{C} b$, then for all C-indiscernible sequence $(b_i)_{i<\omega}$ in tp(b/C) such that $b_i \mathop{\smile\!\!\!\!\!\!\mid}^{a}_{C} (b_j)_{j<i}$ there exists a' such that $a'b_i \equiv_C ab$ for all $i < \omega$. In particular, the following are equivalent, for C algebraically closed and $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_{C} b$.*

(1) $a \mathop{\smile\!\!\!\!\!\!\mid}^{w}_{C} b$;

(2) for all C-indiscernible sequence $(b_i)_{i<\omega}$ in tp(b/C) such that, $b_i \mathop{\smile\!\!\!\!\!\!\mid}^{a}_{C} (b_j)_{j<i}$ and $G(\overline{Cb_i} + \overline{Cb_k}) = G(\overline{Cb_i}) + G(\overline{Cb_k})$ there exists a' such that $a'b_i \equiv_C ab$ for all i;

(3) for some C-indiscernible sequence $(b_i)_{i<\omega}$ in tp(b/C) such that, $b_i \mathop{\smile\!\!\!\!\!\!\mid}^{a}_{C} (b_j)_{j<i}$ and $G(\overline{Cb_i} + \overline{Cb_k}) = G(\overline{Cb_i}) + G(\overline{Cb_k})$ there exists a' such that $a'b_i \equiv_C ab$ for all i.

Proof. The first assertion holds because $\mathop{\smile\!\!\!\!\!\!\mid}^{w}$ satisfies $\mathop{\smile\!\!\!\!\!\!\mid}^{a}$-AMALGAMATION over algebraically closed sets (Theorem 5.2.2). The proof is a classical induction similar to the proof of Lemma 7.1.9 or [CK17, Proposition 4.11].

(1) implies *(2)* is a particular case of the first assertion. *(2)* implies *(3)* follows from the fact that such sequence exists, which follows from FULL EXISTENCE of $\mathop{\smile\!\!\!\!\!\!\mid}^{w}$. We show that *(3)* implies *(1)*. Assume that $a \not\mathop{\smile\!\!\!\!\!\!\mid}^{w}_{C} b$ and let $\Lambda(x, b, c)$ be as in Lemma 4.1.6. If *(3)* holds, then in particular $\{\Lambda(x, b_i, c) \mid i < \omega\}$ is consistent, for some $(b_i)_{i<\omega}$ such that $b_i \equiv_C b$ and $b_i \mathop{\smile\!\!\!\!\!\!\mid}^{a}_{C} b_j$. This contradicts Lemma 4.1.6. ☐

In particular, we have the following combinatorial characterization of $\mathop{\smile\!\!\!\!\!\!\mid}^{w}$ over algebraically closed sets.

Corollary 5.2.4. *The following are equivalent, for C algebraically closed*

(1) $a \mathop{\smile\!\!\!\!\!\!\mid}^{w}_{C} b$;

(2) for all C-indiscernible sequence $(b_i)_{i<\omega}$ in tp(b/C) such that, $b_i \mathop{\smile\!\!\!\!\!\!\mid}^{w}_{C} (b_j)_{j<i}$ there exists a' such that $a'b_i \equiv_C ab$ for all i;

(3) for some C-indiscernible sequence $(b_i)_{i<\omega}$ in tp(b/C) such that, $b_i \mathop{\smile\!\!\!\!\!\!\mid}^{w}_{C} (b_j)_{j<i}$ there exists a' such that $a'b_i \equiv_C ab$ for all i.

Proof. *(1)* implies *(2)* follows from Proposition 5.2.3, and *(2)* implies *(3)* holds since $\mathop{\smile\kern-0.65em\mid}^{w}$ satisfies Full Existence. Assume that *(3)* holds for some a' and indiscernible sequence $(b_i)_{i<\omega}$ such that $b_i \mathop{\smile\kern-0.65em\mid}^{w}_{C} (b_j)_{j<i}$for all $i < \omega$. In particular, $(b_i)_{i<\omega}$ is a Morley sequence in the sense of ACF_p, and $a'b_i \equiv^{\mathrm{ACF}}_{C} ab$ for all $i < \omega$. As $\mathop{\smile\kern-0.65em\mid}^{\mathrm{ACF}}$ is forking independence in the sense of ACF_p, we have $a \mathop{\smile\kern-0.65em\mid}^{ACF}_{C} b$. By Proposition 5.2.3 we have $a \mathop{\smile\kern-0.65em\mid}^{w}_{C} b$. ☐

The Kim-Pillay theorem (see Fact 1.4.6) states that if a relation $\mathop{\smile\kern-0.65em\mid}$ satisfies Invariance, Symmetry, Monotonicity, Base Monotonicity, Transitivity, Full Existence, Local Character, $\mathop{\smile\kern-0.65em\mid}$-amalgamation over models and Finite Character[2], then the theory is simple and this relation is forking independence. From Theorem 5.2.2 and Proposition 4.4.2, the weak independence $\mathop{\smile\kern-0.65em\mid}^{w}$ satisfies all the previous properties except Base Monotonicity. This is similar to the case of $K_{n,m}$-free bipartite graph [CK17, Remark 4.17].
The property $\mathop{\smile\kern-0.65em\mid}$-amalgamation over models is a special case of Stationnarity over algebraically closed sets, hence from Lemma 4.3.3, the strong independence $\mathop{\smile\kern-0.65em\mid}^{st}$ satisfies every property of the Kim-Pillay characterization except Local Character otherwise, ACFG would be simple. Example 7.1.4 shows directly that Local Character is not satisfied by $\mathop{\smile\kern-0.65em\mid}^{st}$, nor by any relation stronger than $\mathop{\smile\kern-0.65em\mid}^{w}$ which satisfies Base Monotonicity. As we saw in Lemma 4.3.3, ACFG is mock stable in the sense of Adler.

5.3 Some structural features of (K, G)

Let $P(X)$ be a polynomial in variables $X = X_1, \dots, X_n$ with coefficients in K. We say that P is *$\mathbb{F}_p$-flat over K* if whenever u is a zero of P in some field extension of K, there exists a non trivial $\mathbb{F}_p$-linear combination of u that falls in K.

Lemma 5.3.1. *Let (K, G) be an $\aleph_0$-saturated model of* ACFG*, and $P(X_1, \dots, X_n)$ a polynomial non-$\mathbb{F}_p$-flat over K. Then for every $I \subset \{1, \dots, n\}$ there exists a zero a of P in K such that $a_i \in G \iff i \in I$.*

Proof. Let $I \subset \{1, \dots, n\}$. As P is non-$\mathbb{F}_p$-flat, there exists a zero t of P in an extension of K such that no non nontrivial $\mathbb{F}_p$-combination of t falls in K. It follows that $(\overline{K(t)}, G + \langle t_i \mid i \in I\rangle)$ is an $\mathscr{L}_G$-extension of (K, G). Indeed $(G + \langle t_i \mid i \in I\rangle) \cap K = G$. Furthermore $t_j \in (G + \langle t_i \mid i \in I\rangle)$ if and only if $j \in I$. As (K, G) is existentially closed in $(\overline{K(t)}, G + \langle t_i \mid i \in I\rangle)$, we have that

$$(K, G) \models \exists x (P(x) = 0 \wedge \bigwedge_{i \in I} x_i \in G \wedge \bigwedge_{j \notin I} x_j \notin G).$$

☐

Lemma 5.3.2. *A polynomial P in $K[X]$ is $\mathbb{F}_p$-flat over K if and only if all its irreducible factors in $K[X]$ are of the form $c(\lambda_1 X_1 + \dots + \lambda_n X_n - b)$ for some $\lambda_1, \dots, \lambda_n$ in $\mathbb{F}_p \setminus \{0\}$ and $b, c \in K$.*

Proof. Assume that P is $\mathbb{F}_p$-flat over K. If $|X| = 1$, then P satisfies the conclusion. Assume that $|X| > 1$. Let $t_2, \dots, t_n$ be algebraically independent over K, and consider $P(X_1, t_2, \dots, t_n)$. This polynomial has zeros in $\overline{K(t_2, \dots, t_n)}$ hence by $\mathbb{F}_p$-flatness each root u satisfies $\lambda_1 u + \lambda_2 t_2 + \dots + \lambda_n t_n = b$ for some non-zero tuple $\lambda_1, \dots, \lambda_n$ from $\mathbb{F}_p$ and $b \in K$. By hypothesis on $t_2, \dots, t_n$ we have that $\lambda_1 \neq 0$. It follows that $X_1 - \lambda_1^{-1}(\lambda_2 t_2 + \dots + \lambda_n t_n - b)$ divides $P(X_1, t_2, \dots, t_n)$ hence $\lambda_1 X_1 + \dots + \lambda_n X_n - b$ divides P, as $K[X_1, t_2, \dots, t_n] \cong K[X]$. If $\lambda_i = 0$ for some i, then the tuple $(0, \dots, t, \dots, 0)$ with t transcendental over K at the i-th coordinate, is a zero of P that contradicts the $\mathbb{F}_p$-flatness. It follows that P is of the desired form. The other direction is trivial. ☐

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[2] This property is trivial for $\mathop{\smile\kern-0.65em\mid}^{w}$ and $\mathop{\smile\kern-0.65em\mid}^{st}$.

Example 5.3.3 ($\mathbb{F}_p$-flatness might depends on p)**.** Consider the polynomial $P = X^2 + Y^2$, with $b \in K$. Then P is $\mathbb{F}_p$-flat over any algebraically closed field if and only if -1 is a square in $\mathbb{F}_p$. From [Fre01, Exercice 1.9.24], when $p > 2$ this is equivalent to $p \in 4\mathbb{Z}+1$. Using Lemmas 5.3.1 and 5.3.2 it follows that whenever $(K, G) \models \mathrm{ACFG}$, $p > 2$,

- if $p \notin 4\mathbb{Z} + 1$ there exists $g \in G$ and $u \in K \setminus G$ such that $g^2 + u^2 = 0$;
- if $p \in 4\mathbb{Z} + 1$ such couple (u, g) does not exists in (K, G), as every couple of solution to $X^2 + Y^2 = 0$ are $\mathbb{F}_p$-linearly dependent.

Proposition 5.3.4. *Let* (K, G) *be a model of* ACFG*. The following holds:*

(1) $K = G \cdot G = G \cdot (K \setminus G) = (K \setminus G) \cdot (K \setminus G)$*;*

(2) G *is stably embedded in* K*;*

(3) For $a \notin \mathbb{F}_p$ *and* $P \in K[X] \setminus (K + \mathbb{F}_p \cdot X)$*, we have* $K = G + aG = (K \setminus G) + aG = G + P(G)$*.*

Proof. (1) For all $b \in K$ the polynomial $XY - b$ is not $\mathbb{F}_p$-flat by Lemma 5.3.2, so we conclude using Lemma 5.3.1.
(2) From *(1)*, every element in K is product of two elements in G, so any $\mathscr{L}_G$-formula $\phi(x, a_1, \dots, a_n)$ is equivalent to $\phi(x, g_1h_1, \dots, g_nh_n)$ with $g_i, h_i \in G$.
(3) For all $P \in K[X] \setminus (K + \mathbb{F}_p \cdot X)$, $b \in K$, the polynomial $Y + P(X) - b$ is not $\mathbb{F}_p$-flat, similarly to *(1)*. ☐

Proposition 5.3.5. *Let* $\zeta_1, \dots, \zeta_n$ *be* $\mathscr{L}_{\mathrm{ring}}$*-definable endomorphisms of* $(K, +)$*,* $\mathbb{F}_p$*-linearly independent. Then*

$$K/(\zeta_1^{-1}(G) \cap \dots \cap \zeta_n^{-1}(G)) \cong K/\zeta_1^{-1}(G) \times \dots \times K/\zeta_n^{-1}(G).$$

Proof. Using the first isomorphism theorem, it is sufficient to prove that the function $\zeta : K \to K/\zeta_1^{-1}(G) \times \dots \times K/\zeta_n^{-1}(G)$ defined by $\zeta(u) = (u + \zeta_1^{-1}(G), \dots, u + \zeta_n^{-1}(G))$ is onto. Let $c_1, \dots, c_n \in K$, we want to show that there exists $c \in K$ such that for all i $\zeta_i(c - c_i) \in G$. Let t be a transcendental element over K, by model completeness of ACF_p, $\zeta_1, \dots, \zeta_n$ are $\mathbb{F}_p$-linearly independent definable endomorphisms of $(\overline{Kt}, +)$. Consider the $\mathscr{L}_G$-structure

$$(\overline{Kt}, G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle).$$

We have $(G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle) \cap K = G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle \cap K$. For $\lambda_1, \dots, \lambda_n \in \mathbb{F}_p$, if $\sum_i \lambda_i \zeta_i(t - c_i) \in K$ then $\sum_i \lambda_i \zeta_i(t) \in K$. By Fact 1.5.12, there is some k such that $t \mapsto (\sum_i \lambda_i \zeta_i(t))^{p^k}$ is polynomial, hence as t is transcendental over K, $(\sum_i \lambda_i \zeta_i)^{p^k} = 0$, so $\sum_i \lambda_i \zeta_i = 0$. As $\zeta_1, \dots, \zeta_n$ are $\mathbb{F}_p$-linearly independent, $\lambda_1 = \dots = \lambda_n = 0$. It follows that $(G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle) \cap K = G$, so $(\overline{Kt}, G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle)$ extends (K, G). As (K, G) is existentially closed in $(\overline{Kt}, G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle)$ we have that $(K, G) \models \exists x \bigwedge_i \zeta_i(x - c_i) \in G$, hence ζ is onto. ☐

If $\zeta_1, \dots, \zeta_n$ are $\mathbb{F}_p$-linearly independent $\mathscr{L}_{\mathrm{ring}}$-definable isomorphisms of $(K, +)$, the previous result can be used to find canonical parameters for the quotient $K/(\zeta_1^{-1}(G) \cap \dots \cap \zeta_n^{-1}(G))$ provided one have canonical parameters for the quotient K/G, see Example 6.0.1.

5.4 Models of ACFG in $\overline{\mathbb{F}_p}$

From Theorem 3.1.2, for any quantifier free $\mathscr{L}_{\mathrm{ring}}$-formula $\phi(x, y)$, there exists an $\mathscr{L}_{\mathrm{ring}}$-formula $\theta_\phi(y)$ such that for $K \models \mathrm{ACF}_p$ sufficiently saturated and b tuple in K such that $K \models \theta_\phi(b)$ if and only if there exists a realisation a of $\phi(x, b)$ which is $\mathbb{F}_p$-linearly independent over $\overline{\mathbb{F}_p(b)}$. By quantifier elimination in ACF_p, the formula θ_ϕ can be choosen quantifier-free.

Lemma 5.4.1. *If $\mathbb{F}_{p^n} \models \theta_\phi(b)$ then for all $m \geq n$ there exists $k > m$ such that*

$$\mathbb{F}_{p^k} \models \exists x \phi(x,b) \wedge x \text{ is } \mathbb{F}_p\text{-linearly independent over } \mathbb{F}_{p^m}.$$

Proof. Assume that $\mathbb{F}_{p^n} \models \theta_\phi(b)$. Then as θ_ϕ is quantifier free, $\overline{\mathbb{F}_p} \models \theta_\phi(b)$. It follows that for some elementary extension K of $\overline{\mathbb{F}_p}$, there is some realisation a of $\phi(x,b)$ which is $\mathbb{F}_p$-linearly independent over $\overline{\mathbb{F}_p}$. In particular for every non trivial polynomial $P(Z,Y) \in \mathbb{F}_p[Z,Y]$ (where Z is a single variable and Y a tuple of variables with $|Y| = |y|$), no nontrivial $\mathbb{F}_p$-linear combination of a is a root of $P(Z,b)$. As $\overline{\mathbb{F}_p} \equiv^{\mathrm{ACF}} K$, the following sentence holds in $\overline{\mathbb{F}_p}$:

$$\forall y(\theta_\phi(y) \to (\exists x \phi(x,y) \wedge \text{"no nontrivial } \mathbb{F}_p\text{-linear combination of } x \text{ is a root of } P(Z,y)\text{"})).$$

In particular, for the polynomial $X^{p^m} - X$ for some m we have

$$\overline{\mathbb{F}_p} \models \exists x \phi(x,b) \wedge \text{no non-trivial } \mathbb{F}_p\text{-linear combination of } x \text{ falls in } \mathbb{F}_{p^m}.$$

Hence for some $k > m, n$ there exists a tuple a from $\mathbb{F}_{p^k}$ such that

$$\overline{\mathbb{F}_p} \models \phi(a,b) \wedge a \text{ is } \mathbb{F}_p\text{-linearly independent over } \mathbb{F}_{p^m}.$$

As $\phi(x,y)$ is quantifier-free, we also have that

$$\mathbb{F}_{p^k} \models \phi(a,b) \wedge a \text{ is } \mathbb{F}_p\text{-linearly independent over } \mathbb{F}_{p^m}.$$

□

Proposition 5.4.2. *For any $n \in \mathbb{N}$ and any G_0 additive subgroup of $\mathbb{F}_{p^n}$ there exists a subgroup G of $\overline{\mathbb{F}_p}$ such that $G \cap \mathbb{F}_{p^n} = G_0$ and $(\overline{\mathbb{F}_p}, G) \models$ ACFG.*

Proof. Start with the following claim.

Claim. Let $n \in \mathbb{N}$, let $s \in \mathbb{N}$, let $k_1, \dots, k_s \in \mathbb{N}$ and let $\phi_1(x^1, y^1), \dots, \phi_s(x^s, y^s)$ be quantifier free formulae in $\mathscr{L}_{\mathrm{ring}}$. For $i \leq s$, let $B_i = \left\{ b \in \mathbb{F}_{p^n}^{|y^i|} \mid b \models \theta_{\phi_i}(y) \right\}$. Then there exists $m > n$ such that for all $i \leq s$ and $b \in B_i$ there exists some $|x^i|$-tuples $a^{i,1}, \dots, a^{i,k_i}$ (depending on b) from $\mathbb{F}_{p^m}$ such that

1. $(a^{i,j}_k)_{i \leq s, j \leq k_i, k \leq |x^i|}$ is a $\mathbb{F}_p$-linearly independent tuple over $\mathbb{F}_{p^n}$
2. $\mathbb{F}_{p^m} \models \phi_i(a^{i,1}, b), \dots, \mathbb{F}_{p^m} \models \phi_i(a^{i,k_i}, b)$.

Proof of the Claim. We do it step by step, as there are only a finite number of tuples to add. Start with $\phi_1(x^1, y^1)$. Take a first $b \in B_1$. As $\mathbb{F}_{p^n} \models \theta_{\phi_1}(b)$, we use Lemma 5.4.1 with $m = n$ to get a first $m_1 > n$ such that there exists $a^1 \in \mathbb{F}_{p^{m_1}}^{|x_1|}$ such that $\models \phi_1(a^1, b)$ and a^1 is $\mathbb{F}_p$-linearly independent over $\mathbb{F}_{p^n}$. Using again Lemma 5.4.1 with $m = m_1$ there exists $m_2 > m_1$ and a second $a^2 \in \mathbb{F}_{p^{m_2}}^{|x^1|}$ such that $\mathbb{F}_{p^{m_2}} \models \phi_1(a^2, b)$ and a^2 is $\mathbb{F}_p$-linearly independent over $\mathbb{F}_{p^{m_1}}$. In particular a^2 is $\mathbb{F}_p$-linearly independent from a^1 over $\mathbb{F}_{p^n}$. So we can construct as many (finitely) solution to $\phi_1(x^1, b)$ as we want which are $\mathbb{F}_p$-linearly independent over $\mathbb{F}_{p^n}$. Once we have enough $\mathbb{F}_p$-linearly independent solutions of $\phi_1(x, b)$, we can do the same trick with another $b' \in B_1$, and add as many (finitely) solution as we want, $\mathbb{F}_p$-linearly independent from one another and from the ones corresponding to b, in a finite extension of $\mathbb{F}_{p^n}$. Once we have done it for all elements of B_1, we do the same with every element $b \in B_2$, continuing to use Lemma 5.4.1 to get solutions of $\phi_2(x^2, b)$ $\mathbb{F}_p$-linearly independent from one another and from the previous ones. As every B_i is finite and they are in finite number, we can finish to add $\mathbb{F}_p$-linearly independent solutions of ϕ_i in a finite number of steps and the claim is proven. □

From Proposition 5.1.1, the axioms for ACFG are given by the following scheme: for all quantifier free $\mathscr{L}_{\text{ring}}$-formula $\phi(x,y)$, for all $0 \leq k \leq |x|$ and $0 \leq k' \leq |y|$

$$\forall y\, ((\theta_\phi(y) \wedge \langle y_1, \dots, y_{k'}\rangle \cap G = \{0\}) \to (\exists x \phi(x,y) \wedge \langle x, y_1, \dots, y_{k'}\rangle \cap G = \langle x_1, \dots, x_k\rangle))$$

with the following convention: $a_1, \dots, a_0 = \emptyset$. We will denote the previous sentence by $\Gamma(\phi, k, k')$. Now we construct by induction a model of ACFG starting from $(\mathbb{F}_{p^n}, G_0)$. Let $(\phi_i(x^i, y^i))_{i<\omega}$ be an enumeration of all quantifier-free formula in $\mathscr{L}_{\text{ring}}$. We construct an increasing sequence $(n_j)_{j<\omega}$ starting with $n_0 = n$ and additive subgroups G_j of $\mathbb{F}_{p^{n_j}}$ such that for all $s < \omega$, for $\phi_1(x^1, y^1), \dots, \phi_s(x^s, y^s)$, for all $1 \leq l \leq s$, for all $0 \leq k \leq |x^l|$ and $0 \leq k' \leq |y^l|$ the following holds for all $|y^l|$-tuples b from $\mathbb{F}_{p^{n_s}}$

If $\mathbb{F}_{p^{n_s}} \models \theta_{\phi_l}(b) \wedge \langle b_1, \dots, b_{k'}\rangle \cap G_s = \{0\}$ then there exists $a^{l,k}$ an $|x^l|$-tuple from $\mathbb{F}_{p^{n_{s+1}}}$ such that $\mathbb{F}_{p^{n_{s+1}}} \models \phi(a^{l,k}, b) \wedge \langle a^{l,k}, b_1, \dots, b_{k'}\rangle \cap G_{s+1} = \langle a_1^{l,k}, \dots, a_k^{l,k}\rangle$. $(\star)$

Assume that for some $s < \omega$ we have $n_0, \dots, n_s$ and $G_0 \subseteq \mathbb{F}_{p^{n_0}}, \dots, G_s \subseteq \mathbb{F}_{p^{n_s}}$ constructed as above. For every $i \leq s$, we define as above $B_i = \left\{ b \in \mathbb{F}_{p^{n_s}}^{|y^i|} \mid b \models \theta_{\phi_i}(y) \right\}$, and we apply the claim with $k_i = |x^i| + 1$, to get some $n_{s+1} > n_s$. For each $1 \leq i \leq s$ and $b \in B_i$ we have $|x^i| + 1$ many $|x^i|$-tuples $a^{i,1}(b), \dots, a^{i,k_i}(b)$ from $\mathbb{F}_{p^{n_{s+1}}}$ all $\mathbb{F}_p$-independents over $\mathbb{F}_{p^{n_s}}$ and such that for all j, we have $\mathbb{F}_{p^{n_{s+1}}} \models \phi_i(a^{i,j}(b), b)$. Now define G_{s+1} to be

$$G_s \oplus \bigoplus_{1\leq i \leq s} \bigoplus_{b \in B_i} \langle a_1^{i,2}(b)\rangle \oplus \langle a_1^{i,3}(b), a_2^{i,3}(b)\rangle \oplus \cdots \oplus \langle a_1^{i,k_i}(b), \dots, \oplus a_{k_i}^{i,k_i}(b)\rangle.$$

We extend G_s by the low triangle of the $(|x_i|+1) \times |x_i|$ matrix $(a_k^{i,j}(b))_{1 \leq j \leq k_i, 1 \leq k \leq |x^i|}$ for each $i < s$ and $b \in B_i$:

$$\begin{pmatrix} a_1^{i,1} & a_2^{i,1} & \dots & a_{|x^i|}^{i,1} \\ a_1^{i,2} & a_2^{i,2} & \dots & a_{|x^i|}^{i,2} \\ a_1^{i,3} & a_2^{i,3} & \dots & a_{|x^i|}^{i,2} \\ \vdots & & \ddots & \\ a_1^{i,k_i} & a_2^{i,k_i} & \dots & a_{|x^i|}^{i,k_i} \end{pmatrix}.$$

Now we have for each $1 \leq i \leq s$ and any $0 \leq k \leq |x^i|$ and $0 \leq k' \leq |y^i|$, if $b \in B_i$, then there exists $a^{i,k}(b) \in \mathbb{F}_{p^{n_{s+1}}}^{|x^i|}$ such that $\mathbb{F}_{p^{n_{s+1}}} \models \phi_i(a^{i,k}(b), b)$. By construction if $\langle b_1, \dots, b_{k'}\rangle \cap G_s = \{0\}$, and by $\mathbb{F}_p$-linear independence of all the $a^{i,k}$, we have $\langle a^{i,k}, b_1, \dots, b_{k'}\rangle \cap G_{s+1} = \langle a_1^{i,k}, \dots, a_k^{i,k}\rangle$. By induction we construct a familly $(\mathbb{F}_{p^{n_i}}, G_i)$ satisfying $(\star)$. Now let

$$G = \bigcup_{i<\omega} G_i \subseteq \overline{\mathbb{F}_p}.$$

By construction, we have that $(\overline{\mathbb{F}_p}, G)$ is a model of ACFG. □

Recall from Section 1.6 that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ endowed with the Chabauty topology is a Cantor space. Let

$$\mathcal{C} = \left\{ G \in Sg(\overline{\mathbb{F}_p}) \mid (\overline{\mathbb{F}_p}, G) \models \mathrm{ACFG} \right\}.$$

Recall that a set is G_δ if it is a countable intersection of open sets.

Proposition 5.4.3. *$\mathcal{C}$ is a dense G_δ of $Sg(\overline{\mathbb{F}_p})$.*

Proof. We first show that it is dense. By Lemma 1.6.1, the topology on $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is generated by balls of the form $\mathcal{B}(G_0, \mathbb{F}_{p^n}) = \left\{ H \in \mathrm{Sg}(\overline{\mathbb{F}_p}) \mid H \cap \mathbb{F}_{p^n} = G_0 \right\}$ where G_0 is a subgroup of $\mathbb{F}_{p^n}$. By Proposition 5.4.2, evey such ball contains an element of $\mathcal{C}$, hence $\mathcal{C}$ is dense. We show that it is a G_δ. First, from Proposition 5.1.1, ACFG is axiomatised by adding to the theory ACF_G the following $\mathscr{L}_G$-sentences, for all tuples of variable $x' \subset x$, $y' \subset y$ and $\mathscr{L}_{\mathrm{ring}}$-formula $\phi(x, y)$

$$\forall y(\langle y'\rangle \cap G = \{0\} \wedge \theta_\phi(y)) \rightarrow (\exists x \phi(x, y) \wedge \langle xy'\rangle \cap G = \langle x'\rangle)$$

which is equivalent to

$$\forall y \exists x \left[\neg\theta_\phi(y)) \vee \langle y'\rangle \cap G \neq \{0\} \vee (\phi(x, y) \wedge \langle xy'\rangle \cap G = \langle x'\rangle)\right].$$

Let $\phi(x, y)$, $x' \subseteq x$ and $y' \subseteq y$ be given. Let b be a $|y|$-tuple, and consider the set

$$\mathscr{O}_b = \bigcup_{a \in \overline{\mathbb{F}_p}^{|x|}, \overline{\mathbb{F}_p} \models \phi(a,b)} \left\{H \mid \langle b'\rangle \cap H \neq \{0\}\right\} \cup \left\{H \mid \langle ab'\rangle \cap H = \langle a'\rangle\right\}.$$

The set $\{H \mid \langle b'\rangle \cap H \neq \{0\}\}$ is equal to $\bigcup_{u \in \langle b'\rangle \setminus \{0\}} \{H \mid u \in H\}$ which is clearly open. From Lemma 1.6.1, $\{H \mid \langle ab'\rangle \cap H = \langle a'\rangle\}$ is also open, so $\mathscr{O}_b$ is open. Now it is an easy checking that

$$\mathcal{C} = \bigcap_{\phi(x,y), x' \subseteq x, y' \subseteq y} \ \bigcap_{b \in \overline{\mathbb{F}_p}^{|y|}, \overline{\mathbb{F}_p} \models \theta_\phi(b)} \mathscr{O}_b.$$

Hence $\mathcal{C}$ is G_δ. □

Remark 5.4.4 (Ultraproduct model of ACFG). From the proof of Proposition 5.4.2, starting from $G_0 \subseteq \mathbb{F}_{p^{n_0}}$, there exists a strictly increasing sequence $(n_i)_{i<\omega}$ of integers and an increasing sequence of groups $G_i \subseteq \mathbb{F}_{p^{n_i}}$ satisfying $(\star)$. Let $\mathscr{U}$ be a nonprincipal ultrafilter on ω, it does not take long to see that the ultraproduct $\prod_{\mathscr{U}}(\overline{\mathbb{F}_p}, G_i))$ is a model of ACFG, in which the group is pseudo-finite. The construction of the $G'_i s$ in the proof of Proposition 5.4.2 is rather artificial. Is there more "natural" generic subgroups of $\overline{\mathbb{F}_p}$? Given an arbitrary set $\{G_i \mid i < \omega\}$ of subgroups of $\overline{\mathbb{F}_p}$ and a non principal ultrafilter $\mathscr{U}$ on ω, how likely is it that $\prod_{\mathscr{U}}(\overline{\mathbb{F}_p}, G_i))$ is a model of ACFG?

Remark 5.4.5 (Characteristic 0). Let $\mathscr{P}$ be the set of prime numbers and $\mathscr{U}$ a non-principal ultrafilter on $\mathscr{P}$. For each $q \in \mathscr{P}$ let G_q be a subgroup of $\overline{\mathbb{F}_q}$ such that $(\overline{\mathbb{F}_q}, G_q)$ is a model of ACFG (here we mean $\mathrm{ACF_qG}$). Recall that $\mathbb{C} \cong \prod_{q \in \mathscr{P}} \overline{\mathbb{F}_q}/\mathscr{U}$. Consider the ultraproduct

$$(\mathbb{C}, V) \cong \prod_{q \in \mathscr{P}} (\overline{\mathbb{F}_q}, G_q)/\mathscr{U}.$$

It is clear that V is a subgroup of $\mathbb{C}$. For each $q \in \mathscr{P}$,

$$Stab_{\overline{\mathbb{F}_q}}(G_q) := \left\{a \in \overline{\mathbb{F}_q} \mid aG_q \subseteq G_q\right\} = \mathbb{F}_q,$$

this follows from Proposition 5.3.4 *(3)*. Hence $F = Stab_{\mathbb{C}}(V)$ is a pseudo-finite subfield of $\mathbb{C}$, and V is an F-vector space. It follows from Proposition 3.2.7 that $(\mathbb{C}, V)$ is *not* existentially closed in the class of $\mathscr{L}_G$-structures consisting of a field of characteristic 0 in which G is an additive subgroup. Nonetheless, some properties such as the ones in Proposition 5.3.4 will be satisfied by $(\mathbb{C}, V)$ (replacing $\mathbb{F}_p$ by F).

Remark 5.4.6. Observe that the proof of Lemma 5.4.1 gives the following: if F is an infinite *locally finite* field[3], and that for some universal $\mathscr{L}_{\mathrm{ring}}$-formula $\phi(x, y)$ there exists an existential formula $\theta_\phi(y)$ such that for all tuple b, we have $F \models \theta_\phi(b)$ if and only if there exists a realisation a of $\phi(x, b)$ in an elementary extension of F such that a is $\mathbb{F}_p$-linearly independent over F; then for all finite subfields $F_0 \subset F_1$ of F, if $F_0 \models \theta_\phi(b)$ there exists a finite subfield F_2 of F and

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[3] A *locally finite* field is a field such that every finitely generated subfield is finite. Equivalently it is embeddable in $\overline{\mathbb{F}_p}$.

a tuple a from F_2 realizing $\phi(x, b)$ which is $\mathbb{F}_p$-linearly independent over F_1. By the same method as in the proof of Theorem 5.4.2, we may construct an increasing sequence of finite fields $(F_i)_{i<\omega}$ and finite subgroups $G_i \subseteq F_i$ such that for an enumeration of universal formula $\phi(x, y)$ and existential formula $\theta_\phi(y)$, if (F_i, G_i) satisfies the premise of the axiom, then the conclusion is satisfied in (F_{i+1}, G_{i+1}). Now consider the theory Psf_c (see Subsection 1.5.2), it is model-complete, hence every formula is equivalent to an existential formula and a universal formula, with some constants. It is then possible to choose constants $c(i)$ in F_i such that $X^n + c_{n-1,n}(i)X^{n-1} + \cdots + c_{0,n}(i)$ is irreducible over F_i, for all n. Then one can check that a non principal ultraproduct of $(F_i, c(i))_{i<\omega}$ is a model of Psf_c, hence the ultraproduct $\prod_{\mathscr{U}}(F_i, c_i, G_i)$ is a model of $\mathrm{Psf}_c G$ (see Example 3.2.3).

CHAPTER 6

Imaginaries in ACFG

Let (K,G) be a saturated model of ACFG. It is easy to see that for all $a \in K \setminus G$, there exists $b \in K \setminus G$ algebraically independent from a over $\mathbb{F}_p$ such that $a-b \in G$ (see Lemma 6.1.1). Let $\alpha = a/G = b/G \in (K,G)^{eq}$. If it exists, a canonical parameter for α in K would be definable over both a and b, hence it would be definable over an element of $\overline{\mathbb{F}_p}$. This would give an embedding of K/G into the countable set $\mathrm{dcl}^{eq}(\emptyset)$ which is absurd in a saturated model (K,G) for cardinality reasons.

Let (K,G) be a model of ACFG, there is a canonical projection

$$\pi : K \to K/G.$$

Consider the 2-sorted structure, $(K, K/G)$ with the $\mathscr{L}_{\mathrm{ring}}$-structure on K, the group structure on K/G (in the language of abelian groups) and the group epimorphism $\pi : K \to K/G$. We forget about the predicate G as it is 0-definable in $(K,K/G)$. The structure $(K,K/G)$ is bi-interpretable with (K,G). We fix (K,G) and $(K,K/G)$ for the rest of this chapter.
In this chapter we show that $(K,K/G)$ has weak elimination of imaginaries, hence imaginaries of (K,G) can be weakly eliminated up to the quotient K/G.

Some definable imaginaries in (K,G) can be easily eliminated in the structure $(K,K/G)$.

Example 6.0.1. Let $\zeta : K \to K$ be a $\mathscr{L}_{\mathrm{ring}}$-definable group endomorphism. Then in $(K,K/G)^{eq}$, every element in $K/\zeta^{-1}(G)$ is interdefinable with an element in K/G. Indeed, for any element $a \in K$ and any automorphism σ of $(K,K/G)$, $\sigma(a) - a \in \zeta^{-1}(G)$ if and only if σ fixes $\pi(\zeta(a))$, hence $\pi(\zeta(a))$ is a canonical parameter for the class of a modulo $\zeta^{-1}(G)$.

Let $\zeta_1, \cdots, \zeta_n$ be $\mathbb{F}_p$-linearly independent $\emptyset$-$\mathscr{L}_{\mathrm{ring}}$-definable group endomorphisms $K \to K$. Let $\pi_\zeta : K \to K/\zeta_1^{-1}(G) \cap \cdots \cap \zeta_n^{-1}(G)$ and consider an element α of the sort $K/\zeta_1^{-1}(G) \cap \cdots \cap \zeta_n^{-1}(G)$ in $(K,K/G)^{eq}$. From Proposition 5.3.5 the natural map

$$K/\zeta_1^{-1}(G) \cap \cdots \cap \zeta_n^{-1}(G) \to K/\zeta_1^{-1}(G) \times \cdots \times K/\zeta_n^{-1}(G)$$

is an isomorphism. Let a be such that $\pi_\zeta(a) = \alpha$. For each $1 \le i \le n$ let $\alpha_i = \pi(\zeta_i^{-1}(a)) \in K/G$. Then the tuple $\alpha_1, \cdots, \alpha_n$ is a canonical parameter for α.

If quotients of the form $K/\zeta^{-1}(G)$ can be fully eliminated, what about quotients of the form $K/\zeta(G)$? In that case the kernel of ζ is a finite vector space, hence a canonical parameter for $\alpha \in K/\zeta(G)$ is a finite set of the form $\pi(a + \ker(\zeta))$ which is not necessarily eliminable in $(K,K/G)$ as shows Example 6.3.5. We even show in Example 6.3.6 that adding canonical parameters for the sort K/G is not sufficient to eliminate all finite imaginaries of the structure $(K, K/G)$.

In this chapter, greek letters Γ, α denote subsets or tuples (which might be infinite) from K/G. Any tuple in the structure $(K, K/G)$ will be denoted by $a\gamma$, with a a tuple from K, γ a tuple from K/G. We also extend π for (finite or infinite) tuples by $\pi(a) := (\pi(a_i))_i$.

Contents

6.1 First steps with imaginaries

Let σ be a field automorphism of K. It is clear that the following are equivalent:

- σ is an $\mathscr{L}^G$-automorphism of K;
- there exists $\tilde{\sigma} : K/G \to K/G$ such that $\pi \circ \sigma = \tilde{\sigma} \circ \pi$.

$$\begin{tikzcd}
(K,G) \arrow[r, "\sigma"] \arrow[d, "\pi"] & (K,G) \arrow[d, "\pi"] \\
K/G \arrow[r, "\tilde{\sigma}"] & K/G
\end{tikzcd}$$

An automorphism of the structure $(K, K/G)$ is a pair $(\sigma, \tilde{\sigma})$ as above. It follows that for a, b, C from K, we have

$$a \equiv_C^{(K,G)} b \iff a \equiv_C^{(K,K/G)} b.$$

In this chapter, the relation $\equiv$ means having the same type in the structure $(K, K/G)$.

Lemma 6.1.1. *Let a, b be two tuples of the same length from K. Let $C, D \subseteq K$ and assume that*

- *$\pi(a)$ is an $\mathbb{F}_p$-independent tuple over $\pi(\overline{C})$*
- *$\pi(b)$ is an $\mathbb{F}_p$-independent tuple over $\pi(\overline{C})$*

Then there exists $a' \equiv_C a$ such that $a' \mathop{\smile\hskip-0.9em\mid}^{\mathrm{ACF}}_{C} D$ and $\pi(a') = \pi(b)$.

Proof. Let $x \mathop{\smile\hskip-0.9em\mid}^{\mathrm{ACF}}_{C} K$ such that $x \equiv_C^{\mathrm{ACF}} a$, and $f : \overline{Cx} \to \overline{Ca}$ a field isomorphism over C sending x to a. Let $G_{Cx} = f^{-1}(G(\overline{Ca}))$. Consider now the subgroup of $\overline{CDbx}$ defined by

$$H = G_{Cx} + G(\overline{CbD}) + \langle x_i - b_i \mid i \leq |x| \rangle.$$

We show that the type in the sense of ACFG defined by the pair $(\overline{CDbx}, H)$ is consistent. As $x \mathop{\smile\hskip-0.9em\mid}^{\mathrm{ACF}}_{C} K$ we have $\overline{CDbx} \cap K = \overline{CDb}$. In order to prove that $H \cap \overline{CDb} = G(\overline{CDb})$, it suffices to show that

$$\overline{CDb} \cap (G_{Cx} + \langle x_i - b_i \mid i \leq |x| \rangle) \subseteq G(\overline{C}).$$

Assume that $g_{Cx} + \sum_i \lambda_i (x_i - b_i) \in \overline{CDb}$, where $g_{Cx} \in G_{Cx}$. It follows that $g_{Cx} + \sum_i \lambda_i x_i \in \overline{CDb}$. On the other hand $g_{Cx} + \sum_i \lambda_i x_i \in \overline{Cx}$. As $x \mathop{\smile\hskip-0.9em\mid}^{\mathrm{ACF}}_{C} bD$ we have $\overline{Cx} \cap \overline{CDb} = \overline{C}$ hence $g_{Cx} + \sum_i \lambda_i x_i \in \overline{C}$. Apply $\pi \circ f$ to get that $\sum_i \lambda_i \pi(a_i) \in \pi(\overline{C})$ hence by hypothesis $\lambda_i = 0$ for all $i \leq |x|$. It follows that $g_{Cx} \in \overline{C}$ and so $g_{Cx} \in G(\overline{C})$. We have showed that $\overline{CDb} \cap (G_{Cx} + \langle x_i - b_i \mid i \leq |x| \rangle) \subseteq G(\overline{C})$. The type is consistent by Proposition 2.3.6, so realised by say a'. As $x \mathop{\smile\hskip-0.9em\mid}^{\mathrm{ACF}}_{C} D$ we have $a' \mathop{\smile\hskip-0.9em\mid}^{\mathrm{ACF}}_{C} D$. In order to show that $a' \equiv_C a$ we have to check that $H \cap \overline{Cx} = G_{Cx}$, this is similar to the argument above, using this time that $\pi(b)$ is $\mathbb{F}_p$-independent over $\pi(\overline{C})$. We have $a'_i - b_i \in G$ hence $\pi(a'_i) = \pi(b_i)$, for all $i \leq |x|$. ☐

Lemma 6.1.2 (Minimal representative)**.** *Let a, C be in K such that $\pi(a)$ is an $\mathbb{F}_p$-independent tuple over $\pi(\overline{C})$. Then there exists a' of same length as a, algebraically independent over Cb such that*

- *$\pi(a) = \pi(a')$*
- *$\pi(\overline{Ca'}) = \langle \pi(\overline{C}) \pi(a) \rangle$*
- *$a' \mathop{\smile\hskip-0.9em\mid}^{\mathrm{ACF}}_{C} b$.*

Proof. It is again a type to realize. Consider x of same length as a and algebraically independent over Cba. Let V be a $\mathbb{F}_p$-vector space complement to $\overline{C} \oplus \langle x \rangle$ in $\overline{Cx}$ and set

$$H = G(\overline{Cab}) + \langle x - a \rangle + V.$$

We check that the pair $(\overline{Cabx}, H)$ defines a consistent type over Cab. First $H \cap \overline{Cab} = G(\overline{Cab}) + (\langle x - a \rangle + V) \cap \overline{Cab}$. For $v \in V$, if $\sum_i \lambda_i (x_i - a_i) + v \in \overline{Cab}$ then $\sum_i \lambda_i x_i + v \in \overline{Cab}$. As $\overline{Cab} \cap \overline{Cx} = \overline{C}$, $\sum_i \lambda_i x_i + v \in \overline{C}$ hence $v = 0$ and, as x is $\mathbb{F}_p$-independent over $\overline{C}$, $\lambda_i = 0$ for all $i \leq |x|$. The type is consistent by Proposition 2.3.6. We show that $H \cap \overline{Cx} = G(\overline{C}) + V$. First $H \cap \overline{Cx} = V + \overline{Cx} \cap (G(\overline{Cab}) + \langle x - a \rangle)$. Let $g + \sum_i \lambda_i (x_i - a_i) \in (G(\overline{Cab}) + \langle x - a \rangle) \cap \overline{Cx}$, then $g + \sum_i \lambda_i a_i \in \overline{Cab} \cap \overline{Cx} = \overline{C}$ and so applying π gives $\sum_i \lambda_i \pi(a_i) \in \pi(\overline{C})$ hence $\lambda_i = 0$ for all $i \leq |x|$. It follows that $\overline{Cx} \cap (G(\overline{Cab}) + \langle x - a \rangle) = G(\overline{C})$ hence $H \cap \overline{Cx} = G(\overline{C}) + V$. Assume that a' realises this type, it is clear that $\pi(a) = \pi(a')$ and $a' \mathop{\smile\!\!\!\!\!\!|}\limits^{\mathrm{ACF}}_{C} b$. By construction there exists $V' \subseteq \overline{Ca'}$ such that $\overline{Ca'} = \overline{C} \oplus \langle a \rangle \oplus V'$ and $G(\overline{Ca'}) = G(\overline{C}) \oplus V'$, so it follows that $\pi(\overline{Ca'}) = \pi(C) \oplus \langle \pi(a') \rangle$. ☐

In particular if α is an $\mathbb{F}_p$-independent tuple over $\pi(\overline{C})$ then there exists some algebraically independent tuple a over C such that $\pi(a) = \alpha$ and $\pi(\overline{Ca}) = \langle \pi(\overline{C})\alpha \rangle$. We call such a tuple a *minimal representative* of α over C. Lemma 6.1.2 states that minimal representatives always exists and that they can be taken independent in the sense of fields from any parameters.

Corollary 6.1.3. *Let α and β be tuples in K/G of the same length, γ tuple from K/G and $C \subseteq K$. If α and β are $\mathbb{F}_p$-independent tuples over $\langle \pi(\overline{C})\gamma \rangle$ then $\alpha \equiv_{C\gamma} \beta$.*

Proof. We may assume that γ is linearly independent over $\pi(\overline{C})$ and let r_γ be a minimal representative of γ over C. Let a and b be representatives of α and β over $\overline{Cr_\gamma}$. Using Lemma 6.1.1, there exists $a' \equiv_{Cr_\gamma} a$ such that $\pi(a') = \pi(b) = \beta$. Let σ be an automorphism of $(K, K/G)$ over Cr_γ sending a on a'. It is clear that σ fixes γ and sends α to β hence $\alpha \equiv_{C\gamma} \beta$. ☐

Remark 6.1.4. A consequence of Corollary 6.1.3 is that the induced structure on K/G is the one of a pure $\mathbb{F}_p$-vector space.

We will describe the algebraic closure acl in the structure $(K, K/G)$. It is classical that every formula in the language of $(K, K/G)$ (or of $(K, G)^{eq}$) without parameters and with free variables in the home sort K is equivalent to an $\mathscr{L}^G$-formula. In particular $\mathrm{acl}(C) \cap K = \overline{C}$ for all $C \subseteq K$.

Corollary 6.1.5. *Let $C \subseteq K$ and $\gamma \subseteq K/G$, then*

- $\mathrm{acl}(C\gamma) \cap K = \overline{C}$
- $\mathrm{acl}(C\gamma) \cap K/G = \langle \pi(\overline{C})\gamma \rangle$.

Proof. For the first assertion, we may assume that γ is an $\mathbb{F}_p$-independent tuple over $\pi(\overline{C})$. Let u be in $\mathrm{acl}(C\gamma) \cap K$ witnessed by an algebraic formula $\phi(x, c, \gamma)$ with $c \in C$. Using twice Lemma 6.1.2, let r_γ be a minimal representative of γ over C, and r'_γ a minimal representative of γ over C such that $r'_\gamma \mathop{\smile\!\!\!\!\!\!|}\limits^{\mathrm{ACF}}_{C} r_\gamma$. As u satisfies $\phi(x, c, \pi(r_\gamma))$ and $\phi(x, c, \pi(r'_\gamma))$, u belongs to $\overline{Cr_\gamma} \cap \overline{Cr'_\gamma} = \overline{C}$ (note that we don't use the minimality here). The reverse inclusion being trivial, it follows that $\mathrm{acl}(C\gamma) \cap K = \overline{C}$.

For the second assertion, assume that $\alpha \notin \langle \pi(\overline{C})\gamma \rangle$. By Corollary 6.1.3, any element in $K/G \setminus \langle \pi(\overline{C})\gamma \rangle$ has the same type as α over $C\gamma$ hence $\alpha \notin \mathrm{acl}(C\gamma)$. The reverse inclusion being trivial, it follows that $\mathrm{acl}(C\gamma) \cap K/G = \langle \pi(\overline{C})\gamma \rangle$. ☐

6.2 Independence in $(K, K/G)$

Recall the weak independence in (K, G):

$$a \mathop{\smile\hspace{-0.9em}|}\limits_{C}^{w} b \iff a \mathop{\smile\hspace{-0.9em}|}\limits_{C}^{\mathrm{ACF}} b \text{ and } G(\overline{Ca} + \overline{Cb}) = G(\overline{Ca}) + G(\overline{Cb})$$

It is an easy checking that under the assumption that $\overline{Ca} \cap \overline{Cb} = \overline{C}$ the following two assertions are equivalent:

- $G(\overline{Ca} + \overline{Cb}) = G(\overline{Ca}) + G(\overline{Cb})$
- $\pi(\overline{Ca}) \cap \pi(\overline{Cb}) = \pi(\overline{C})$

We define the following relation in $(K, K/G)$:

$$a\alpha \mathop{\smile\hspace{-0.9em}|}\limits_{C\gamma}^{w} b\beta \iff a \mathop{\smile\hspace{-0.9em}|}\limits_{C}^{\mathrm{ACF}} b \text{ and } \langle \pi(\overline{Ca})\alpha\gamma \rangle \cap \langle \pi(\overline{Cb})\beta\gamma \rangle = \langle \pi(\overline{C})\gamma \rangle$$

It is the right candidate for Kim-independence in $(K, K/G)$. We study only the restriction of this relation to sets $a\alpha, b\beta, C\gamma$ with $\alpha\beta\gamma \subseteq \pi(\overline{Ca}) \cap \pi(\overline{Cb})$. This restriction can be described only in terms of the structure (K, G) as we will see now.

An infinite tuple λ of elements of $\mathbb{F}_p$ is *almost trivial* if $\lambda_i = 0$ for cofinitely many i's. If γ is an infinite tuple, an element $u \in \langle \gamma \rangle$ is an almost trivial linear combination of γ_i's, i.e. there exists λ almost trivial such that $u = \sum_i \lambda_i \gamma_i$. Given two tuples a and b, the tuple consisting of the coordinates $a_i - b_i$ is denoted by $a - b$.

Lemma 6.2.1. *Let a, b be tuples such that γ is a (finite or infinite) tuple from $\pi(\overline{a}) \cap \pi(\overline{b})$. Assume that $\overline{a} \cap \overline{b} = \overline{C}$, then the following are equivalent:*

(1) $\pi(\overline{a}) \cap \pi(\overline{b}) = \langle \pi(\overline{C})\gamma \rangle$

(2) $G(\overline{a} + \overline{b}) = G(\overline{a}) + G(\overline{b}) + \langle r^a - r^b \rangle$ *for some (all) representatives r^a, r^b of γ in $\overline{a}$ and $\overline{b}$ respectively.*

Proof. (1) implies *(2)*. Let $u_a \in \overline{a}$ and $u_b \in \overline{b}$ such that $u_a - u_b \in G$. Then $\pi(u_a) = \pi(u_b) \in \pi(\overline{C}) + \langle \gamma \rangle$ so there exists $u_c \in \overline{C}$ and $\lambda \in \mathbb{F}_p^{|\gamma|}$ such that for some (any) representatives r^a and r^b of γ in $\overline{a}$ and $\overline{b}$ respectively, there exists $g_a \in G(\overline{a})$, $g_b \in G(\overline{b})$ and an almost trivial sequence $\lambda \in \mathbb{F}_p^{|\gamma|}$ with

$$\begin{aligned} u_a &= g_a + u_c + \sum_i \lambda_i r_i^a \\ u_b &= g_b + u_c + \sum_i \lambda_i r_i^b. \end{aligned}$$

It follows that $u_a - u_b \in G(\overline{a}) + G(\overline{b}) + \langle r^a - r^b \rangle$.
(2) implies *(1)*. If $u_a \in \overline{a}$ and $u_b \in \overline{b}$ are such that $\pi(u_a) = \pi(u_b)$, then $u_a - u_b \in G(\overline{a} + \overline{b})$ hence $u_a - u_b = g_a + g_b + \sum_i \lambda_i (r_i^a - r_i^b)$ (for an almost trivial sequence $\lambda \in \mathbb{F}_p^{|\gamma|}$). It follows that $u_a - g_a - \sum_i \lambda_i r_i^a \in \overline{a} \cap \overline{b} = \overline{c}$, so $\pi(u_a) \in \pi(\overline{C}) + \langle \gamma \rangle$. ☐

Lemma 6.2.2 (Maximal representative). *Let γ be a tuple $\mathbb{F}_p$-independent over $\pi(\overline{C})$ and d a tuple from K such that $\pi(d) = \gamma$. Then there exists $(K', G') \succ (K, G)$ and a tuple r_γ of length $|\gamma|$ in K', algebraically independent over K such that*

$$G(\overline{Kr_\gamma}) = G(K) + \langle r_\gamma - d \rangle.$$

Furthermore the following hold for all tuples a, b from K containing C such that $\gamma \in \pi(\overline{a}) \cap \pi(\overline{b})$:

(1) if $C = \overline{C}$ then $a \equiv_{C\gamma} b$ if and only if $a \equiv_{\overline{Cr_\gamma}} b$;

(2) $a \mathop{\smile\!\!\!\!\!\mid}\nolimits^{w}_{C\gamma} b$ if and only if $a \mathop{\smile\!\!\!\!\!\mid}\nolimits^{w}_{Cr_\gamma} b$.

Proof. Let x be an algebraically independent tuple over K of size $|d|$, and define H on $K(x)$ to be $G(K) + \langle x - d\rangle$. It is easy to see that $(\overline{K(x)}, H)$ defines a consistent type over K so let r_γ be a realization of this type in an elementary extension (K', G') of (K, G). We may assume that (K', G') is κ-saturated and κ-homogeneous for some big enough κ.

Claim. if $C = \overline{C}$ and $r'_\gamma \equiv_{C\gamma} r_\gamma$ with $r'_\gamma \mathop{\smile\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_{C} b$ and $G(\overline{Cbr'_\gamma}) = G(\bar{b}) + \langle r'_\gamma - r^b\rangle$ for some $r^b \in \pi^{-1}(\gamma) \cap \bar{b}^{|\gamma|}$, then any $\mathscr{L}^G$-isomorphism over $C\gamma$ that sends an enumeration R'_γ of $\overline{Cr'_\gamma}$ to an enumeration R_γ of $\overline{Cr_\gamma}$ (and sends r'_γ to r_γ) extends to an $\mathscr{L}^G$-isomorphism between $\overline{R'_\gamma b}$ and $\overline{R_\gamma b}$ which fixes b.

Proof of the Claim. . Let σ be an automorphism of $(K', K'/G')$ over $C\gamma$ sending r'_γ to r_γ. Then it sends any enumeration R'_γ of $\overline{Cr'_\gamma}$ to an enumeration R_γ of $\overline{Cr_\gamma}$. We may assume that $b = \bar{b}$. By stationarity of the type $tp^{\mathrm{ACF}}(b/C)$, the field isomorphism $\sigma \restriction \overline{CR'_\gamma}$ extends to $\tilde{\sigma} : \overline{bR'_\gamma} \to \overline{bR_\gamma}$ with $\tilde{\sigma}$ fixing b. We show that $\tilde{\sigma}$ is an $\mathscr{L}^G$-isomorphism. First observe that since $G(\overline{Kr_\gamma}) = G(K) + \langle r_\gamma - r^b\rangle$ then $G(\overline{br_\gamma}) = G(b) + \langle r_\gamma - r^b\rangle$. As $\tilde{\sigma}$ fixes b and sends r'_γ to r_γ it is clear that $\tilde{\sigma}$ send $G(\overline{br'_\gamma})$ to $G(\overline{br_\gamma})$ so $\tilde{\sigma}$ is an $\mathscr{L}^G$-isomorphism. Now this isomorphism extends to an automorphism of (K', G') and an automorphism of $(K', K'/G')$ that fixes γ as it send r'_γ to r_γ. □

(1). Assume that $a \equiv_{C\gamma} b$ and let σ be an automorphism of $(K', K'/G')$ over $C\gamma$ sending a on b. As before, we have that $G(\overline{ar_\gamma}) = G(\bar{a}) + \langle r_\gamma - r^a\rangle$ and $G(\overline{br_\gamma}) = G(\bar{b}) + \langle r_\gamma - r^b\rangle$, for some (any) representatives r^a, r^b of γ in $\bar{a}$, $\bar{b}$ respectively. Let R_γ be an enumeration of $\overline{Cr_\gamma}$ and $R'_\gamma = \sigma(R_\gamma)$, $r'_\gamma = \sigma(r_\gamma)$. As $r_\gamma \mathop{\smile\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_{C} a$, we have $r'_\gamma \mathop{\smile\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_{C} b$. Furthermore $G(\overline{ar_\gamma}) = G(\bar{a}) + \langle r_\gamma - r^a\rangle$ and $aR_\gamma \equiv_{C\gamma} bR'_\gamma$, then $G(\overline{Cbr'_\gamma}) = G(\bar{b}) + \langle r'_\gamma - r^b\rangle$. By the claim, $\sigma^{-1} \restriction \overline{Cr'_\gamma}$ extends $\overline{Cr'_\gamma b}$ with the identity on b hence $R_\gamma \equiv_{Cb\gamma} R'_\gamma$. It follows that $aR_\gamma \equiv_{C\gamma} bR_\gamma$. The other direction is trivial.

(2). From left to right. It is clear that $a \mathop{\smile\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_{Cr_\gamma} b$. We want to show that $G(\overline{ar_\gamma} + \overline{br_\gamma}) = G(\overline{ar_\gamma}) + G(\overline{br_\gamma})$. Observe that $G(\overline{abr_\gamma}) = G(\overline{ab}) + \langle r^a - r_\gamma\rangle$ for any tuple r^a from $\bar{a}$ with $\pi(r^a)$. Let $u \in \overline{ar_\gamma}$ and $v \in \overline{br_\gamma}$. If $u + v \in G$ there exists $g_{ab} \in G(\overline{ab})$ and $\lambda \in \mathbb{F}_p^{|x|}$ such that $u + v = g_{ab} + \sum_i \lambda_i(r_i^a - r_{\gamma i})$ for an almost trivial tuple λ. It follows that $g_{ab} \in (\overline{ar_\gamma} + \overline{br_\gamma}) \cap \overline{ab} = \bar{a} + \bar{b}$ by Lemma 1.5.11. As $a \mathop{\smile\!\!\!\!\!\mid}\nolimits^{w}_{C\gamma} b$ and using Lemma 6.2.1, we have that $G(\bar{a} + \bar{b}) = G(\bar{a}) + G(\bar{b}) + \langle r^a - r^b\rangle$. We deduce that $g_{ab} = g_a + g_b + \sum_i \mu_i(r_i^a - r_i^b)$, for an almost trivial tuple μ. For all i, $r_i^a - r_{\gamma i} \in G(\overline{ar_\gamma})$ and $r_{\gamma i} - r_i^b \in G(\overline{br_\gamma})$ hence $g_{ab} = g_a + g_b + \sum_i \mu_i(r_i^a - r_{\gamma i}) + \sum_i \mu_i(r_{\gamma i} - r_i^b) \in G(\overline{ar_\gamma}) + G(\overline{br_\gamma})$. It follows that $u + v \in G(\overline{ar_\gamma}) + G(\overline{br_\gamma})$. The other inclusion being trivial we have $G(\overline{ar_\gamma} + \overline{br_\gamma}) = G(\overline{ar_\gamma}) + G(\overline{br_\gamma})$.

From right to left. First, $r_\gamma \mathop{\smile\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_{C} b$ hence by Transitivity and Monotonicity $a \mathop{\smile\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_{C} b$. By hypothesis, $G(\overline{ar_\gamma} + \overline{br_\gamma}) = G(\overline{ar_\gamma}) + G(\overline{br_\gamma})$. Furthermore $G(\overline{ar_\gamma}) = G(\bar{a}) + \langle r_\gamma - r^a\rangle$ and $G(\overline{br_\gamma}) = G(\bar{b}) + \langle r_\gamma - r^b\rangle$. It is easy to see that

$$(G(\bar{a}) + G(\bar{b}) + \langle r_\gamma - r^a\rangle + \langle r_\gamma - r^b\rangle) \cap (\bar{a} + \bar{b}) = G(\bar{a}) + G(\bar{b}) + \langle r^b - r^a\rangle.$$

It follows that $a \mathop{\smile\!\!\!\!\!\mid}\nolimits^{w}_{C\gamma} b$. □

Remark 6.2.3. Let $\mathop{\smile\!\!\!\!\!\mid}\nolimits^{ST}$ be the following relation, defined for $\gamma \in \pi(\overline{Ca}) \cap \pi(\overline{Cb})$:

$$a \underset{C\gamma}{\mathop{\smile\!\!\!\!\!\mid}^{ST}} b \iff a \underset{C}{\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ACF}}} b \text{ and } G(\overline{Cab}) = G(\overline{Ca}) + G(\overline{Cb}) + \langle r_\gamma^a - r_\gamma^b\rangle$$

for some (any) representatives r_γ^a, r_γ^b of γ in $\overline{Ca}$, $\overline{Cb}$ respectively.

A *maximal representative of* γ *over* C *with respect to* b is a representative r_γ such that $r_\gamma \mathop{\smile\!\!\!\!\!\!|}^{ST}_{C\gamma} b$. The previous result implies that this relation satisfies FULL EXISTENCE and STATIONNARITY over algebraically closed sets. This relation clearly extends the strong independence in (K, G).

Theorem 6.2.4. *The relation* $\mathop{\smile\!\!\!\!\!\!|}^{w}$ *satisfies the following properties.*

1. *(Full Existence) Let* $a, b, C = \overline{C}$ *in* K *and* $\gamma \in K/G$ *such that* $\gamma \in \pi(\overline{Ca}) \cap \pi(\overline{Cb})$ *and* γ $\mathbb{F}_p$*-independent over* $\pi(\overline{C})$*. Then there exists* $a' \equiv_{C\gamma} a$ *such that* $a' \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} b$*.*
2. *(Transitivity) If* $a\alpha \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} b\beta$ *and* $a\alpha \mathop{\smile\!\!\!\!\!\!|}^{w}_{Cb\gamma\beta} d\delta$ *then* $a\alpha \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} bd\beta\delta$
3. *(Independence theorem) Let* $c_1, c_2, a, b, C = \overline{C}$ *in* K *and* $\gamma \in K/G$ *such that* $\gamma \in \pi(\overline{Ca}) \cap \pi(\overline{Cb}) \cap \pi(\overline{Cc_1}) \cap \pi(\overline{Cc_2})$ *and* γ $\mathbb{F}_p$*-independent over* $\pi(\overline{C})$*.*

   If $c_1 \equiv_{C\gamma} c_2$ *and* $c_1 \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} a, c_2 \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} b, a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} b$*, then there exists* c *such that* $c \equiv_{Ca\gamma} c_1$*,* $c \equiv_{Cb\gamma} c_2$ *and* $c \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} a, b$*.*

Proof. *Transitivity* is just checking from the definition of $\mathop{\smile\!\!\!\!\!\!|}^{w}$. For *Full Existence*, assume the hypothesis and let r_γ be a maximal representative as in Lemma 6.2.2. By FULL EXISTENCE of $\mathop{\smile\!\!\!\!\!\!|}^{w}$ in (K, G) there exists $a' \equiv_{Cr_\gamma} a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^{w}_{Cr_\gamma} b$. Using again Lemma 6.2.2, $a' \equiv_{C\gamma} a$ and $a' \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} b$. For *Independence theorem*, we use the same strategy. Assume the hypothesis and let r_γ be a maximal representative of γ as in Lemma 6.2.2. From Lemma 6.2.2, we have that $c_1 \equiv_{\overline{Cr_\gamma}} c_2$ and $c_1 \mathop{\smile\!\!\!\!\!\!|}^{w}_{\overline{Cr_\gamma}} a$, $c_2 \mathop{\smile\!\!\!\!\!\!|}^{w}_{\overline{Cr_\gamma}} b$, $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{\overline{Cr_\gamma}} b$. As $\mathop{\smile\!\!\!\!\!\!|}^{w}$ in (K, G) satisfies $\mathop{\smile\!\!\!\!\!\!|}^{a}$-AMALGAMATION over algebraically closed sets there exists c such that $c \equiv_{\overline{Cr_\gamma}a} c_1$, $c \equiv_{\overline{Cr_\gamma}b} c_2$ and $c \mathop{\smile\!\!\!\!\!\!|}^{w}_{\overline{Cr_\gamma}} a, b$. It follows that $c \equiv_{Ca\gamma} c_1$, $c \equiv_{Cb\gamma} c_2$, and by Lemma 6.2.2, $c \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} a, b$. □

Remark 6.2.5. Notice that $\mathop{\smile\!\!\!\!\!\!|}^{w}$ satisfies $\mathop{\smile\!\!\!\!\!\!|}^{a}$-amalgamation over algebraically closed fields in (K, G). In Theorem 6.2.4, we can weaken the hypothesis $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} b$ to $a \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} b$ because if $a \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} b$ and $r \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} ab$, then $a \mathop{\smile\!\!\!\!\!\!|}^{a}_{Cr} b$ (this result is contained in the proof of Lemma 7.2.2).

6.3 Weak elimination of imaginaries in $(K, K/G)$

The following Lemma is a rewriting of the classical argument for the proof of elimination of imaginaries that appears for instance in [CP98] and [KR18]. It is similar to [CK17, Proposition 4.25], the only difference being that in our case, $\mathop{\smile\!\!\!\!\!\!|}$ is defined only on some subsets, and the base set might contain imaginaries, but the proof is the same.

Lemma 6.3.1. *Let* $\mathscr{M}$ *be a* κ*-homogeneous and* κ*-saturated structure. Let* $E \subset \mathscr{M}^{eq}$*. Assume that there exists a binary relation* $\mathop{\smile\!\!\!\!\!\!|}_{E}$ *on some tuples from* $\mathscr{M}$ *such that*

- *(Invariance) If* $a \mathop{\smile\!\!\!\!\!\!|}_{E} b$ *and* $ab \equiv_E a'b'$ *then* $a' \mathop{\smile\!\!\!\!\!\!|}_{E} b'$
- *(Extension) If* $a \mathop{\smile\!\!\!\!\!\!|}_{E} b$ *and* d *tuple from* $\mathscr{M}$ *then there exists* $a' \equiv_{Eb} a$ *and* $a' \mathop{\smile\!\!\!\!\!\!|}_{E} bd$
- *(Independent consistency) If* $a_1 \mathop{\smile\!\!\!\!\!\!|}_{E} a_2$*,* $b \mathop{\smile\!\!\!\!\!\!|}_{E} a_2$ *and* $a_2 \equiv_E b$*, then there exists* a *such that* $a \equiv_{Ea_1} a_2$*,* $a \equiv_{Ea_2} b$*.*

Let $e \in \mathscr{M}^{eq}$*. If there exists a* 0*-definable function* f *in* $\mathscr{M}^{eq}$ *and* a_1, a_2 *in* $\mathscr{M}$ *such that* $f(a_1) = f(a_2) = e$ *and* $a_1 \mathop{\smile\!\!\!\!\!\!|}_{E} a_2$ *then* $e \in \mathrm{dcl}^{eq}(E)$*.*

Proof. If e is not in $\mathrm{dcl}^{eq}(E)$, then there exists $e' \neq e$ such that $e' \equiv_E e$. Let σ be an automorphism of $\mathscr{M}^{eq}$ over E sending e on e'. Let $b_1b_2 = \sigma(a_1a_2)$. By Invariance, $b_1 \mathop{\smile\!\!\!\!\!\!|}_{E} b_2$ and $f(b_1) = f(b_2) = e'$. By Extension there exists $b \equiv_{Eb_1} b_2$ such that $b \mathop{\smile\!\!\!\!\!\!|}_{E} a_2$. By Independent Consistency, there exists a such that $a \equiv_{Ea_1} a_2$, $a \equiv_{Ea_2} b$. From $a \equiv_{Ea_1} a_2$ follows that $f(a) = f(a_1) = e$ and from $a \equiv_{Ea_2} b$ follows that $f(a) \neq e$, a contradiction. □

Remark 6.3.2. Recall that Extension follows from Full Existence, Symmetry and Transitivity. Independent consistency is a consequence of the independence theorem. It follows from Theorem 6.2.4 that for all $C = \overline{C}$ and γ $\mathbb{F}_p$-independent over $\pi(\overline{C})$, the restriction of $\mathop{\smile\hskip-0.9em^{|}\ }^{w}_{C\gamma}$ to tuples a such that $\gamma \in \pi(\overline{Ca})$ satisfies the hypothesis of the previous Lemma.

The following classical fact follows from a group theoretic Lemma due to P.M. Neumann ([Neu76]). It appears first in [EH93, Lemma 1.4].

Fact 6.3.3. *Let $\mathscr{M}$ be a saturated model, X a 0-definable set, $e \in \mathscr{M}$, $E = \operatorname{acl}(e) \cap X$ and a tuple a from X. Then there is a tuple b from X such that*

$$a \equiv_{Ee} b \text{ and } \operatorname{acl}(Ea) \cap \operatorname{acl}(Eb) \cap X = E.$$

Theorem 6.3.4. *Let $e \in (K, G)^{eq}$ then there exists a tuple $c\gamma$ from $(K, K/G)$ such that $c\gamma \in \operatorname{acl}^{eq}(e)$ and $e \in \operatorname{dcl}^{eq}(c\gamma)$. It follows that $(K, K/G)$ has weak elimination of imaginaries.*

Proof. We work in $(K, G)^{eq}$, seeing $(K, K/G)$ as a 0-definable subset. Suppose that e is an imaginary element, there is a tuple a from K and a 0-definable function f such that $e = f(a)$. We set $C\langle\pi(C)\gamma\rangle = \operatorname{acl}^{eq}(e) \cap (K, K/G)$. We may assume that γ is $\mathbb{F}_p$-linearly independent over $\pi(C)$. As $\gamma \subseteq \operatorname{acl}^{eq}(e) \cap K/G \subseteq \operatorname{acl}^{eq}(a) \cap K/G$ we have that $\operatorname{acl}^{eq}(Ca\gamma) \cap (K, K/G) = \overline{Ca}\pi(\overline{Ca})$ and $\gamma \subseteq \pi(\overline{Ca})$. By Fact 6.3.3 there exists $b \equiv_{C\gamma e} a$ such that

$$\operatorname{acl}^{eq}(Ca\gamma) \cap \operatorname{acl}^{eq}(Cb\gamma) \cap (K, K/G) = C\langle\pi(C)\gamma\rangle.$$

Again, $\operatorname{acl}^{eq}(Cb\gamma) \cap (K, K/G) = \overline{Cb}\pi(\overline{Cb})$ and $\gamma \subseteq \pi(\overline{Cb})$. Furthermore $f(b) = e$ and

$$(\overline{Ca}\pi(\overline{Ca})) \cap (\overline{Cb}\pi(\overline{Cb})) = C\langle\pi(C)\gamma\rangle.$$

We construct a sequence $(a_i)_{i<\omega}$ such that

$$a_{n+1} \mathop{\smile\hskip-0.9em^{|}\ }\limits^{w}_{C\gamma a_n} a_1, \dots, a_{n-1} \text{ and } a_n a_{n+1} \equiv_{C\gamma} ab.$$

Start by $a_1 = a$ and $a_2 = b$. Assume that $a_1, \dots, a_n$ has already been constructed. We have that $a_{n-1} \equiv_{C\gamma} a_n$ so let σ be a $c\gamma$-automorphism of the monster such that $\sigma(a_{n-1}) = a_n$. By Full Existence (Theorem 6.2.4) there exists $a_{n+1} \equiv_{Ca_n\gamma} \sigma(a_n)$ such that $a_{n+1} \mathop{\smile\hskip-0.9em^{|}\ }^{w}_{Ca_n\gamma} a_1, \dots, a_{n-1}$. It follows that

$$a_n a_{n+1} \equiv_{C\gamma} a_n \sigma(a_n) \equiv_{C\gamma} a_{n-1} a_n.$$

Let $(a_i)_{i<\omega}$ be such a sequence. In particular the following holds for all $i < j < k$

$$a_k \mathop{\smile\hskip-0.9em^{|}\ }\limits^{\mathrm{ACF}}_{Ca_j} a_i,\ \overline{Ca_i} \cap \overline{Ca_j} = \overline{C} \text{ and } \pi(\overline{Ca_i}) \cap \pi(\overline{Ca_j}) = \langle\pi(C)\gamma\rangle.$$

By Ramsey and compactness we may assume that $(a_i)_{i<\omega}$ is indiscernible over $C\gamma$. As the three properties above holds for the whole sequence, it is in the Erenfeucht-Mostowski type of the sequence, and hence is still true for the indiscernible sequence. Note that $f(a_i) = e$. We have that $(a_i)_{i<\omega}$ is totally indiscernible over C in the sense of ACF hence $a_1a_2a_3 \equiv_C^{\mathrm{ACF}} a_1a_3a_2$. Furthermore we have $a_1 \mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{ACF}}_{Ca_2} a_3$, hence by INVARIANCE $a_1 \mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{ACF}}_{Ca_3} a_2$. By elimination of imaginaries in ACF it follows that $a_1 \mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{ACF}}_{C} a_2$, since $\overline{Ca} \cap \overline{Cb} = \overline{C}$. As $\pi(\overline{Ca_1}) \cap \pi(\overline{Ca_2}) = \langle\pi(C)\gamma\rangle$, we have that

$$a_1 \mathop{\smile\hskip-0.9em^{|}\ }\limits^{w}_{C\gamma} a_2.$$

As $f(a_1) = f(a_2) = e$, we deduce from Lemma 6.3.1 that $e \in \operatorname{dcl}^{eq}(C\gamma)$. $\square$

Example 6.3.5 (($(K, K/G)$ does not eliminate finite imaginaries)**.** The structure on K/G is the one of an $\mathbb{F}_p$-vector space (with twisted algebraic and definable closures, $\mathrm{acl}(\alpha) = \langle \pi(\overline{\mathbb{F}_p})\alpha \rangle$ and $\mathrm{dcl}(\alpha) = \langle \pi(\mathrm{dcl}(\mathbb{F}_p))\alpha \rangle$). This follows from Corollaries 6.1.3 and 6.1.5. Consider the unordered pair $\{\alpha, \beta\}$ for two singletons $\alpha, \beta \in K/G$, linearly independent over $\pi(\overline{\mathbb{F}_p})$. Assume that there exists a tuple $d\gamma$ such that for all automorphism σ of $(K, K/G)$

$$\sigma(d\gamma) = d\gamma \iff \sigma(\{\alpha, \beta\}) = \{\alpha, \beta\}\,.$$

As $d\gamma$ and $\alpha\beta$ are interalgebraic, we have first that $d \subset \overline{\mathbb{F}_p}$ and hence $\alpha, \beta \in \mathrm{acl}(\gamma) \cap K/G = \mathrm{dcl}(\gamma) \cap K/G = \langle \gamma \rangle$. As α, β are linearly independent over $\mathrm{acl}(\emptyset)$, we have $\alpha\beta \equiv_\emptyset \beta\alpha$ so let σ be an automorphism of $(K, K/G)$ sending $\alpha\beta$ on $\beta\alpha$. As σ fixes γ, it fixes $\langle \gamma \rangle$ hence $\alpha = \beta$, a contradiction.

Example 6.3.6 ($K \times (K/G)^{eq}$ does not eliminate finite imaginaries)**.** Let t be a transcendental element over $\mathbb{F}_p$. We assume that $G(\overline{\mathbb{F}_p(t)}) = \overline{\mathbb{F}_p(t)}$ (in a model (K, G) of ACFG such that $G(\overline{\mathbb{F}_p}) = \overline{\mathbb{F}_p}$). Let $\alpha, \beta \in K/G$ be $\mathbb{F}_p$-independent, and let e be the unordered pair $\left\{\sqrt{t}\alpha, -\sqrt{t}\beta\right\}$. We have the following:

(1) $\mathrm{dcl}^{eq}(e) \cap K = \mathrm{dcl}(t)$

(2) $\mathrm{dcl}^{eq}(e) \cap (K/G)^{eq} = \mathrm{dcl}^{eq}(\{\alpha, \beta\}) \cap (K/G)^{eq}$

(1) The right to left inclusion is clear. Let $u \in \mathrm{dcl}^{eq}(e) \cap K$, in particular $u \in \mathrm{dcl}^{eq}(t, \alpha\beta) \cap K \subseteq \mathrm{acl}^{eq}(t, \alpha\beta) \cap K = \overline{\mathbb{F}(t)}$. Assume that $u \notin \mathrm{dcl}(t)$. There exists $u' \neq u$ with $u' \equiv_t u$. Let α', β' such that $u'\alpha'\beta' \equiv_t u\alpha\beta$. As α, β and α', β' are $\mathbb{F}_p$-lineary independent over $\pi(\overline{\mathbb{F}(t, u)}) = \pi(\overline{\mathbb{F}(t)}) = \{0\}$, we have that $\alpha\beta \equiv_{\overline{\mathbb{F}_p(t)}} \alpha'\beta'$ (Corollary 6.1.3). It follows that $u' \equiv_{t,\alpha,\beta} u$ hence $u' \equiv_e u$ so $u \notin \mathrm{dcl}^{eq}(e)$, a contradiction.
(2) The right to left inclusion is clear. Let $\{\gamma_1, \dots, \gamma_n\}$ be an element of $\mathrm{dcl}^{eq}(e) \cap (K/G)^{eq}$. For all i, γ_i is algebraic over $t\alpha\beta$, by Corollary 6.1.5 $\gamma_i \in \langle \pi(\overline{\mathbb{F}_p(t)}), \alpha, \beta \rangle = \langle \alpha, \beta \rangle$. It follows that permutations of the set $\left\{\sqrt{t}\alpha, -\sqrt{t}\beta\right\}$ that permutes $\{\gamma_1, \dots, \gamma_n\}$ are exactly permutations of the set $\{\alpha, \beta\}$ that permutes $\{\gamma_1, \dots, \gamma_n\}$ hence $\{\gamma_1, \dots, \gamma_n\} \in \mathrm{dcl}^{eq}(\{\alpha, \beta\})$. In fact, such a set $\{\gamma_1, \dots, \gamma_n\}$ is the union of two sets of the same cardinal (possibly intersecting), every element in one set is of the form $\lambda\alpha + \mu\beta$ and has a "dual" element $\mu\alpha + \lambda\beta$ in the other set.

If e is interdefinable with an element from $K \times (K/G)^{eq}$, by *(1)* and *(2)*, we may assume that $e \in \mathrm{dcl}^{eq}(t\,\{\alpha, \beta\})$. By hypothesis $\alpha\beta \equiv_{\overline{\mathbb{F}_p(t)}} \beta\alpha$, hence an automorphism sending $\sqrt{t}, -\sqrt{t}\alpha\beta$ to $\sqrt{t}, -\sqrt{t}\beta\alpha$ fixes $t\,\{\alpha, \beta\}$ and moves e to $\left\{\sqrt{t}\beta, -\sqrt{t}\alpha\right\}$, hence $e \notin \mathrm{dcl}^{eq}(t\,\{\alpha, \beta\})$, a contradiction.

CHAPTER 7

Forking and thorn-forking in ACFG

In this chapter, we give a description of forking and thorn-forking in the theory ACFG. We also link these notions with other classical relations or other independence relations encountered in the previous chapters. The results of this chapter are summarized by the diagram Figure 7.1, in which all arrows are strict.

Contents

$$\begin{tikzcd}
 & & & & \mathop{\smile\hskip-0.9em^|\ }^{\mathrm{ACF}} \arrow[d, equal] & \\
 & & \mathop{\smile\hskip-0.9em^|\ }^{\text{þ}} \restriction K \arrow[d, equal] & & \mathop{\smile\hskip-0.9em^|\ }^{a\,m} \arrow[dr] & \\
\mathop{\smile\hskip-0.9em^|\ }^{st} \arrow[r] & \mathop{\smile\hskip-0.9em^|\ }^{wsm} \arrow[r] & \mathop{\smile\hskip-0.9em^|\ }^{wm} \arrow[r] \arrow[d, equal] & \mathop{\smile\hskip-0.9em^|\ }^{w} \arrow[ur] \arrow[d, equal] \arrow[dr] & & \mathop{\smile\hskip-0.9em^|\ }^{a} \\
 & & \mathop{\smile\hskip-0.9em^|\ }^{d} \arrow[d, equal] & \mathop{\smile\hskip-0.9em^|\ }^{K} & \mathop{\smile\hskip-0.9em^|\ }^{aeq}\restriction K. \arrow[ur] & \\
 & & \mathop{\smile\hskip-0.9em^|\ }^{f} & & &
\end{tikzcd}$$

Figure 7.1: Interactions of independence relations in ACFG.

7.1 Forcing base monotonicity and extension

In this subsection, given a ternary relation $\mathop{\smile\!\!\!\!\!\!\!\mid}$ in an arbitrary theory, we introduce the relations $\mathop{\smile\!\!\!\!\!\!\!\mid}^m$ and $\mathop{\smile\!\!\!\!\!\!\!\mid}^*$, following the work of Adler in [Adl09a].

Definition 7.1.1 (Monotonised)**.** Let $\mathop{\smile\!\!\!\!\!\!\!\mid}$ be any ternary relation, we define $\mathop{\smile\!\!\!\!\!\!\!\mid}^m$ to be the relation defined by

$$A \mathop{\smile\!\!\!\!\!\!\!\mid}^m_C B \iff \forall D \subseteq CB \;\; A \mathop{\smile\!\!\!\!\!\!\!\mid}_{CD} BC.$$

We call $\mathop{\smile\!\!\!\!\!\!\!\mid}^m$ the *monotonised* of $\mathop{\smile\!\!\!\!\!\!\!\mid}$.

Note that the relation $\mathop{\smile\!\!\!\!\!\!\!\mid}^M$ in [Adl09a, Section 4] is the relation $\mathop{\smile\!\!\!\!\!\!\!\mid}^{a\,m}$ in our context.

Lemma 7.1.2. *The relation $\mathop{\smile\!\!\!\!\!\!\!\mid}^m$ satisfies* Base Monotonicity*. Furthermore, for each of the following point*

- Invariance
- Monotonicity
- Transitivity

if $\mathop{\smile\!\!\!\!\!\!\!\mid}$ satisfies it then so does $\mathop{\smile\!\!\!\!\!\!\!\mid}^m$.

Proof. Let A, B, C, D such that $A \mathop{\smile\!\!\!\!\!\!\!\mid}^m_C BD$. Then for all $D' \subseteq \mathrm{acl}(BCD)$ we have that $A \mathop{\smile\!\!\!\!\!\!\!\mid}_{CD'} B$ so in particular for all $D' \subseteq \mathrm{acl}(BCD)$ containing D we have $A \mathop{\smile\!\!\!\!\!\!\!\mid}_{CD'} B$ hence for all $D'' \subseteq \mathrm{acl}(BCD)$ we have $A \mathop{\smile\!\!\!\!\!\!\!\mid}_{CDD''} B$ hence $A \mathop{\smile\!\!\!\!\!\!\!\mid}^m_{CD} B$. To prove that Invariance is preserved, note that there exists an isomorphism $\sigma : ABC \to A'B'C'$ which extends to $\mathrm{acl}(ABC) \to \mathrm{acl}(A'B'C')$ and so induces an isomorphism $ABCD \to A'B'C'\sigma(D)$ for all $D \subseteq \mathrm{acl}(BC)$. For Monotonicity, it is an easy checking. For Transitivity Assume that $B \mathop{\smile\!\!\!\!\!\!\!\mid}^m_C A$ and $A' \mathop{\smile\!\!\!\!\!\!\!\mid}^m_{CB} A$, and take $D \subseteq \mathrm{acl}(AC)$. We have in particular that $B \mathop{\smile\!\!\!\!\!\!\!\mid}_{CD} A$ and $A' \mathop{\smile\!\!\!\!\!\!\!\mid}_{CBD} A$ hence using Transitivity of $\mathop{\smile\!\!\!\!\!\!\!\mid}$ we have $A'B \mathop{\smile\!\!\!\!\!\!\!\mid}_{CD} A$. This holds for any $D \subseteq \mathrm{acl}(AC)$ hence $A'B \mathop{\smile\!\!\!\!\!\!\!\mid}^m_C A$. □

Let $\mathop{\smile\!\!\!\!\!\!\!\mid}$, $\mathop{\smile\!\!\!\!\!\!\!\mid}'$ be two ternary relations, such that $\mathop{\smile\!\!\!\!\!\!\!\mid}'$ is stronger than $\mathop{\smile\!\!\!\!\!\!\!\mid}$. If $\mathop{\smile\!\!\!\!\!\!\!\mid}'$ satisfies Base Monotonicity then $\mathop{\smile\!\!\!\!\!\!\!\mid}'$ is stronger than $\mathop{\smile\!\!\!\!\!\!\!\mid}^m$. Note that $\mathop{\smile\!\!\!\!\!\!\!\mid}$ may be symmetric and $\mathop{\smile\!\!\!\!\!\!\!\mid}^m$ not (see Corollary 7.2.3). However in some cases, the monotonised is symmetric, as shows the following example.

Example 7.1.3. We work here in ACF. We have

$$A \mathop{\smile\!\!\!\!\!\!\!\mid}^{a\,m}_C B \iff A \mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_C B.$$

Indeed the right to left implication follows from $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{ACF}} \to \mathop{\smile\!\!\!\!\!\!\!\mid}^a$ and the fact that $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{ACF}}$ satisfies Base Monotonicity. From left to right, assume that $A \mathop{\not\smile\!\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_C B$, we may assume that A, B, C are algebraically closed, and $C = A \cap B$. There exists $b_1, \dots, b_s \in B$ algebraically independent over C such that for $D = \{b_2, \dots, b_s\}$, then we have $b_1 \in (\overline{AD} \cap B) \setminus \overline{CD}$ so $A \mathop{\not\smile\!\!\!\!\!\!\!\mid}^{a\,m}_C B$.

This result translates as follows: in ACF, $\mathop{\smile\!\!\!\!\!\!\!\mid}^f = \mathop{\smile\!\!\!\!\!\!\!\mid}^{a\,m}$. It raises the following question: when do we recover forking independence from the monotonised of the relation $\mathop{\smile\!\!\!\!\!\!\!\mid}^a$? Does the Symmetry of the monotonised of a symmetric relation imply nice features on the theory? Observe that the proof above shows that in any pregeometry (S, cl), the independence relation associated with the pregeometry is obtained by forcing Base Monotonicity on the relation $A \mathop{\smile\!\!\!\!\!\!\!\mid}_C B \iff \mathrm{cl}(AC) \cap \mathrm{cl}(BC) = \mathrm{cl}(C)$.

The following example shows that the monotonised does not preserve Local Character. Also it implies that $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{st}$ doesn't satisfy Local Character since $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{st} \to \mathop{\smile\!\!\!\!\!\!|}\nolimits^{wm}$.

Example 7.1.4. In ACFG, the relation $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{wm}$ does not satisfy Local Character.
Let κ be any uncountable cardinal and consider the set $A = \{t_i, t_i' \mid i < \kappa\}$ and an element t such that $t(t_i, t_i')_{i<\kappa}$ are algebraically independent over K. Let $F = \mathbb{F}_p(t, A)$ and define H over F as $G(\overline{\mathbb{F}_p}) + \langle t \cdot t_i + t_i' \mid i < \kappa\rangle$. The pair (F, H) defines a consistent type over $\emptyset$, as $\overline{\mathbb{F}_p} \cap H = G(\overline{\mathbb{F}_p})$ and $F \cap K = \overline{\mathbb{F}_p}$, so we assume that t, A are realisation of the type in K. By contradiction suppose that there exists $A_0 \subset A$ with $|A_0| \leq \aleph_0$ such that $t \mathop{\smile\!\!\!\!\!\!|}\nolimits^{wm}_{A_0} A$. By definition, for all $D \subseteq A$ we have $t \mathop{\smile\!\!\!\!\!\!|}\nolimits^{w}_{A_0 D} A$. Let $D = \{t_i \mid i < \kappa\} \setminus A_0$. We have that

$$G(\overline{tDA_0} + \overline{A}) = G(\overline{tDA_0}) + G(\overline{A}).$$

We compute the $\mathbb{F}_p$-dimension over $G(\overline{\mathbb{F}_p})$ on each side of the previous equation. On one hand, we have $t \cdot t_i + t_i' \in G(\overline{tDA_0} + \overline{A})$ for all $i < \kappa$, as they are $\mathbb{F}_p$-linearly independent over $\overline{\mathbb{F}_p}$ we have $\mathbb{F}_p\text{-}\dim(G(\overline{tDA_0} + \overline{A})/G(\overline{\mathbb{F}_p})) \geq \kappa$. For all $i < \kappa$, $t \cdot t_i + t_i' \in G(\overline{tDA_0})$ if and only if $t_i' \in \overline{tDA_0}$ if and only if $t_i' \in A_0$, because if t_i' is algebraic over $t, A_0, t_1, \ldots, t_k$ then t is in A_0 otherwise this contradicts that t, A are algebraically independent. We conclude that $\mathbb{F}_p\text{-}\dim(G(\overline{tDA_0})/G(\overline{\mathbb{F}_p})) \leq |A_0| \leq \aleph_0$. As $G(\overline{A}) = G(\overline{\mathbb{F}_p})$ we have that $\mathbb{F}_p\text{-}\dim([G(\overline{tDA_0}) + G(\overline{A})]/G(\overline{\mathbb{F}_p})) \leq \aleph_0$ so the equality cannot hold.

Definition 7.1.5 (Adler, [Adl09a] Section 3)**.** For $\mathop{\smile\!\!\!\!\!\!|}$ any ternary relation, $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}$ is defined as follows:

$$A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B \iff \forall \hat{B} \supseteq B\ \exists A' \equiv_{BC} A\ A' \mathop{\smile\!\!\!\!\!\!|}_{C} \hat{B}.$$

Fact 7.1.6 ([Adl09a] Lemma 3.1)**.** *If $\mathop{\smile\!\!\!\!\!\!|}$ satisfies* Invariance *and* Monotonicity *then $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}$ satisfies* Invariance, Monotonicity *and* Extension. *Furthermore, for each of the following point*

- Base Monotonicity
- Transitivity
- Full Existence

if $\mathop{\smile\!\!\!\!\!\!|}$ satisfies it then so does $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}$.

Recall from Section 1.2 that $a \mathop{\smile\!\!\!\!\!\!|}\nolimits^{u}_{C} b$ if and only if $tp(a/Cb)$ is finitely satisfiable in C.

Remark 7.1.7. Let $(b_i)_{i<\kappa}$ be a C-indiscernible infinite sequence with $\kappa > \omega$. Then for all $\geq \alpha \geq \omega$

$$b_{<\beta} \mathop{\smile\!\!\!\!\!\!|}^{u}_{Cb_{<\alpha}} b_\beta.$$

Furthermore, for κ big enough, the sequence $(b_i)_{i<\kappa}$ is indiscernible over $\mathrm{acl}(C)$ (see [Cas11, Corollary 1.7, 2.]).

Remark 7.1.8. By Lemma 1.2.5 and Fact 7.1.6, if $\mathop{\smile\!\!\!\!\!\!|}$ satisfies Invariance, Monotonicity, then $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}$ satisfies Invariance, Monotonicity, Extension and Closure over algebraically closed sets. If $\mathop{\smile\!\!\!\!\!\!|}$ satisfies also Base Monotonicity, then so does $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}$ hence $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}$ satisfies Closure over any sets. In particular, by Lemma 7.1.2 if $\mathop{\smile\!\!\!\!\!\!|}$ satisfies Invariance and Monotonicity, then $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{m\,*}$ satisfies Invariance, Monotonicity, Closure, Base Monotonicity, Extension. Assume that $\mathop{\smile\!\!\!\!\!\!|}$ satisfies Full Existence and Transitivity, then $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}$ satisfies the following $a \mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}_{C} b \to \mathrm{acl}(Ca) \mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}_{C} b$. Indeed, assume that $a \mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}_{C} b$, then by Fact 7.1.6, $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}$ satisfies Full Existence so we have $\mathrm{acl}(Ca) \mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}_{Ca} b$. By Fact 7.1.6, $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}$ also satisfies Transitivity, hence $\mathrm{acl}(Ca) \mathop{\smile\!\!\!\!\!\!|}\nolimits^{*}_{C} b$. By Lemma 7.1.2 and Fact 7.1.6, if $\mathop{\smile\!\!\!\!\!\!|}$ satisfies

Transitivity then so does $\mathop{\smile\!\!\!\!\!\!|}^{m\,*}$. It follows that if $\mathop{\smile\!\!\!\!\!\!|}$ satisfies Invariance, Monotonicity, Transitivity and if $\mathop{\smile\!\!\!\!\!\!|}^{m}$ satisfies Full Existence, then

$$a \mathop{\smile\!\!\!\!\!\!|}_{C} b \to \operatorname{acl}(Ca) \mathop{\smile\!\!\!\!\!\!|}^{m\,*}_{\operatorname{acl}(C)} \operatorname{acl}(Cb).$$

Lemma 7.1.9. *Let $\mathop{\smile\!\!\!\!\!\!|}$ be a ternary relation, which satisfies*

- *Invariance, Monotonicity;*
- *$\mathop{\smile\!\!\!\!\!\!|}^{u}$-amalgamation over algebraically closed sets.*

Then $\mathop{\smile\!\!\!\!\!\!|}^{m\,*} \to \mathop{\smile\!\!\!\!\!\!|}^{f}$.

Proof. We show that $\mathop{\smile\!\!\!\!\!\!|}^{m\,*} \to \mathop{\smile\!\!\!\!\!\!|}^{d}$, the result follows from the fact that $\mathop{\smile\!\!\!\!\!\!|}^{f} = \mathop{\smile\!\!\!\!\!\!|}^{d\,*}$ (Section 1.2). By Lemma 7.1.2, Fact 7.1.6, Remark 7.1.8, and the hypothesis on $\mathop{\smile\!\!\!\!\!\!|}$, $\mathop{\smile\!\!\!\!\!\!|}^{m\,*}$ satisfies Invariance, Monotonicity, Base Monotonicity, Extension and Closure. Assume $a \mathop{\smile\!\!\!\!\!\!|}^{m\,*}_{C} b$, for any a, b, C. Let $(b_i)_{i<\kappa}$ be a C-indiscernible sequence with $b = b_0$, for a big enough κ. By Remark 7.1.7, $b_{<i} \mathop{\smile\!\!\!\!\!\!|}^{u}_{Cb_{<\omega}} b_i$ for all $i \geq \omega$. By Fact 1.2.3, and Lemma 1.2.5, $\mathop{\smile\!\!\!\!\!\!|}^{u}$ satisfies Closure and Monotonicity, hence $b_{<i} \mathop{\smile\!\!\!\!\!\!|}^{u}_{\operatorname{acl}(Cb_{<\omega})} b_i$. Also $(b_i)_{i\geq\omega}$ is $Cb_{<\omega}$-indiscernible, so if κ is big enough, by Remark 7.1.7 we have that $b_i \equiv_{\operatorname{acl}(Cb_{<\omega})} b_\omega$. There exists a C-automorphisme sending b to b_ω hence there exists some a_ω such that $a_\omega b_\omega \equiv_C ab$. By Invariance, we have $a_\omega \mathop{\smile\!\!\!\!\!\!|}^{m\,*}_{C} b_\omega$, so by Closure we have $a_\omega \mathop{\smile\!\!\!\!\!\!|}^{m\,*}_{\operatorname{acl}(C)} \operatorname{acl}(Cb_\omega)$, hence by Extension there exists a'_ω such that $a'_\omega \equiv_{\operatorname{acl}(Cb_\omega)} a_\omega$ and $a'_\omega \mathop{\smile\!\!\!\!\!\!|}^{m\,*}_{\operatorname{acl}(C)} b_\omega b_{<\omega}$. It follows from Closure and Base Monotonicity that

$$a'_\omega \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl}(Cb_{<\omega})} b_\omega.$$

We also have

$$a'_\omega b_\omega \equiv_C a_\omega b_\omega \equiv_C ab.$$

For each $i \geq \omega$ there exists an $\operatorname{acl}(Cb_{<\omega})$-automorphism σ_i sending b_ω to b_i, so setting $a'_i = \sigma_i(a'_\omega)$ we have:

$$\forall i \geq \omega \quad a'_i b_i \equiv_{\operatorname{acl}(Cb_{<\omega})} a'_\omega b_\omega \text{ and } a'_i \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl}(Cb_{<\omega})} b_i.$$

We show that there exists a'' such that $a'' b_i \equiv_{\operatorname{acl}(Cb_{<\omega})} a_\omega b_\omega$ for all $\omega \leq i < \omega + \omega$. By induction and compactness, it is sufficient to show that for all $\omega \leq i < \omega + \omega$, there exists a''_i such that for all $\omega \leq k \leq i$ we have $a''_i b_k \equiv_{\operatorname{acl}(Cb_{<\omega})} a_\omega b_\omega$ and $a''_i \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl}(Cb_{<\omega})} b_{\leq i}$. For the case $i = \omega$ take $a''_\omega = a'_\omega$. Assume that a''_i has been constructed, we have

$$a'_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl}(Cb_{<\omega})} b_{i+1} \text{ and } b_{\leq i} \mathop{\smile\!\!\!\!\!\!|}^{u}_{\operatorname{acl}(Cb_{<\omega})} b_{i+1} \text{ and } a''_i \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl}(Cb_{<\omega})} b_{\leq i}.$$

As $a'_{i+1} \equiv_{\operatorname{acl}(Cb_{<\omega})} a''_i$, by $\mathop{\smile\!\!\!\!\!\!|}^{u}$-amalgamation over algebraically closed sets, there exists a''_{i+1} such that

(1) $a''_{i+1} b_{i+1} \equiv_{\operatorname{acl}(Cb_{<\omega})} a'_{i+1} b_{i+1}$

(2) $a''_{i+1} b_{\leq i} \equiv_{\operatorname{acl}(Cb_{<\omega})} a''_i b_{\leq i}$

(3) $a''_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl}(Cb_{<\omega})} b_{\leq i+1}$.

By induction and compactness there exists a'' be such that $a'' b_i \equiv_{\operatorname{acl}(Cb_{<\omega})} a_\omega b_\omega$ for all $\omega \leq i < \omega + \omega$. By indiscernibility of $(b_i)_{i<\kappa}$ there exists a''' such that for all $i < \kappa$ $a''' b_i \equiv_C ab$, hence $a \mathop{\smile\!\!\!\!\!\!|}^{d}_{C} b$. ☐

Remark 7.1.10. It is important to observe that since $\mathop{\smile\hskip-0.9em ^| \ }^u$ is not in general a symmetric relation, the parameters a and b in the statement of $\mathop{\smile\hskip-0.9em ^| \ }^u$-AMALGAMATION do not play a symmetrical role. If a relation satisfies $\mathop{\smile\hskip-0.9em ^| \ }^u$-amalgamation, we mean that $tp(c_1/Ca)$ and $tp(c_2/Cb)$ can be amalgamated whenever $a \mathop{\smile\hskip-0.9em ^| \ }^u_C b$ or $b \mathop{\smile\hskip-0.9em ^| \ }^u_C a$.

Proposition 7.1.11. *Let $\mathop{\smile\hskip-0.9em ^| \ }$ be a relation such that*

(1) $\mathop{\smile\hskip-0.9em ^| \ }$ is weaker than $\mathop{\smile\hskip-0.9em ^| \ }^d$;

(2) $\mathop{\smile\hskip-0.9em ^| \ }$ satisfies INVARIANCE*,* MONOTONICITY*, $\mathop{\smile\hskip-0.9em ^| \ }^u$-*AMALGAMATION *over algebraically closed sets;*

(3) $\mathop{\smile\hskip-0.9em ^| \ }^m$ satisfies EXTENSION *over algebraically closed sets;*

Then $\mathop{\smile\hskip-0.9em ^| \ }^m = \mathop{\smile\hskip-0.9em ^| \ }^f = \mathop{\smile\hskip-0.9em ^| \ }^d$.

Proof. The relation $\mathop{\smile\hskip-0.9em ^| \ }^d$ satisfies BASE MONOTONICITY by Fact 1.2.3 hence from *(1)* we have $\mathop{\smile\hskip-0.9em ^| \ }^d \to \mathop{\smile\hskip-0.9em ^| \ }^m$. By hypothesis *(3)*, $\mathop{\smile\hskip-0.9em ^| \ }^m = \mathop{\smile\hskip-0.9em ^| \ }^{m*}$, hence by *(2)* and Lemma 7.1.9 we have $\mathop{\smile\hskip-0.9em ^| \ }^d = \mathop{\smile\hskip-0.9em ^| \ }^m = \mathop{\smile\hskip-0.9em ^| \ }^f$. ☐

7.2 Forking in ACFG

We show that forking in ACFG is obtained by forcing the property BASE MONOTONICITY on Kim-independence.

We work in a big model (K, G) of ACFG.

Lemma 7.2.1. *Let A, B, C be three additive subgroups of K, then $A \cap (B + C) = A \cap [B + C \cap (A + B)]$.*

Proof. Let $a \in A \cap (B + C)$. There exist $b \in B$ and $c \in C$, such that $a = b + c$. Then $c = a - b \in C \cap (A + B)$ hence $a \in A \cap [B + C \cap (A + B)]$. The other inclusion is trivial. ☐

Lemma 7.2.2 (Mixed Transitivity on the left)**.** *Let A, B, C, D be algebraically closed sets, with A, B, D containing C and $B \subseteq D$. If $A \mathop{\smile\hskip-0.9em ^| \ }^{wm}_C B$ and $A \mathop{\smile\hskip-0.9em ^| \ }^{st}_B D$ then $A \mathop{\smile\hskip-0.9em ^| \ }^{wm}_C D$.*

Proof. Let A, B, C, D be as in the hypothesis. Let $E \subseteq D$ containing C, we want to show that $A \mathop{\smile\hskip-0.9em ^| \ }^w_E D$. We may assume that E is algebraically closed. We clearly have $A \mathop{\smile\hskip-0.9em ^| \ }^{\mathrm{ACF}}_E D$, so we have to show that

$$G(\overline{AE} + D) = G(\overline{AE}) + G(D).$$

From $A \mathop{\smile\hskip-0.9em ^| \ }^{\mathrm{ACF}}_C E, B$ we have $\overline{AE} \cap \overline{AB} \mathop{\smile\hskip-0.9em ^| \ }^{\mathrm{ACF}}_E E, B$ and $\overline{AE} \cap \overline{AB} \mathop{\smile\hskip-0.9em ^| \ }^{\mathrm{ACF}}_B E, B$. By elimination of imaginaries in ACF, $\overline{AE} \cap \overline{AB} \mathop{\smile\hskip-0.9em ^| \ }^{\mathrm{ACF}}_{E \cap B} E, B$. By Lemma 1.5.11, it follows that $\overline{AE} \cap \overline{AB} = \overline{A(E \cap B)}$.

Claim. $(\overline{AE} + D) \cap (\overline{AB} + D) = \overline{A(E \cap B)} + D$.

Proof of the claim. By modularity, we have that $(\overline{AE} + D) \cap (\overline{AB} + D) = D + \overline{AE} \cap (\overline{AB} + D)$. By Lemma 7.2.1 we have that

$$\overline{AE} \cap (\overline{AB} + D) = \overline{AE} \cap \left(\overline{AB} + (\overline{AE} + \overline{AB}) \cap D\right).$$

By Lemma 1.5.11, we have $(\overline{AE} + \overline{AB}) \cap D = E + B$, hence

$$\begin{aligned}
\overline{AE} \cap (\overline{AB} + D) &= \overline{AE} \cap (\overline{AB} + E + B) \\
&= \overline{AE} \cap (\overline{AB} + E) \\
&= \overline{AE} \cap \overline{AB} + E \text{ by modularity} \\
&= \overline{A(E \cap B)} + E.
\end{aligned}$$

It follows that $(\overline{AE}+D)\cap(\overline{AB}+D)=\overline{A(E\cap B)}+D+E=\overline{A(E\cap B)}+D$. $\square$

By hypothesis, $G(\overline{AD})=G(\overline{AB})+G(D)$, so, by the claim

$$G(\overline{AE}+D)=G(\overline{AE}+D)\cap(G(\overline{AB})+G(D))=G\left(\overline{A(E\cap B)}+D\right)\cap G(\overline{AB})+G(D).$$

Furthermore $G\left(\overline{A(E\cap B)}+D\right)\cap G(\overline{AB})=G\left(\overline{A(E\cap B)}+D\cap\overline{AB}\right)=G\left(\overline{A(E\cap B)}+B\right)$. As $A\mathop{\smile\!\!\!\!\!\!|}^{wm}_{C}B$ we have $G(\overline{A(E\cap B)}+B)=G(\overline{A(E\cap B)})+G(B)$. We conclude that

$$G(\overline{AE}+D)=G(\overline{A(E\cap B)})+G(B)+G(D)=G(\overline{A(E\cap B)})+G(D).$$

$\square$

Corollary 7.2.3. *In* ACFG, $\mathop{\smile\!\!\!\!\!\!|}^{wm}$ *satisfies* EXTENSION. *In particular, in* $\mathop{\smile\!\!\!\!\!\!|}^{wm}=\mathop{\smile\!\!\!\!\!\!|}^{f}=\mathop{\smile\!\!\!\!\!\!|}^{d}$.

Proof. Assume that $a\mathop{\smile\!\!\!\!\!\!|}^{wm}_{C}b$ and d is given. By FULL EXISTENCE of $\mathop{\smile\!\!\!\!\!\!|}^{st}$ there exists $a'\equiv_{Cb}a$ such that $a'\mathop{\smile\!\!\!\!\!\!|}^{st}_{Cb}d$. Also $a'\mathop{\smile\!\!\!\!\!\!|}^{wm}_{C}b$ hence by Lemma 7.2.2 $a'\mathop{\smile\!\!\!\!\!\!|}^{wm}_{C}b,d$, which shows EXTENSION for $\mathop{\smile\!\!\!\!\!\!|}^{wm}$. In particular $\mathop{\smile\!\!\!\!\!\!|}^{w}$ satisfies hypothesis *(3)* of Proposition 7.1.11. We check that it satisfies the rest of the hypotheses of Proposition 7.1.11. *(1)* follows from Corollary 5.2.4. From Theorem 5.2.2, $\mathop{\smile\!\!\!\!\!\!|}^{w}$ satisfies the properties INVARIANCE, MONOTONICITY and $\mathop{\smile\!\!\!\!\!\!|}^{u}$-AMALGAMATION over algebraically closed sets (since $\mathop{\smile\!\!\!\!\!\!|}^{u}\to\mathop{\smile\!\!\!\!\!\!|}^{a}$, by Fact 1.2.3), so $\mathop{\smile\!\!\!\!\!\!|}^{w}$ satisfies *(2)*. $\square$

7.3 Thorn-Forking in ACFG

Let (K,G) be a monster model of ACFG. Let $\mathop{\smile\!\!\!\!\!\!|}^{aeq}$ be the relation $\mathop{\smile\!\!\!\!\!\!|}^{a}$ in the sense of $(K,G)^{eq}$ (Section 1.2). The thorn-forking independence relation $\mathop{\smile\!\!\!\!\!\!|}^{þ}$ is the relation defined over subsets of $(K,G)^{eq}$ by $\mathop{\smile\!\!\!\!\!\!|}^{þ}=(\mathop{\smile\!\!\!\!\!\!|}^{aeq})^{m*}$. We will only consider the restrictions of $\mathop{\smile\!\!\!\!\!\!|}^{aeq}$ and $\mathop{\smile\!\!\!\!\!\!|}^{þ}$ to the home sort, which we denote respectively by $\mathop{\smile\!\!\!\!\!\!|}^{aeq}\upharpoonright K$ and $\mathop{\smile\!\!\!\!\!\!|}^{þ}\upharpoonright K$. By Corollary 6.1.5 and Theorem 6.3.4, for $a,b,C\subset K$

$$a\mathop{\smile\!\!\!\!\!\!|}^{aeq}_{C}b\iff\overline{Ca}\cap\overline{Cb}=\overline{C}\text{ and }\pi(\overline{Ca})\cap\pi(\overline{Cb})=\pi(\overline{C}).$$

Fact 7.3.1 ([Adl09a] Theorem 4.3)**.** *The following are equivalent.*

- *T is rosy*
- *$\mathop{\smile\!\!\!\!\!\!|}^{þ}$ in T^{eq} satisfies* LOCAL CHARACTER.

Proposition 7.3.2. *Let (K,G) be a model of* ACFG. *Then $\mathop{\smile\!\!\!\!\!\!|}^{þ}\upharpoonright K=\mathop{\smile\!\!\!\!\!\!|}^{wm}=\mathop{\smile\!\!\!\!\!\!|}^{f}=\mathop{\smile\!\!\!\!\!\!|}^{d}$. In particular* ACFG *is not rosy.*

Proof. Assume that $a\mathop{\smile\!\!\!\!\!\!|}^{þ}_{C}b$. In particular $a\mathop{\smile\!\!\!\!\!\!|}^{aeq\,m}_{C}b$ so for all $C\subseteq D\subseteq\overline{Cb}$ we have $\overline{Da}\cap\overline{Cb}=\overline{D}$ hence by Example 7.1.3 we have

$$a\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C}b.$$

On the other hand, we have $\pi(\overline{Ca})\cap\pi(\overline{Cb})=\pi(\overline{C})$, hence by Section 6.1

$$a\mathop{\smile\!\!\!\!\!\!|}^{w}_{C}b.$$

It follows that $\mathop{\smile\!\!\!\!\!\!|}^{þ}\upharpoonright K\to\mathop{\smile\!\!\!\!\!\!|}^{wm}$. By Fact 1.2.3, $\mathop{\smile\!\!\!\!\!\!|}^{d}\to\mathop{\smile\!\!\!\!\!\!|}^{aeq}\upharpoonright K$, hence as $\mathop{\smile\!\!\!\!\!\!|}^{f}$ satisfies BASE MONOTONICITY and EXTENSION it follows that $\mathop{\smile\!\!\!\!\!\!|}^{f}\to\mathop{\smile\!\!\!\!\!\!|}^{þ}\upharpoonright K$. Hence by Corollary 7.2.3 we conclude that $\mathop{\smile\!\!\!\!\!\!|}^{þ}\upharpoonright K=\mathop{\smile\!\!\!\!\!\!|}^{wm}=\mathop{\smile\!\!\!\!\!\!|}^{f}=\mathop{\smile\!\!\!\!\!\!|}^{d}$. As ACFG is not simple, $\mathop{\smile\!\!\!\!\!\!|}^{f}$ does not satisfy LOCAL CHARACTER, so $\mathop{\smile\!\!\!\!\!\!|}^{þ}\upharpoonright K$ does not satisfy LOCAL CHARACTER hence neither does $\mathop{\smile\!\!\!\!\!\!|}^{þ}$. By Fact 7.3.1, ACFG is not rosy. $\square$

Remark 7.3.3. There is another way of proving that ACFG is not rosy which does not use the description of forking in ACFG but only the fact that $\mathop{\smile\hskip-0.9em|}^{\text{þ}} \restriction K \to \mathop{\smile\hskip-0.9em|}^{wm}$. Indeed $\mathop{\smile\hskip-0.9em|}^{wm}$ does not satisfy Local Character from Example 7.1.4 hence neither does $\mathop{\smile\hskip-0.9em|}^{\text{þ}} \restriction K$ and hence neither does $\mathop{\smile\hskip-0.9em|}^{\text{þ}}$.

Remark 7.3.4. It is worth mentioning that in the definition of $\mathop{\smile\hskip-0.9em|}^{\text{þ}}$, the relation $\mathop{\smile\hskip-0.9em|}^{aeq}$ cannot be replaced by $\mathop{\smile\hskip-0.9em|}^{a}$. Indeed, in the structure (K, G), by Example 7.1.3 $\mathop{\smile\hskip-0.9em|}^{a\,m} = \mathop{\smile\hskip-0.9em|}^{\mathrm{ACF}}$ and then as Extension clearly holds for $\mathop{\smile\hskip-0.9em|}^{\mathrm{ACF}}$, we have $\mathop{\smile\hskip-0.9em|}^{a\,M*} = \mathop{\smile\hskip-0.9em|}^{\mathrm{ACF}}$. This relation satisfies Local Character. This means that $\mathop{\smile\hskip-0.9em|}^{a\,M*}$ is not the restriction of $\mathop{\smile\hskip-0.9em|}^{aeq\ m*}$ to the home sort. This is what Adler mention in [Adl09a, Example 4.5].

7.4 Forking and thorn-forking in other generic constructions

Forking and dividing. In the three following examples:

(1) Generic $\mathscr{L}$-structure $T^{\emptyset}_{\mathscr{L}}$ [KR17, Proposition 3.18];

(2) Generic $K_{n,m}$-free bipartite graph [CK17, Corollary 4.12];

(3) omega-free PAC fields [Cha02, Theorem 3.3];

we also have that forking and dividing coincides for types, and coincides with the monotonised of Kim-independence. In *(1)* and *(2)* the strategy is the following: first prove that $\mathop{\smile\hskip-0.9em|}^{d} = \mathop{\smile\hskip-0.9em|}^{Km}$ and then show that $\mathop{\smile\hskip-0.9em|}^{d}$ satisfies Extension. The latter is obtained using Full Existence of the *strong* independence relation and a similar mixed transitivity result. This is discussed in [KR18, Subsection 3.3]. We followed a close strategy: using Lemma 7.1.9 (based on the approach of *(3)*), have that $\mathop{\smile\hskip-0.9em|}^{wM*}$ strengthens $\mathop{\smile\hskip-0.9em|}^{d}$. Then we use a mixed transitivity result and Full Existence of the strong independence to show that $\mathop{\smile\hskip-0.9em|}^{Km}$ satisfies Extension. These results suggest that Proposition 7.1.11 can be used to show that in other examples of NSOP$_1$ theories, forking and dividing agrees on types, for instance in Steiner triple system [BC18], or bilinear form over an infinite dimensional vector space over an algebraically closed field [Gra99] [CR16].

Strong independence and Mixed Transitivity. There is also a notion of *strong independence* in the three previous examples which is symmetric and stationary over algebraically closed sets. Concerning *(3)* the strong independence satisfies also the other axioms for mock stability [KK11, Example 0.1 (3)]. In *(2)*, it also satisfies Full Existence, Monotonicity and Transitivity [CK17, Proposition 4.20]. In *(1)*, it is defined in [KR18, Remark 3.19], as a remark, to state a mixed transitivity result, but nothing about it is proven. It is likely that *(1)* and *(2)*, are also mock stable, witnessed by the strong independence. Informally, the strong independence is in general defined to hold between two sets when they are the most unrelated to each other with respect to the ambient theory. Another way of seeing this relation is by saying that the two sets can be somehow "*freely amalgamated*". The definition given in [KR17, Remark 3.19] make this precise, for $C \subseteq A \cap B$, we have $A \mathop{\smile\hskip-0.9em|}^{\otimes}_{C} B$ if and only if the substructure spanned by ABC is isomorphic to the fibered coproduct of the structures spanned by A and B over the substructure spanned by C. This definition coincides with our definition of strong independence in ACFG.

Question 1. *Is there a model-theoretic definition of the strong independence that encompasses the strong independence in the three examples above and in* ACFG*?*

The mixed transitivity result (Lemma 7.2.2) is starting to be reccurent in NSOP$_1$ examples. It holds in example *(1)* ([KR18, Remark 3.19]) and in *(2)* ([CK17, Lemma 4.23]). Note that a similar mixed transitivity appears in a SOP_3 (hence SOP_1) example: the generic K_n-free graph ([Con17a]), this was observed in [KR18, Remark 3.19].

The mixed transitivity result holds as well in omega-free PAC fields. Let $\mathop{\smile\hspace{-0.9em}|}^{w}$ be the *weak* independence and $\mathop{\smile\hspace{-0.9em}|}^{st}$ the *strong* independence in the sens of [Cha02, (1.2)]. Then for all A, B, C, D acl-closed in an omega free PAC field, with $C \subseteq A \cap B$ and $B \subseteq D$ we have:

$$\text{If } A \mathop{\smile\hspace{-0.9em}|}^{wm}_{C} B \text{ and } A \mathop{\smile\hspace{-0.9em}|}^{st}_{B} D \text{ then } A \mathop{\smile\hspace{-0.9em}|}^{wm}_{C} D.$$

This is contained in the proof[1] of [Cha02, (3.1) Proposition].

Thorn-forking. The three other examples are also not rosy. For *(1)*, it is [KR17, Subsection 3.3], for *(2)*, it is [CK17, Proposition 4.28] and for *(3)*, it is [Cha08, Subsection 3.5]. Also, for both *(1)* and *(2)* we have $\mathop{\smile\hspace{-0.9em}|}^{f} = \mathop{\smile\hspace{-0.9em}|}^{d} = \mathop{\smile\hspace{-0.9em}|}^{þ}$, and they both weakly eliminate imaginaries.

The following questions have been asked for the last two or three years by specialists in regards to the observations above.

Question 2. (Q_1) *Does forking equals dividing for types in every* NSOP_1 *theory?*

(Q_2) *Does the mixed transitivity result holds in every* NSOP_1 *theory?*

(Q_3) *Is there an* NSOP_1 *not simple rosy theory?*

Remark 7.4.1. In omega-free PAC fields [Cha02], the strong independence $\mathop{\smile\hspace{-0.9em}|}^{st}$ and the weak independence $\mathop{\smile\hspace{-0.9em}|}^{w}$ are linked by the following relation for A, B, C acl-closed, $A \cap B = C$:

$$A \mathop{\smile\hspace{-0.9em}|}^{st}_{C} B \iff \text{for all } C \subseteq D \subseteq A \text{ and } C \subseteq D' \subseteq B \;\; A \mathop{\smile\hspace{-0.9em}|}^{w}_{DD'} B.$$

In ACFG this is not the case. Let (K, G) be a model of ACFG and for conveniance assume that $G(\overline{\mathbb{F}_p}) = \{0\}$. Let t and t' be algebraically independent over $\mathbb{F}_p$, let $u = t \cdot t'$. Assume that $G(\overline{\mathbb{F}_p(t,t')}) = \langle u \rangle$. Then by Lemma 1.5.8, $u \notin \overline{\mathbb{F}_p(t)} + \overline{\mathbb{F}_p(t')}$, so $G(\overline{\mathbb{F}_p(t)}) + G(\overline{\mathbb{F}_p(t')}) = \{0\}$ so $t \not\mathop{\smile\hspace{-0.9em}|}^{st} t'$. We show that for all $D \subseteq \overline{\mathbb{F}_p(t)}$ and $D' \subseteq \overline{\mathbb{F}_p(t')}$ we have $t \mathop{\smile\hspace{-0.9em}|}^{w}_{DD'} t'$. Let D and D' be as such. There are three cases to consider (the middle case is symmetric):

$$\begin{array}{llll}
t \cdot t' \in \overline{D't} \text{ and } t \cdot t' \in \overline{Dt'} & G(\overline{D't}) = \langle u \rangle & G(\overline{Dt'}) = \langle u \rangle & G(\overline{D't} + \overline{Dt'}) = \langle u \rangle \\
t \cdot t' \in \overline{D't} \text{ and } t \cdot t' \notin \overline{Dt'} & G(\overline{D't}) = \langle u \rangle & G(\overline{Dt'}) = \{0\} & G(\overline{D't} + \overline{Dt'}) = \langle u \rangle \\
t \cdot t' \notin \overline{D't} \text{ and } t \cdot t' \notin \overline{Dt'} & G(\overline{D't}) = \{0\} & G(\overline{Dt'}) = \{0\} & G(\overline{D't} + \overline{Dt'}) = \{0\}
\end{array}$$

In every cases we have $G(\overline{D't}+\overline{Dt'}) = G(\overline{D't})+G(\overline{Dt'})$. As $t \mathop{\smile\hspace{-0.9em}|}^{\mathrm{ACF}}_{DD'} t'$ is clear we have $t \mathop{\smile\hspace{-0.9em}|}^{w}_{DD'} t'$.

Summary on independence relations in ACFG. Every arrow in Figure 7.2 is strict, from that point of view, ACFG is different from *(1)*, *(2)* and *(3)*.

Denote by $A \mathop{\smile\hspace{-0.9em}|}^{wsM}_{C} B$ the relation for all $C \subseteq D \subseteq \overline{AC}$ and $C \subseteq D' \subseteq \overline{BC}$ $A \mathop{\smile\hspace{-0.9em}|}^{w}_{DD'} B$. Remark 7.4.1 states that $\mathop{\smile\hspace{-0.9em}|}^{st}$ is strictly stronger than $\mathop{\smile\hspace{-0.9em}|}^{KsM}$, in *(3)*, this is not the case. In *(1)*, we have that $\mathop{\smile\hspace{-0.9em}|}^{a} = \mathop{\smile\hspace{-0.9em}|}^{aeq} = \mathop{\smile\hspace{-0.9em}|}^{K}$ is strictly weaker than $\mathop{\smile\hspace{-0.9em}|}^{a\,m} = \mathop{\smile\hspace{-0.9em}|}^{d} = \mathop{\smile\hspace{-0.9em}|}^{f} = \mathop{\smile\hspace{-0.9em}|}^{þ}$. In *(2)*, $\mathop{\smile\hspace{-0.9em}|}^{a} = \mathop{\smile\hspace{-0.9em}|}^{aeq}$ is strictly weaker than $\mathop{\smile\hspace{-0.9em}|}^{K}$ and $\mathop{\smile\hspace{-0.9em}|}^{Km} = \mathop{\smile\hspace{-0.9em}|}^{d} = \mathop{\smile\hspace{-0.9em}|}^{f} = \mathop{\smile\hspace{-0.9em}|}^{þ}$.

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[1] In the proof of [Cha02, (3.1) Proposition], D contains B, ψ is over C and $F \cap (C\psi(D))^s = C\psi(D)$, hence $\psi(D)$ and C satisfies condition *(I3)* over B, so $A_1 = \psi(A_0)$ and C satisfies condition *(I3)* over E. As A_1 and C satisfies condition *(I1)* over E, A_1 and C are strongly independent over E. Also A_1 and B satisfy condition *(I1)* and *(I2)* over E. The rest of the proof consist in proving that A_1 and C satisfy condition *(I2)* over E.

$$\begin{tikzcd}
 & & & & \mathop{\vert\hskip-0.55em\smile}^{\mathrm{ACF}} \arrow[d, equal] & \\
 & & \mathop{\vert\hskip-0.55em\smile}^{\text{þ}} \restriction K \arrow[d, equal] & & \mathop{\vert\hskip-0.55em\smile}^{a\,m} \arrow[dr] & \\
\mathop{\vert\hskip-0.55em\smile}^{st} \arrow[r] & \mathop{\vert\hskip-0.55em\smile}^{wsm} \arrow[r] & \mathop{\vert\hskip-0.55em\smile}^{wm} \arrow[r] \arrow[d, equal] & \mathop{\vert\hskip-0.55em\smile}^{w} \arrow[ur] \arrow[dr] \arrow[d, equal] & & \mathop{\vert\hskip-0.55em\smile}^{a} \\
 & & \mathop{\vert\hskip-0.55em\smile}^{d} \arrow[d, equal] & \mathop{\vert\hskip-0.55em\smile}^{K} & \mathop{\vert\hskip-0.55em\smile}^{aeq} \restriction K. \arrow[ur] & \\
 & & \mathop{\vert\hskip-0.55em\smile}^{f} & & &
\end{tikzcd}$$

Figure 7.2: Interactions of independence relations in ACFG.

Part B

Expanding the integers by p-adic valuations

« Fruits en nougat ! Flan ! Paon ! Roinsoles ! Bœuf en daube !
Sur les cuivres, déjà, glisse l'argent de l'aube !
Étouffe en toi le dieu qui chante, Ragueneau !
L'heure du luth viendra, — c'est l'heure du fourneau ! »

CHAPTER 8

Quantifier elimination and dp-rank

For a prime number p, let $v_p : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ be the p-adic valuation, namely, $v_p(a) = \sup\{k \in \mathbb{N} : p^k | a\}$. Let $\emptyset \neq P \subseteq \mathbb{N}$ be a (possibly infinite) set of primes, and let L_P be the language $\{+, 0\} \cup \{|_p : p \in P\}$, where each $|_p$ is a binary relation. We expand $(\mathbb{Z}, +, 0)$ to an L_P-structure $\mathcal{Z}_P$ by interpreting $a|_p b$ as $v_p(a) \leq v_p(b)$ for each $p \in P$. We denote by $T_P := Th(\mathcal{Z}_P)$. For convenience, we enumerate P by $P = \{p_\alpha : \alpha < |P|\}$, and p without a subscript usually denotes some $p \in P$. If $P = \{p\}$ we write T_p instead of $T_{\{p\}}$, etc.

In this chapter, we prove (see Theorem 8.2.1) that T_P eliminates quantifiers in a natural definitional expansion: $L_P^E = L_P \cup \{-, 1\} \cup \{D_n : n \geq 1\}$ where $-$ and 1 are interpreted in the obvious way, and for each $n \geq 1$, D_n is an unary relation symbol interpreted as $\{na : a \in \mathbb{Z}\}$.

Using quantifier elimination, we are able to determine the dp-rank of T_P, and we prove (Theorem 8.3.2) that for $P \neq \emptyset$, dp-rk$(T_P) = |P|$. In particular, for a single prime p we have that T_p is dp-minimal, i.e. dp-rank$(T_p) = 1$.

Contents

8.1 Axioms and basic sentences of T_P

For convenience, in this section and in section 8.2 we work with the valuation functions v_p instead of the relations $|_p$. Let us define a multi-sorted language L_P^M for the valuations v_p on $(\mathbb{Z}, +, 0)$ for $p \in P$ as follows: let Z be the main sort with a function symbol $+$ and a constant symbol 0, interpreted as in $(\mathbb{Z}, +, 0)$. For each $p \in P$ we add a distinct sort Γ_p together with the symbols $<_p$, 0_p, S_p and ∞_p, interpreted as a distinct copy of $(\mathbb{N} \cup \{\infty\}, <, 0, S, \infty)$ where S is the successor function. Finally, we add a function symbol $v_p : Z \to \Gamma_p$, interpreted as the p-adic valuation[1]. When confusion is possible, we denote by $\mathbf{v}_p$ the usual valuation in the metatheory, to distinguish it from the function symbol v_p. We omit the subscript p in $<_p$, 0_p, S_p, ∞_p and Γ_p when no confusion is possible.

We use the following standard notation. Let $k \in \mathbb{N}$ be a nonnegative integer.

- In the Z sort, $\underline{k}$ denotes $\underbrace{1 + 1 + \cdots + 1}_{k \text{ times}}$ if $k > 0$ and 0 if $k = 0$. Also, $\underline{-k}$ denotes $-\underline{k}$.
- For an element a from Z, ka denotes $\underbrace{a + a + \cdots + a}_{k \text{ times}}$ if $k > 0$ and 0 if $k = 0$, $(-k)a$ denotes $-(ka)$, similarly for a variable x in place of a.
- For an element γ from Γ_p, $\gamma + \underline{k}$ denotes $\underbrace{S(S(\ldots(\gamma)\ldots))}_{k \text{ times}}$, similarly for a variable u in place of γ, and $\underline{k}$ is an abbreviation for $0 + \underline{k}$.

The group $(\mathbb{Z}, +, 0)$ with valuations v_p for $p \in P$ can be seen as an L_P-structure and an L_P^M-structure which are interdefinable (with imaginaries) so they essentially define the same sets. We will therefore not distinguish between the L_P-structure and the L_P^M-structure on $(\mathbb{Z}, +, 0)$, except when dealing with dp-rank, where we always refer to the one-sorted language L_P.

For quantifier elimination we define $L_P^{M,E} = L_P^M \cup \{-, 1\} \cup \{D_n : n \geq 1\}$ as before. Quantifier elimination in L_P^E follows from quantifier elimination in $L_P^{M,E}$. We will, therefore, prove quantifier elimination for the theory T_P considered as an $L_P^{M,E}$-theory.

For $a \in \mathbb{Z}$ and $p \in P$, let $(a_i)_{i \in \mathbb{N}}$ be the p-adic representation of a, i.e. $a = \sum_{i \in \mathbb{N}} a_i p^i$ and each a_i is in $\{0, \ldots, p-1\}$. For $\gamma \in \mathbb{N}$, the *prefix of a of length γ* is the sequence sequence $(a_i)_{i<\gamma}$. The *ball* of radius γ and center a is the set of all integers with same prefix of length γ as a.

Proposition 8.1.1. *The following sentences are true in $\mathcal{Z}_P$ and therefore are in T_P:*

(1) Any axiomatization for $Th(\mathbb{Z}, +, -, 0, 1, \{D_n\}_{n \geq 1})$ in the Z sort.

(2) For each p, any axiomatization of $Th(\mathbb{N} \cup \{\infty\}, <, 0, S, \infty)$ in the sort $(\Gamma_p, <_p, 0_p, S_p, \infty_p)$.

(3) For each p : $\forall x(v_p(x) \geq 0 \wedge (v_p(x) = \infty \leftrightarrow x = 0))$.

(4) For each p : $\forall x, y(v_p(x + y) \geq min(v_p(x), v_p(y)))$.

(5) For each p : $\forall x, y(v_p(x) \neq v_p(y) \to v_p(x + y) = min(v_p(x), v_p(y)))$.

(6) For each p and $0 \neq n \in \mathbb{Z}$: $\forall x(v_p(nx) = v_p(x) + \underline{\mathbf{v}_p(n)})$.

(7) For each p : $v_p(\underline{p}) = 1$.

---

[1] It could be interesting to consider the language with just one sort $(N, <, 0, S, \infty)$ for valuation, instead of one for each $p \in P$. Since different valuations are allowed to interact with each other, the resulting structures might be much more complicated.

(8) For each p and $k \in \mathbb{N}$: Every ball in v_p of radius γ consists of exactly p^k disjoint balls of radius $\gamma + k$.

Proof. (1)-(7) are obvious. For (8), let $a \in \mathbb{Z}$ and $\gamma \in \mathbb{N}$. The ball in v_p of radius γ around a is the set of integers such that, in p-adic representation, their prefix of length γ is the same as the prefix of a of length γ. There are p possibilities for each digit, so p^k possibilities for the k digits with indices $\gamma, \ldots, \gamma + k - 1$, which exactly correspond to the balls of radius $\gamma + k$ contained in the original ball. □

Let T'_P be the theory implied by the axioms (1)-(8). All of the following propositions are first order, and we prove them using only T'_P. Let $\mathcal{M}$ be some fixed model of T'_P, with $\mathcal{Z}$ the Z-sort and Γ_p the Γ_p-sort.

Lemma 8.1.2. *For each p:*

(1) $\forall x, y(v_p(x - y) \geq min(v_p(x), v_p(y)))$.

(2) $\forall u \forall y \exists x(v_p(x - y) = u)$. In particular, v_p is surjective.

(3) For each $n \neq 0$, $v_p(\underline{n}) = \underline{\mathbf{v}_p(n)}$.

(4) For each $k \geq 1$: $\forall x(v_p(x) \geq \underline{k} \leftrightarrow D_{p^k}(x))$.

Proof. We only prove item (2), the others are easy to check. By Axiom (8) with $k = 1$, there are x_1, x_2 such that $v_p(x_1 - y) \geq u$, $v_p(x_2 - y) \geq u$, and $v_p(x_1 - x_2) < u + \underline{1}$. Hence by (1) above, $u + \underline{1} > v_p(x_1 - x_2) = v_p((x_1 - y) - (x_2 - y)) \geq \min(v_p(x_1 - y), v_p(x_2 - y)) \geq u$. So either $v_p(x_1 - y) = u$ or $v_p(x_2 - y) = u$. □

The following lemmas are easy exercises.

Lemma 8.1.3.

(1) Let $n_1, \ldots, n_l \in \mathbb{N}$, and let $N \in \mathbb{N}$ be such that $n_i | N$ for all $1 \leq i \leq l$. Let $b_1, \ldots, b_n$ be element of $\mathcal{Z}$. Then every boolean combination of formulas of the form $D_{n_i}(k_i x - b_i)$ is equivalent to a disjunction (possibly empty, i.e. a contradiction) of formulas of the form $D_N(x - \underline{r}_j)$, where for each j, $r_j \in \{0, 1, \ldots, N - 1\}$.

(2) Let $m \in \mathbb{N}$ and let $m', k \in \mathbb{N}$ be such that $m = p^k \cdot m'$ and $gcd(m', p) = 1$. Let $r \in \mathbb{Z}$, and let $r_1 = r \bmod m'$, $r_2 = r \bmod p^k$. Then the formula $D_m(x - \underline{r})$ is equivalent to $D_{m'}(x - \underline{r}_1) \wedge (v_p(x - \underline{r}_2) \geq k)$.

Lemma 8.1.4. *For a_1 and a_2 in $\mathcal{Z}$.*

(1) For every $k \geq 1$, the formula $v_p(x - a_1) < v_p(x - a_2) + \underline{k}$ is equivalent to

$$v_p(x - a_2) < v_p(a_2 - a_1) \vee v_p(x - a_2) > v_p(a_2 - a_1) \vee v_p(x - a_1) < v_p(a_2 - a_1) + \underline{k}.$$

(2) For every $k \geq 0$, the formula $v_p(x - a_1) + \underline{k} < v_p(x - a_2)$ is equivalent to $v_p(x - a_2) > v_p(a_2 - a_1) + \underline{k}$.

Lemma 8.1.5. *For a fixed $p \in P$, a_0, a_1 in $\mathcal{Z}$ and $\gamma_0, \gamma_1 \in \Gamma_p$.*

(1) Every formula of the form $v_p(x - a_0) \geq \gamma_0 \wedge v_p(x - a_1) < \gamma_1$ where $\gamma_0 \geq \gamma_1$, is either inconsistent (if $v_p(a_0 - a_1) \geq \gamma_1$) or equivalent to just $v_p(x - a_0) \geq \gamma_0$ (if $v_p(a_0 - a_1) < \gamma_1$).

(2) Every formula of the form $v_p(x - a_0) \geq \gamma_0 \wedge v_p(x - a_1) < \gamma_1$ where $\gamma_0 < \gamma_1$ and $v_p(a_0 - a_1) < \gamma_0$ is equivalent to just $v_p(x - a_0) \geq \gamma_0$.

Lemma 8.1.6. *Every two balls in Γ_p are either disjoint, or one is contained in the other. More generally, for $(a_i)_i \in \mathcal{Z}$, $(\gamma_i)_i \in \Gamma_p$, every conjunction of formulas of the form $v_p(x - a_i) \geq \gamma_i$ is either inconsistent, or equivalent to a single formula $v_p(x - a_{i_0}) \geq \gamma_{i_0}$, where $\gamma_{i_0} = max\{\gamma_i\}$.*

Definition 8.1.7. For $a, b \in Z$, $\gamma, \delta \in \Gamma_p$, define $(a, \gamma) \leq_p (b, \delta)$ if $\gamma \leq \delta$ and $v_p(a - b) \geq \gamma$. Define $(a, \gamma) \sim_p (b, \delta)$ if $(a, \gamma) \leq_p (b, \delta)$ and $(a, \gamma) \geq_p (b, \delta)$.

$(a, \gamma) \leq_p (b, \delta)$ means that $\gamma \leq \delta$ and, in p-adic representation, the prefix of a of length γ is contained in the prefix of b of length δ. This is equivalent to saying that the ball of radius γ around a (namely, $\{x : v_p(x - a) \geq \gamma\}$) contains the ball of radius δ around b.

Note that $\leq_p$ and $\sim_p$ are defined by quantifier-free formulas, and so do not depend on the model containing the elements under consideration.

Lemma 8.1.8. *The parameters a_i are in $\mathcal{Z}$ and γ_i are in Γ_p for some $p \in P$.*

(1) Every formula of the form $v_p(x - a_0) \geq \gamma_0 \wedge \bigwedge_{m=1}^{n} v_p(x - a_m) < \gamma_m$ is equivalent to the formula $v_p(x - a_0) \geq \gamma_0 \wedge \bigwedge_{m \in C} v_p(x - a_m) < \gamma_m$, for every $C \subseteq \{1, \dots, n\}$ such that $\{(a_m, \gamma_m) : m \in C\}$ contains at least one element from each $\sim_p$-equivalence class of $\leq_p$-minimal elements among $\{(a_m, \gamma_m) : 1 \leq m \leq n\}$ (i.e. representatives for all the maximal balls). In particular, this is true for C consisting of one element from each such class, i.e. for C an antichain.

(2) Assume that $(a_0, \gamma_0), \dots, (a_n, \gamma_n)$ are such that for all $1 \leq m \leq n$ we have $\gamma_m > \gamma_0$, $v_p(a_m - a_0) \geq \gamma_0$, and $k_m := \gamma_m - \gamma_0$ is a standard integer. Assume further that $\{(a_m, \gamma_m) : 1 \leq m \leq n\}$ is an antichain with respect to $\leq_p$. Then every formula of the form $v_p(x - a_0) \geq \gamma_0 \wedge \bigwedge_{m=1}^{n} v_p(x - a_m) < \gamma_m$ is equivalent to a formula of the form $\bigvee_{i=1}^{l} v_p(x - b_i) \geq \gamma_N$ with N such that $\gamma_N = \max\{\gamma_m : 1 \leq m \leq n\}$, where for all i, $v_p(b_i - a_0) \geq \gamma_0$ and for $i \neq j$, $v_p(b_i - b_j) < \gamma_N$, and where $l = p^{k_N} - \sum_m p^{k_N - k_m} \geq 0$ (it may be that $l = 0$, i.e. a contradiction). In particular, l does not depend on the model $\mathcal{M}$ of T'_P containing the a_i's and γ_i's.

Proof. We prove *(1)*. Let C be such. For each $1 \leq m \leq n$ there is an m' such that $(a_{m'}, \gamma_{m'}) \leq (a_m, \gamma_m)$ and $(a_{m'}, \gamma_{m'})$ is minimal among the (a_i, γ_i)'s. So $\forall x (v_p(x - a_{m'}) < \gamma_{m'} \rightarrow v_p(x - a_m) < \gamma_m)$. As $\{(a_i, \gamma_i) : i \in C\}$ contains one element from each $\sim$-equivalence class of $\leq$-minimal elements, we may assume $m' \in C$.
We prove *(2)*. Assume without loss of generality that $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$. Let $b_0, \dots, b_{p^{k_n}-1}$ be the $x_0, \dots, x_{p^k - 1}$ from Axiom 8 for k_n, γ_0, a_0. Then $v_p(x - a_0) \geq \gamma_0$ is equivalent to $\bigvee_{i=0}^{p^{k_n}-1} (v_p(x - b_i) \geq \gamma_n)$. For every $m \geq 1$, let $c_{m,0}, \dots, c_{m,p^{k_n - k_m}-1}$ be the $x_0, \dots, x_{p^k - 1}$ from Axiom 8 for $k_n - k_m$, γ_m, a_m. Then $v_p(x - a_m) \geq \gamma_m$ is equivalent to $\bigvee_{i=0}^{p^{k_n - k_m}-1} (v_p(x - c_{m,i}) \geq \gamma_n)$. For every m, $v_p(a_0 - a_m) \geq \gamma_0$, so for every $0 \leq i \leq p^{k_n - k_m} - 1$, $v_p(c_{m,i} - a_0) \geq \gamma_0$. Hence by the choice of $\{b_j\}_j$, there is a unique $s_{m,i} < p^{k_n}$ such that $v_p(c_{m,i} - b_{s_{m,i}}) \geq \gamma_n$. So $v_p(x - a_m) \geq \gamma_m$ is equivalent to $\bigvee_{i=0}^{p^{k_n - k_m}-1} (v_p(x - b_{s_{m,i}}) \geq \gamma_n)$.

By the choice of $\{c_{m,i}\}_i$, $\bigwedge_{i \neq j} (v_p(c_{m,i} - c_{m,j}) < \gamma_n)$, so also $\bigwedge_{i \neq j} (v_p(b_{s_{m,i}} - b_{s_{m,j}}) < \gamma_n)$. In particular, $i \mapsto s_{m,i}$ is injective for a fixed m, hence $F_m := \{s_{m,i} : 0 \leq i \leq p^{k_n - k_m} - 1\}$ is of size $p^{k_n - k_m}$.

The sets $\{F_m\}_{m=1}^{n}$ must be mutually disjoint. Otherwise, there are $m_1 < m_2$ and i, j such that $s_{m_1,i} = s_{m_2,j}$. Since $v_p(c_{m_1,i} - b_{s_{m_1,i}}) \geq \gamma_n$ and $v_p(c_{m_2,j} - b_{s_{m_2,j}}) \geq \gamma_n$ we get $v_p(c_{m_1,i} - c_{m_2,j}) \geq \gamma_n \geq \gamma_{m_1}$. Since $v_p(c_{m_1,i} - a_{m_1}) \geq \gamma_{m_1}$ and $v_p(c_{m_2,j} - a_{m_2}) \geq \gamma_{m_2} \geq \gamma_{m_1}$, we get $v_p(a_{m_1} - a_{m_2}) \geq \gamma_{m_1}$, a contradiction to the antichain assumption.

Let $F := \bigcup_{m=1}^{n} F_m$. By the above, $\mid F \mid = \sum_m p^{k_n - k_m}$ and

$$\forall x \left((v_p(x - a_0) \geq \gamma_0 \wedge \bigwedge_{m=1}^{n} v_p(x - a_m) < \gamma_m) \leftrightarrow (\bigvee_{i \notin F} v_p(x - b_i) \geq \gamma_n)) \right).$$

□

Lemma 8.1.9. *For all elements $a_i, a_{i,j}$ in $\mathcal{Z}$ and γ_i in Γ_p for some $p \in P$, we have the following.*

(1) If b is a solution to $v_p(x - a_0) \geq \gamma_0 \wedge \bigwedge_{i=1}^n v_p(x - a_i) < \gamma_i$ and $v_p(b' - b) \geq \gamma := max\{\gamma_0, \dots, \gamma_n\}$ then b' is also a solution.

(2) Every formula of the form $v_p(x - a_0) \geq \gamma_0 \wedge \bigwedge_{m=1}^n v_p(x - a_m) < \gamma_m$ where for each $1 \leq m \leq n$, $\gamma_m \geq \gamma_0 + \underline{n}$, has a solution.

(3) If $p_1, \dots, p_l \in P$ are different primes not dividing m and $\gamma_i \in \Gamma_{p_i}$, then every formula of the form
$(\bigwedge_{k=1}^l v_{p_k}(x - a_k) \geq \gamma_k) \wedge D_m(x - r)$ has an infinite number of solutions.

(4) If $p_1, \dots, p_l \in P$ are different primes not dividing m and $\gamma_{k,j} \in \Gamma_{p_k}$, then every formula of the form

$$\bigwedge_{k=1}^{l} \left(v_{p_k}(x - a_{k,0}) \geq \gamma_{k,0} \wedge \bigwedge_{i=1}^{n_k} v_{p_k}(x - a_{k,i}) < \gamma_{k,i} \right) \wedge D_m(x - r)$$

where for each $1 \leq k \leq l$ and $1 \leq i \leq n_k$, $\gamma_{k,i} \geq \gamma_{k,0} + \underline{n}_k$, has an infinite number of solutions. In particular, this holds if each $\gamma_{k,i} - \gamma_{k,0}$ is a nonstandard integer.

Proof. The proofs of *(1)* and *(3)* are left as an easy exercice. We prove *(2)*. By Axiom 8 for $k = n$, there are $b_0, \dots, b_{p^n - 1}$ such that for all i, $v_p(b_i - a_0) \geq \gamma_0$, and for all $i \neq j$, $v_p(b_i - b_j) < \gamma_0 + \underline{n}$. Then some b_i must satisfy $\bigwedge_{m=1}^n v_p(x - a_m) < \gamma_m$, otherwise, since $p^n > n$, by the Pigeonhole Principle there are $i \neq j$ and m such that $v_p(b_i - a_m) \geq \gamma_m$ and $v_p(b_j - a_m) \geq \gamma_m$, and therefore also $v_p(b_i - b_j) \geq \gamma_m \geq \gamma_0 + \underline{n}$, a contradiction.
We prove *(4)*. For each $1 \leq k \leq l$, by *(2)* the formula $v_{p_k}(x - a_{k,0}) \geq \gamma_{k,0} \wedge (\bigwedge_{i=1}^{n_k} v_{p_k}(x - a_{k,i}) < \gamma_{k,i})$ has a solution b_k. Let $\gamma_k := \max\{\gamma_{k,0}, \dots, \gamma_{k,n_k}\}$. By *(3)* the formula $(\bigwedge_{k=1}^l v_{p_k}(x - b_k) \geq \gamma_k) \wedge D_m(x - r)$ has an infinite number of solutions $\{b'_j\}_{j \geq 1}$. By *(1)*, every b'_j is a solution to

$$\bigwedge_{k=1}^{l} \left(v_{p_k}(x - a_{k,0}) \geq \gamma_{k,0} \wedge \bigwedge_{i=1}^{n_k} v_{p_k}(x - a_{k,i}) < \gamma_{k,i} \right) \wedge D_m(x - r)$$

□

8.2 Quantifier elimination in T_P

Theorem 8.2.1. *For every nonempty set P of primes, the theory T_P eliminates quantifiers in the language L_P^E.*

Proof. As mentioned previously, we will in fact prove quantifier elimination for $T'_P \subseteq T_P$. It is enough to prove that for all models $\mathcal{M}_1$ and $\mathcal{M}_2$ of T'_P, with a common substructure A, and for all formulas $\phi(x)$ in a single variable x over A which are a conjunction of atomic or negated atomic formulas, we have $\mathcal{M}_1 \models \exists x \phi(x) \Rightarrow \mathcal{M}_2 \models \exists x \phi(x)$. Let $\mathcal{M}_1$, $\mathcal{M}_2$, A and $\phi(x)$ be such, and let $b \in \mathcal{M}_1$ be such that $\mathcal{M}_1 \models \phi(b)$.

As v_p is surjective for all $p \in P$, we may assume that x is of the Z sort. Since ϕ contains only finitely many symbols from L_P, we may assume for simplicity of notation that P is finite. So $\phi(x)$ is equivalent[2] to a conjunction of formulas of the forms:

(1) $n_i x = a_i$, for some $n_i \neq 0$.

(2) $n_i x \neq a_i$, for some $n_i \neq 0$.

(3) $D_{m_i}(n_i x - a_i)$, for some $n_i \neq 0$.

(4) $\neg D_{m_i}(n_i x - a_i)$, for some $n_i \neq 0$.

(5) $v_{p_\alpha}(n_{i,1}x - a_{i,1}) < v_{p_\alpha}(n_{i,2}x - a_{i,2}) + \underline{k}_i$, for some $p_\alpha \in P$, $n_{i,1} \neq 0$ or $n_{i,2} \neq 0$, and $k_i \in \mathbb{N}$.

(6) $v_{p_\alpha}(n_{i,1}x - a_{i,1}) + \underline{k}_i < v_{p_\alpha}(n_{i,2}x - a_{i,2})$, for some $p_\alpha \in P$, $n_{i,1} \neq 0$ or $n_{i,2} \neq 0$, and $k_i \in \mathbb{N}$.

(7) $v_{p_\alpha}(n_i x - a_i) \geq \gamma_i$, for some $p_\alpha \in P$ and $n_i \neq 0$.

(8) $v_{p_\alpha}(n_i x - a_i) < \gamma_i$, for some $p_\alpha \in P$ and $n_i \neq 0$.

By multiplicativity of the valuations we may assume that for all formulas of forms (5) or (6), either $n_{i,1} = n_{i,2}$, $n_{i,1} = 0$ or $n_{i,2} = 0$. Therefore, by Lemma 8.1.4, we may assume that every formula of form (5) or (6) is equivalent to a formula of form (7) or (8).

By Lemma 8.1.3, the conjunction of all the formulas of the forms (3) or (4) is equivalent to a formula of the form

$$\bigvee_j \left(D_{m_j}(x - r_j) \wedge \bigwedge_{\alpha < |P|} v_{p_\alpha}(x - s_{j,\alpha}) \geq \underline{k}_{j,\alpha} \right)$$

where for all j and α, $\gcd(m_j, p_\alpha) = 1$. As $\mathcal{M}_1 \vDash \phi(b)$, this disjunction is not empty. Let $D_m(x - r) \wedge \bigwedge_{\alpha<|P|} v_{p_\alpha}(x - s_\alpha) \geq \underline{k}_\alpha$ be one of the disjuncts which are satisfied by b. It is enough to find $b' \in \mathcal{M}_2$ which satisfies this disjunct, along with all the formulas of other forms. Note that $v_{p_\alpha}(x - s_\alpha) \geq \underline{k}_\alpha$ is of form (7), so altogether we want to find $b' \in \mathcal{M}_2$ which satisfies a conjunction of formulas of the forms:

(1) $n_i x = a_i$, $n_i \neq 0$.

(2) $n_i x \neq a_i$, $n_i \neq 0$.

(3) $D_m(x - r)$, where for all $\alpha < |P|$, $\gcd(m, p_\alpha) = 1$ (only a single such formula).

(4) $v_{p_\alpha}(n_i x - a_i) \geq \gamma_i$, $\alpha < |P|$, $n_i \neq 0$.

(5) $v_{p_\alpha}(n_i x - a_i) < \gamma_i$, $\alpha < |P|$, $n_i \neq 0$.

It is standard that we may assume that the conjunction does not contain formulas of the form (1). For each formula of the form (2), there is at most one element which does not satisfy it. So it is enough to prove that there are infinitely many elements in $\mathcal{M}_2$ which satisfy all the formulas of forms (3), (4) or (5).

Let $n := \prod_i n_i$. By multiplicativity of the valuations, the conjunction of formulas of forms (3), (4) or (5) is equivalent to the conjunction of:

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[2] The negation of a formula of form (5) is $v_{p_\alpha}(n_{i,1}x - a_{i,1}) \geq v_{p_\alpha}(n_{i,2}x - a_{i,2}) + \underline{k}$, which is equivalent to $v_{p_\alpha}(n_{i,2}x - a_{i,2}) + \underline{k-1} < v_{p_\alpha}(n_{i,1}x - a_{i,1})$ if $k > 0$, which is of form (6), and to $v_{p_\alpha}(n_{i,2}x - a_{i,2}) < v_{p_\alpha}(n_{i,1}x - a_{i,1}) + 1$ if $k = 0$, which is of form (5). Similarly for the negation of a formula of form (6). Also, (7) and (8) are in essence special cases of (5) or (6), but they are required because in A the valuation may be not surjective.

(1) $v_{p_\alpha}(nx - \frac{n}{n_i}a_i) \geq \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.

(2) $v_{p_\alpha}(nx - \frac{n}{n_i}a_i) < \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.

(3) $D_{nm}(nx - nr)$.

By substituting $y = nx$, it is equivalent to satisfy:

(1) $v_{p_\alpha}(y - \frac{n}{n_i}a_i) \geq \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.

(2) $v_{p_\alpha}(y - \frac{n}{n_i}a_i) < \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.

(3) $D_{nm}(y - nr)$.

(4) $D_n(y)$.

Notice that formula (4) is already implied by formula (3). Again by Lemma 8.1.3, we may exchange $D_{nm}(y - nr)$ by a formula $D_{m'}(y - r')$, where for all $\alpha < |P|$, $\gcd(m', p_\alpha) = 1$. Also, by Lemma 8.1.6 we may assume that for each $\alpha < |P|$, there is only one formula of form (1). Altogether, it is enough to prove that in $\mathcal{M}_2$ there are infinitely many elements which satisfy the conjunction of the following formulas:

(1) $v_{p_\alpha}(x - a_{\alpha,0}) \geq \gamma_{\alpha,0}$ for all $\alpha < |P|$.

(2) $v_{p_\alpha}(x - a_{\alpha,i}) < \gamma_{\alpha,i}$ for all $\alpha < |P|$, $1 \leq i \leq n_\alpha$. $\circledast$

(3) $D_m(x - r)$, where for all $\alpha < |P|$, $\gcd(m, p_\alpha) = 1$ (only a single such formula).

By Lemma 8.1.5 (and since this formula is consistent in $\mathcal{M}_1$) we may assume that for all $\alpha < |P|$, $1 \leq i \leq n_\alpha$ we have $\gamma_{\alpha,0} < \gamma_{\alpha,i}$ and $v_{p_\alpha}(a_{\alpha,0} - a_{\alpha,i}) \geq \gamma_{\alpha,0}$. By Lemma 8.1.8 *(1)*, we may assume that for each $\alpha < |P|$, the set

$$\{(a_{\alpha,i}, \gamma_{\alpha,i}) \,:\, 1 \leq i \leq n_\alpha\,,\ \gamma_{\alpha,i} - \gamma_{\alpha,0} \text{ is a standard integer}\}$$

is an antichain with respect to $\leq_{p_\alpha}$ (Definition 8.1.7).

For each $\alpha < |P|$, let $S_\alpha = \{0 \leq i \leq n_\alpha \,:\, \gamma_{\alpha,i} - \gamma_{\alpha,0} \text{ is a standard integer}\}$ and $\gamma'_{\alpha,0} = max\{\gamma_{\alpha,i} \,:\, i \in S_\alpha\}$. For $s = 1, 2$ and for each $\alpha < |P|$, by Lemma 8.1.8 *(2)* the conjunction $v_{p_\alpha}(x - a_{\alpha,0}) \geq \gamma_{\alpha,0} \wedge \bigwedge_{i \in S_\alpha} v_{p_\alpha}(x - a_{\alpha,i}) < \gamma_{\alpha,i}$ is equivalent in $\mathcal{M}_s$ to a formula of the form $\bigvee_{i=1}^{l_\alpha} v_{p_\alpha}(x - a^s_{\alpha,0,i}) \geq \gamma'_{\alpha,0}$, where for all i, $a^s_{\alpha,0,i} \in \mathcal{M}_s$ and l_α does not depend on s. Note that $a^s_{\alpha,0,i}$ may not be in A. Furthermore, by Lemma 8.1.8 *(2)*, $v_{p_\alpha}(a^s_{\alpha,0,i} - a_{\alpha,0}) \geq \gamma_{\alpha,0}$ and for $i \neq j$, $v_{p_\alpha}(a^s_{\alpha,0,i} - a^s_{\alpha,0,j}) < \gamma'_{\alpha,0}$.

Together, the conjunction of the formulas in $\circledast$ is equivalent in $\mathcal{M}_s$ to the disjunction $\psi_s = \bigvee_{k=1}^{l} \psi_{s,k}$, where for each k, $\psi_{s,k}$ is the conjunction of the following formulas:

(1) $v_{p_\alpha}(x - a^s_{\alpha,0,k}) \geq \gamma'_{\alpha,0}$, for all $\alpha < |P|$.

(2) $v_{p_\alpha}(x - a_{\alpha,i}) < \gamma_{\alpha,i}$, for all $\alpha < |P|$, $i \notin S_\alpha$ (so $\gamma_{\alpha,0} < \gamma_{\alpha,i}$ and $\gamma_{\alpha,i} - \gamma_{\alpha,0}$ is not a standard integer).

(3) $D_m(x - r)$, where for all $\alpha < |P|$, $\gcd(m, p_\alpha) = 1$ (only a single such formula).

Furthermore, $l = \prod_{\alpha < |P|} l_\alpha$ does not depend on s.

Since ψ_1 is consistent in $\mathcal{M}_1$ (satisfied by nb), the disjunction for $s = 1$ is not empty, i.e., $l \geq 1$. And since l does not depend on s, the disjunction for $s = 2$ is also not empty. Consider one such disjunct, $\psi_{2,k}$. By Lemma 8.1.9 *(4)*, it has an infinite number of solutions. This completes the proof. $\square$

Corollary 8.2.2. *T_P' is a complete theory. Hence $T_P' = T_P$.*

Proof. By quantifier elimination, it is enough to show that T_P' decides every atomic sentence. These are just the sentences equivalent to one of the forms: $\underline{n}_1 = \underline{n}_2$ in any sort, $\underline{k}_1 <_p \underline{k}_2$ in Γ_p, $D_m(\underline{n})$ in the Z sort and $v_p(\underline{n}_1) < v_p(\underline{n}_2)$ in the Z sort, all of which are clearly decided by T_P'. □

Remark 8.2.3. Suppose $\mathcal{M} \models T_P$ and $\phi(x)$ is a consistent formula in a single variable with parameters from $\mathcal{M}$. Then by quantifier elimination and Lemmas 8.1.3 and 8.1.4, $\phi(x)$ is equivalent to a disjunction of formulas, which are either of the form $x = a$ or of the form

$$D_m(x-r) \wedge \bigwedge_j nx \neq a_j \wedge \bigwedge_{p\in F} \left(v_p(n_p x - a_{p,0}) \geq \gamma_{p,0} \wedge \bigwedge_{i=1}^{l_p} v_p(n_p x - a_{p,i}) < \gamma_{p,i} \right),$$

where $F \subseteq P$ is finite. Moreover, one may assume $\gcd(n_p, p) = 1$.

For p a single prime number and $\mathcal{M} \models T_p$, the following lemma says that the definable subgroups of $(\mathcal{M}, +)$ are only those of the form $m\mathcal{M} \cap \{a \in \mathcal{M} : v(a) \geq \gamma\}$, for $m \in \mathbb{Z}$ and $\gamma \in \Gamma$ and for each such defining formula, there are only finitely many possible m's when varying the parameters of the formula.

Lemma 8.2.4 (Uniformly definable subgroups)**.** *For a single prime p, let $\phi(x,y)$ be any L_p^M-formula, and let $\theta(y)$ be the formula for "$(\phi(x,y), +)$ is a subgroup". Then there are $n_1, \ldots, n_k \geq 1$, having $\gcd(n_i, p) = 1$ for each i, such that the following sentence is true in T_p:*

$$\forall y \left(\theta(y) \to \bigvee_{i=1}^{k} \exists w \forall x (\phi(x,y) \leftrightarrow (D_{n_i}(x) \wedge (v_p(x) \geq v_p(w))) \right).$$

Proof. It is enough to work in $\mathbb{Z}$. By quantifier elimination (and Lemma 8.1.3 (2)), $\phi(x,y)$ is equivalent to a formula of the form $\bigvee_i \bigwedge_j \phi_{i,j}(x,y)$, where for each i,j, $\phi_{i,j}(x,y)$ is one of the following:

1. $t_{i,j}(x,y) = 0$, where $t_{i,j}(x,y)$ is a $\{+,-,1\}$-term, i.e., of the form $k_{i,j}x + l_{i,j}y + \underline{r}_{i,j}$ for $k_{i,j}, l_{i,j}, r_{i,j} \in \mathbb{Z}$.
2. $t_{i,j}(x,y) \neq 0$, where $t_{i,j}(x,y)$ is a $\{+,-,1\}$-term.
3. $v(t_{i,j}(x,y)) \geq v(s_{i,j}(x,y))$, where $t_{i,j}(x,y)$, $s_{i,j}(x,y)$ are $\{+,-,1\}$-terms (note that $v(t_{i,j}(x,y)) < v(s_{i,j}(x,y))$ is equivalent to $v(p \cdot t_{i,j}(x,y)) \leq v(s_{i,j}(x,y))$, which is of the same form).
4. $D_{m_{i,j}}(t_{i,j}(x,y))$, where $t_{i,j}(x,y)$ is a $\{+,-,1\}$-term and $\gcd(m_{i,j}, p) = 1$.

For each i, let $J_i = \{j : \phi_{i,j}(x,y) \text{ is of the form } D_{m_{i,j}}(t_{i,j}(x,y))\}$, and let $m_i = \prod_{j\in J_i} m_{i,j}$. As in the proof of Lemma 8.1.3 (1), the satisfaction of the formula $D_{m_{i,j}}(t_{i,j}(x,y))$ depends only on the reminders of x and y mod $m_{i,j}$, which are determined by the reminders of x and y mod m_i. So there is a set $R_i \subseteq \{0,1,\ldots,m_i-1\}^2$ such that $\bigwedge_{j\in J_i} \phi_{i,j}(x,y)$ is equivalent to $\bigvee_{(r,s)\in R_i}(D_{m_i}(x-\underline{r}) \wedge D_{m_i}(y-\underline{s}))$. Therefore, $\phi(x,y)$ is equivalent to a formula of the form $\bigvee_i (D_{m_i}(x-\underline{r}_i) \wedge D_{m_i}(y-\underline{s}_i) \wedge \bigwedge_j \phi_{i,j}(x,y))$, where $\gcd(m_i,p) = 1$ and for each i,j, $\phi_{i,j}(x,y)$ is one of the following:

1. $t_{i,j}(x,y) = 0$, where $t_{i,j}(x,y)$ is a $\{+,-,1\}$-term.
2. $t_{i,j}(x,y) \neq 0$, where $t_{i,j}(x,y)$ is a $\{+,-,1\}$-term.
3. $v(t_{i,j}(x,y)) \geq v(s_{i,j}(x,y))$, where $t_{i,j}(x,y)$, $s_{i,j}(x,y)$ are $\{+,-,1\}$-terms.

For each i, let $\phi_i(x,y)$ be the i'th disjunct, i.e., the formula $D_{m_i}(x-\underline{r}_i) \wedge D_{m_i}(y-\underline{s}_i) \wedge \bigwedge_j \phi_{i,j}(x,y)$.

Let $b \in \mathbb{Z}$ be such that $\phi(\mathbb{Z},b)$ is a subgroup. If $\phi(\mathbb{Z},b)$ is finite, it must be $\{0\}$. To account for this case, we may take $n_1 = 1$, and for $w = 0$ we have that $\phi(x,b)$ is equivalent to $D_{n_1}(x) \wedge (v_p(x) \geq v_p(0))$. If $\phi(\mathbb{Z},b)$ is infinite, then $\phi(\mathbb{Z},b) = n\mathbb{Z}$ for some $n \geq 1$. Moreover, there must be an i_0 such that $\phi_{i_0}(\mathbb{Z},b)$ is infinite. So $D_{m_{i_0}}(b-\underline{s}_{i_0})$ holds, hence $\phi_{i_0}(x,b)$ is equivalent to just $D_{m_{i_0}}(x-\underline{r}_{i_0}) \wedge \bigwedge_j \phi_{i_0,j}(x,b)$. As $\phi(\mathbb{Z},b)$ is infinite, it is clear that no formula $\phi_{i_0,j}(x,y)$ is of the form (1), hence $\phi_{i_0}(x,b)$ is equivalent to $D_{m_{i_0}}(x-\underline{r}_{i_0}) \wedge \bigwedge_j \phi_{i_0,j}(x,b)$, where for each j, $\phi_{i_0,j}(x,b)$ is one of the following:

(1) $k_{i_0,j}x \neq c_{i_0,j}$.

(2) $v(k'_{i_0,j}x - c'_{i_0,j}) \geq v(k''_{i_0,j}x - c''_{i_0,j})$.

Applying Lemma 8.1.4 to formulas as in (2), we may assume that $\phi_{i_0}(x,b)$ is equivalent to $D_{m_{i_0}}(x-\underline{r}_{i_0}) \wedge \bigwedge_j \phi_{i_0,j}(x,b)$, where for each j, $\phi_{i_0,j}(x,b)$ is one of the following:

(1) $k_{i_0,j}x \neq c_{i_0,j}$.

(2) $v(k_{i_0,j}x - c_{i_0,j}) \geq \gamma_{i_0,j}$.

(3) $v(k_{i_0,j}x - c_{i_0,j}) < \gamma_{i_0,j}$.

The formula $v(k_{i_0,j}x - c_{i_0,j}) \geq \gamma_{i_0,j}$ defines a coset of $p^{\gamma_{i_0,j}}\mathbb{Z}$, and the formula $v(k_{i_0,j}x - c_{i_0,j}) < \gamma_{i_0,j}$ defines a finite union of cosets of $p^{\gamma_{i_0,j}}\mathbb{Z}$. Let

$$J = \{j \,:\, \phi_{i_0,j}(x,b) \text{ is of form 2 or 3}\}$$

and let $\delta = \max\{\gamma_{i_0,j} \,:\, j \in J\}$. Then for every $j \in J$, every coset of $p^{\gamma_{i_0,j}}\mathbb{Z}$ is a finite union of cosets of $p^\delta\mathbb{Z}$. So $\bigcap_{j\in J} \phi_{i_0,j}(\mathbb{Z},b)$ is a finite intersection of finite unions of cosets of $p^\delta\mathbb{Z}$, and hence is itself just a finite union of cosets of $p^\delta\mathbb{Z}$ (since every two cosets are either equal or disjoint). Therefore, $\phi_{i_0}(\mathbb{Z},b)$ is a set of the form $U\backslash F$, where F is a finite set (the set of points excluded by the inequalities $k_{i_0,j}x \neq c_{i_0,j}$), and U is a finite union of the form $\bigcup_{j=1}^N ((m_{i_0}\mathbb{Z} + r_{i_0}) \cap (p^\delta\mathbb{Z} + c_j))$. For each j, $(m_{i_0}\mathbb{Z} + r_{i_0}) \cap (p^\delta\mathbb{Z} + c_j)$ is a coset of $m_{i_0}p^\delta\mathbb{Z}$ (it is not empty, since $\gcd(m_{i_0},p) = 1$), so U is of the form $\bigcup_{j=1}^N (m_{i_0}p^\delta\mathbb{Z} + d_j)$. As $\phi_{i_0}(\mathbb{Z},b)$ is infinite, this union is not empty.

Now, $(m_{i_0}p^\delta\mathbb{Z} + d_1)\backslash F \subseteq U\backslash F = \phi_{i_0}(\mathbb{Z},b) \subseteq \phi(\mathbb{Z},b) = n\mathbb{Z}$, so n divides $m_{i_0}p^\delta$ since F is finite. Write $n = n'p^\gamma$ with $\gcd(n',p) = 1$. Then $n' | m_{i_0}$, and in particular, $n' \leq m_{i_0}$. So $\phi(x,b)$ is equivalent to $D_n(x)$, which is equivalent to $D_{n'}(x) \wedge v(x) \geq \gamma$, and $n' \leq m_{i_0}$. Recall that i_0 depends on b, but there are only finitely many i's, so $m = \max\{m_i\}$ exists, and hence, for any b such that $\phi(x,b)$ is a subgroup, there is an $n' \leq m$ with $\gcd(n',p) = 1$, and there is a γ such that $\phi(x,b)$ is equivalent to $D_{n'}(x) \wedge v(x) \geq \gamma$, and we are done. ☐

8.3 dp-rank of T_P

Quantifier elimination now enables us to determine the dp-rank of T_P. Recall from Section 1.4.3 the definition of dp-rk. In this section, we work in the one-sorted language L_P^E.

Proposition 8.3.1. *For any prime p, T_p is dp-minimal (in the one-sorted language L_P^E).*

Proof. We set $L = L_p^E$ and $T = T_p$. Let L^- contain the symbols of L, except for the divisibility relations $\{D_n\}_{n\geq 1}$. Let $\mathcal{Z}^-$ be the reduct of $\mathcal{Z}_p$ to L^-. Let $\mathbb{Q}_p^-$ be $\mathbb{Q}_p$ as an L^--structure. It is a reduct of the structure $(\mathbb{Q}_p, +, -, \cdot, 0, 1, |_p)$, which is dp-minimal (see [DGL11, Theorem 6.6]), and therefore is also dp-minimal. Note that $\mathcal{Z}^-$ is a substructure of $\mathbb{Q}_p^-$.

Let $L' = L \cup \{Z\}$. Interpret Z in $\mathbb{Q}_p$ as $\mathbb{Z}$, and interpret each D_n such that $D_n \cap \mathbb{Z}$ is the usual divisibility relation and $D_n \cap (\mathbb{Q}_p \backslash \mathbb{Z}) = \emptyset$, thus making it an L'-structure $\mathbb{Q}_p'$. Let $\mathcal{M}$ be an ω_1-saturated model of $Th(\mathbb{Q}_p')$, and let $A = Z(\mathcal{M})$ be the interpretation of Z in it. Then A is an ω_1-saturated model of T.

Suppose that T is not dp-minimal. Then, by Fact 1.4.16, there is an ict-pattern of length 2, hence there are formulas $\phi(x, y)$, $\psi(x, z)$ in L with $|x| = 1$, and elements $(b_i : i < \omega)$, $(c_j : j < \omega)$, $(a_{i,j} : i, j < \omega)$ in A such that $\phi(a_{i,j}, b_{i'})$ if and only if $i = i'$ and $\psi(a_{i,j}, c_{j'})$ if and only if $j = j'$. By Theorem 8.2.1 we may assume that ϕ, ψ are quantifier-free and in disjunctive normal form. Let N be the largest n such that D_n appears in ϕ or ψ. Color each pair (i, j) such that $i > j$ by $a_{i,j}$ mod $N!$. By Ramsey Theorem, we may assume that all the elements $a_{i,j}$ with $i > j$ have the same residue modulo $N!$, and so modulo all $n \leq N$.

Write ϕ as $\bigvee_k \bigwedge_l (\phi'_{k,l} \wedge \phi''_{k,l})$ and ψ as $\bigvee_k \bigwedge_l (\psi'_{k,l} \wedge \psi''_{k,l})$, where $\phi'_{k,l}$, $\psi'_{k,l}$ are atomic or negated atomic L^--formulas and $\phi''_{k,l}$, $\psi''_{k,l}$ are atomic or negated atomic formulas containing no relations other than $\{D_n\}_{n \geq 1}$. For each k, denote by ϕ_k, ψ_k the formulas $\bigwedge_l (\phi'_{k,l} \wedge \phi''_{k,l})$ and $\bigwedge_l (\psi'_{k,l} \wedge \psi''_{k,l})$ respectively.

For every $i > j$ we have $\phi(a_{i,j}, b_i)$, so there is a $k_{i,j}$ such that $\phi_{k_{i,j}}(a_{i,j}, b_i)$. Again by Ramsey Theorem, we may assume that all the $k_{i,j}$'s are equal to some k_0, so for every $i > j$ we have $\phi_{k_0}(a_{i,j}, b_i)$. For every $i' \neq i$ we have $\neg\phi(a_{i',j}, b_i)$, so in particular $\neg\phi_{k_0}(a_{i',j}, b_i)$. Similarly, we may assume that for some k_1, for every $i > j$ we have $\psi_{k_1}(a_{i,j'}, c_j)$ if and only if $j = j'$.

Let ϕ'_k, ψ'_k be the formulas obtained from ϕ_k, ψ_k respectively, by deleting all the formulas $\phi''_{k,l}$, $\psi''_{k,l}$. So ϕ'_k, ψ'_k are L^--formulas.

For every $m \in \mathbb{N}$, let $I_m = \{m+1, \ldots, 2m\}$, $J_m = \{1, \ldots, m\}$. For every $(i, j) \in I_m \times J_m$, we have $\phi_{k_0}(a_{i,j}, b_i)$ and therefore also $\phi'_{k_0}(a_{i,j}, b_i)$. Let $i \neq i' \in I_m$, and suppose for a contradiction that $\phi'_{k_0}(a_{i',j}, b_i)$, i.e. $\bigwedge_l (\phi'_{k_0,l}(a_{i',j}, b_i))$. But we know that $\neg\phi_{k_0}(a_{i',j}, b_i)$, so for some l_0 we have $\neg\phi'_{k_0,l_0}(a_{i',j}, b_i) \vee \neg\phi''_{k_0,l_0}(a_{i',j}, b_i)$. Therefore, we get $\neg\phi''_{k_0,l_0}(a_{i',j}, b_i)$. But from $\phi_{k_0}(a_{i,j}, b_i)$ we also get $\phi''_{k_0,l_0}(a_{i,j}, b_i)$. Together, this contradicts the fact that all the elements $a_{i,j}$ with $i > j$ have the same residue modulo all $n \leq N$.

Altogether, in A, for every $(i, j) \in I_m \times J_m$ we have $\phi'_{k_0}(a_{i,j}, b_{i'})$ if and only if $i = i'$, and similarly also $\psi'_{k_1}(a_{i,j}, c_{j'})$ if and only if $j = j'$. Since ϕ'_{k_0}, ψ'_{k_1} are quantifier-free, and A is a substructure of $\mathcal{M}$, this holds also in $\mathcal{M}$. As m is arbitrary, this contradicts the dp-minimality of $Th(\mathbb{Q}_p^-)$. □

Theorem 8.3.2. *For every nonempty set P of primes, dp-rank$(T_P) = |P|$.*

Proof. dp-rank$(T_P) \leq |P|$ follows from Proposition 8.3.1 and Lemma 1.4.17 for $L_P^E = \bigcup_{\alpha < |P|} L_{p_\alpha}^E$. For $\alpha < |P|$ let $\phi_\alpha(x, y)$ be the formula $x|_{p_\alpha} y \wedge y|_{p_\alpha} x$ (i.e. $v_{p_\alpha}(x) = v_{p_\alpha}(y)$), and for $\alpha < |P|$, $i \in \mathbb{N}$ let $a_{\alpha,i}$ be such that $v_{p_\alpha}(a_{\alpha,i}) = i$. Let $F \subseteq |P|$ be finite. By Lemma 8.1.9 *(4)*, for every $\eta : F \to \mathbb{N}$ there is a b_η such that for every $\alpha \in F$, $v_{p_\alpha}(b_\eta) = v_{p_\alpha}(a_{\alpha,\eta(\alpha)})$. If P is finite, just take $F = |P|$. Otherwise, by compactness, there are such b_η for $F = |P|$ as well. These $\phi_\alpha(x, y)$, $a_{\alpha,i}$ and b_η form an ict-pattern of length $|P|$, hence, by Fact 1.4.16 dp-rank$(T_P) \geq |P|$. □

CHAPTER 9

Minimality phenomena

In this chapter, we show that there is no intermediate structures between $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, +, 0, <)$, and between $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, +, 0, |_p)$. Those are two *minimal* expansions of $(\mathbb{Z}, +, 0)$. We also introduce a fine notion of *reduct* which allows us to extend these minimality results to elementary extensions. We finish by some counter-examples of minimality in elementary extensions.

Contents

9.1 Minimality and Conant's example

Definition 9.1.1. Let L_1 and L_2 be two first-order languages, and let $\mathcal{M}_1$ be an L_1-structure and $\mathcal{M}_2$ an L_2-structure, both with the same underlying universe M. Let $A \subseteq M$ be a set of parameters.

(1) We say that $\mathcal{M}_1$ is an A-*reduct* of $\mathcal{M}_2$, and $\mathcal{M}_2$ is an A-*expansion* of $\mathcal{M}_1$, if for every $n \geq 1$, every subset of M^n which is L_1-definable over $\emptyset$ (equivalently, over A) is also L_2-definable over A. When $A = M$ we just say that $\mathcal{M}_1$ is a *reduct* of $\mathcal{M}_2$, and $\mathcal{M}_2$ is an *expansion* of $\mathcal{M}_1$. We will mostly use this with either $A = \emptyset$ or $A = M$.

(2) We say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are A-*interdefinable* if $\mathcal{M}_1$ is an A-reduct of $\mathcal{M}_2$ and $\mathcal{M}_2$ is an A-reduct of $\mathcal{M}_1$. When $A = M$ we just say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are *interdefinable*.

(3) Let $A \subseteq B \subseteq M$ be another set of parameters. We say that $\mathcal{M}_1$ is a B-*proper* A-*reduct* of $\mathcal{M}_2$, and $\mathcal{M}_2$ is a B-*proper* A-*expansion* of $\mathcal{M}_1$, if $\mathcal{M}_1$ is an A-reduct of $\mathcal{M}_2$, but $\mathcal{M}_2$ is not a B-reduct of $\mathcal{M}_1$. When $B = M$ we just say *proper* instead of B-proper. We will mostly use this with either $B = M$ or $B = \emptyset$.

Let $\mathcal{M}_1$ be an L_1-structure and $\mathcal{M}_2$ an L_2-structure, both with the same underlying universe M, and suppose that $\mathcal{M}_1$ is a $\emptyset$-reduct of $\mathcal{M}_2$. Then we can replace L_2 by $L_2 \cup L_1$, interpreting each L_1-symbol in $\mathcal{M}_2$ as it is interpreted in $\mathcal{M}_1$. As we have not added new $\emptyset$-definable sets, this new structure is $\emptyset$-interdefinable with the original $\mathcal{M}_2$. Therefore we may always assume for simplicity of notation that $L_1 \subseteq L_2$ and $\mathcal{M}_1 = \mathcal{M}_2|_{L_1}$.

A-reducts are preserved by elementary extensions and elementary substructures containing A, in the following sense:

Lemma 9.1.2. *Let $\mathcal{M} \prec \mathcal{N}$ be two L-structures with universes M and N respectively. Let $A \subseteq M$ and let $\mathcal{N}'$ be an A-reduct of $\mathcal{N}$ with language L'. Let $\mathcal{M}'$ be the structure obtained by restricting the relations and functions of $\mathcal{N}'$ to M. Then:*

(1) $\mathcal{M}'$ is well-defined, it is an A-reduct of $\mathcal{M}$, and $\mathcal{M}' \prec \mathcal{N}'$.

(2) $\mathcal{N}'$ is an A-proper A-reduct of $\mathcal{N}$ if and only if $\mathcal{M}'$ is an A-proper A-reduct of $\mathcal{M}$.

(3) $\mathcal{N}'$ is a proper A-reduct of $\mathcal{N}$ if and only if $\mathcal{M}'$ is a proper A-reduct of $\mathcal{M}$.

The proof of the previous Lemma is trivial.

Remark 9.1.3. Lemma 9.1.2 is not necessarily true if $A \not\subseteq M$. If $\mathcal{N}'$ contains a constant $c \notin M$, or a n-ary function f such that $f(M^n) \not\subseteq M$, then $\mathcal{M}'$ is not well-defined. Even when it is well-defined, the rest is still not necessarily true. For example, let $\mathcal{M} = (\mathbb{Z}, +, 0, 1, <)$, and let $\mathcal{N} = (N, +, 0, 1, <)$ be a nontrivial elementary extension of $\mathcal{M}$. Let $b \in N$ be a positive infinite element, and let $\mathcal{N}' = (N, +, 0, 1, [0, b])$. Then $\mathcal{M}' = (\mathbb{Z}, +, 0, 1, \mathbb{N}) \not\prec \mathcal{N}'$ (as $[0, b]$ contains an element $x = b$ such that $x \in [0, b]$ but $x + 1 \notin [0, b]$). Also, $\mathcal{M}'$ is interdefinable with $\mathcal{M}$, but we will see that $\mathcal{N}'$ is a proper reduct of $\mathcal{N}$.

Definition 9.1.4. Let $\mathcal{F}$ be a family of first-order structures, and let $\mathcal{M} \in \mathcal{F}$. We say that $\mathcal{M}$ is A-*minimal* in $\mathcal{F}$ if there are no A-proper A-reducts of $\mathcal{M}$ in $\mathcal{F}$. We say that $\mathcal{M}$ is A-*maximal* in $\mathcal{F}$ if there are no A-proper A-expansions of $\mathcal{M}$ in $\mathcal{F}$. When $A = M$ we just say that $\mathcal{M}$ is *minimal* or *maximal*, respectively.

Based on a result by Palacín and Sklinos [PS18], Conant and Pillay proved in [CP18] the following:

Fact 9.1.5 ([CP18] Theorem 1.2). *$(\mathbb{Z},+,0,1)$ has no proper stable expansions of finite dp-rank.*

In other words, $(\mathbb{Z},+,0,1)$ is maximal among the stable structures of finite dp-rank.

Remark 9.1.6. This theorem is no longer true if we replace $(\mathbb{Z},+,0,1)$ by an elementarily equivalent structure $(N,+,0,1)$. Let $(N,+,0,1,|_p)$ be a nontrivial elementary extension of $(\mathbb{Z},+,0,1,|_p)$, let $b\in N$ be such that $\gamma := v_p(b)$ is nonstandard, and let $B=\{a\in N : b|_p a\} = \{a\in N : v_p(a)\geq\gamma\}$. Then $(N,+,0,1,B)$ is a proper expansion of $(N,+,0,1)$ of dp-rank 1, and in Proposition 9.3.2 we show that it is also stable.

In this section, we give another proof of the following fact, due to Conant.

Fact 9.1.7 ([Con18] Theorem 1.1). *$(\mathbb{Z},+,0,1,<)$ is minimal among the proper expansions of $(\mathbb{Z},+,0,1)$.*

Remark 9.1.8. This is no longer true if we replace $(\mathbb{Z},+,0,1,<)$ by an elementarily equivalent structure. Let $(N,+,0,1,<)$ be a nontrivial elementary extension of $(\mathbb{Z},+,0,1,<)$, let $b\in N$ be a positive infinite element, and let $B=[0,b]$. Then $(N,+,0,1,B)$ is a proper expansion of $(N,+,0,1)$, and in Proposition 9.3.5 we show that it is also a proper reduct of $(N,+,0,1,<)$. Note that the formula $y-x\in B$ defines the ordering on B, so this structure is unstable. We will see (Remark 9.1.11) that every structure which is a proper expansion of $(N,+,0,1)$ and a reduct of $(N,+,0,1,<)$, and which has a definable one-dimensional set which is not definable in $(N,+,0,1)$, defines a set of the form $[0,b]$ for a positive infinite b. Hence a stable intermediate structure between $(N,+,0,1,<)$ and $(N,+,0,1)$, if such exists, cannot contain new definable sets of dimension one.

Nevertheless, a weaker version of Fact 9.1.7 does hold as well for elementarily equivalent structures.

Corollary 9.1.9. *Let $(N,+,0,1,<)$ be an elementary extension of $(\mathbb{Z},+,0,1,<)$. Then $(N,+,0,1,<)$ is $\emptyset$-minimal among the $\emptyset$-proper $\emptyset$-expansions of $(N,+,0,1)$.*

Proof of Corollary 9.1.9 from Fact 9.1.7. As $(\mathbb{Z},+,0,1,<)$ is a $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$, by Fact 9.1.7 it is obviously minimal among the proper $\emptyset$-expansions of $(\mathbb{Z},+,0,1)$. In $(\mathbb{Z},+,0,1)$, every element is $\emptyset$-definable, so a proper $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$ is the same as a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$. Now if $\mathcal{N}$ is a $\emptyset$-proper $\emptyset$-reduct of $(\mathbb{Z},+,0,1,<)$, and a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$, then also in $\mathcal{N}$ every element is $\emptyset$-definable, so $\mathcal{N}$ is a proper reduct of $(\mathbb{Z},+,0,1,<)$. Hence $(\mathbb{Z},+,0,1,<)$ is $\emptyset$-minimal among the $\emptyset$-proper $\emptyset$-expansions of $(\mathbb{Z},+,0,1)$. By Lemma 9.1.2, we conclude. ☐

Lemma 1.4.3 allows us to give a simple proof for the unstable case of Corollary 9.1.9:

Theorem 9.1.10 (Conant, Unstable case of Corollary 9.1.9). *Let $(N,+,0,1,<)$ be an elementary extension of $(\mathbb{Z},+,0,1,<)$. Then $(N,+,0,1,<)$ is $\emptyset$-minimal among the* unstable *$\emptyset$-proper $\emptyset$-expansions of $(N,+,0,1)$.*

Proof. Let $\mathcal{N}$ be any unstable structure with universe N, which is a $\emptyset$-proper $\emptyset$-expansion of $(N,+,0,1)$ and a $\emptyset$-reduct of $(N,+,0,1,<)$. We show that $\mathcal{N}$ is $\emptyset$-interdefinable with $(N,+,0,1,<)$. It is enough to show that $x\geq 0$ is definable over $\emptyset$ in $\mathcal{N}$. Let L be the language of $\mathcal{N}$, $L^-=\{+,0,1\}$ and $L_<=\{+,0,1,<\}$. We may expand all these languages by adding the symbols $\{-\}\cup\{D_n : n\geq 1\}$, as all of them are already definable over $\emptyset$ in all three languages. As $\mathcal{N}$ is a $\emptyset$-expansion of $(N,+,0,1)$ and a $\emptyset$-reduct of $(N,+,0,1,<)$, we may replace L with $L\cup L^-$ and $L_<$ with $L_<\cup L\cup L^-$ without adding new $\emptyset$-definable sets to any structure. So we may assume that $L^-\subseteq L\subseteq L_<$.

Let $\mathcal{M}$ be a monster model for $Th(\mathbb{Z},+,0,1,<)$, so $\mathcal{M}|_L$ is a monster for $Th(\mathcal{N})$. As $(N,+,0,1)$ is stable but $\mathcal{N}$ is not, by Lemma 1.4.3 there exist an L-formula $\phi(x,y)$ over $\emptyset$ with $|x|=1$ and $b\in\mathcal{M}$ such that $\phi(x,b)$ is not L^--definable with parameters in $\mathcal{M}$. By quantifier elimination in $Th(\mathbb{Z},+,0,1,<)$ and Lemma 8.1.3 (1) (which is a theorem of $Th(\mathbb{Z},+,0,1)$), $\phi(x,b)$ is equivalent to a formula of the form

$$\bigvee_i (D_{m_i}(x-k_i) \wedge x \in [c_i, c'_i])$$

where $c_i, c'_i \in M \cup \{-\infty, +\infty\}$ and $[c_i, c'_i]$ denotes the closed interval except if one of the bound is infinite, in which case it is open on the infinite side. Let $m = \prod_i m_i$. As each formula of the form $D_{m_i}(x-k)$ is equivalent to a disjunction of formulas of the form $D_m(x-k')$, we can rewrite this as

$$\bigvee_i (D_m(x-k_i) \wedge x \in [c_i, c'_i])$$

(with possibly different k_i's and numbering). By grouping together disjuncts with the same k_i, we can rewrite this as

$$\bigvee_i (D_m(x-k_i) \wedge \bigvee_j x \in [c_{i,j}, c'_{i,j}])$$

where for $i_1 \neq i_2$, $k_{i_1} \not\equiv k_{i_2} \bmod m$. As this formula is equivalent to $\phi(x,b)$, which is not L^--definable with parameters in $\mathcal{M}$, there must be an i_0 such that $D_m(x-k_{i_0}) \wedge \bigvee_j x \in [c_{i_0,j}, c'_{i_0,j}]$ is not L^--definable with parameters in $\mathcal{M}$. This latter formula, which we denote by $\phi_{i_0}(x,b)$, is equivalent to $\phi(x,b) \wedge D_m(x-k_{i_0})$, and so is L-definable. Let $\psi(x,b)$ be the formula $\phi_{i_0}(mx+k_{i_0}, b)$. Then $\psi(x,b)$ is L-definable and equivalent to just $\bigvee_j mx + k_{i_0} \in [c_{i_0,j}, c'_{i_0,j}]$. This substitution is reversible as $\phi_{i_0}(x,b)$ is equivalent to $D_m(x-k_{i_0}) \wedge \psi(\frac{x-k_{i_0}}{m}, b)$, therefore also $\psi(x,b)$ is not L^--definable with parameters in $\mathcal{M}$. Each formula of the form $mx+k \in [c,c']$ is equivalent to the formula $x \in [\lfloor \frac{c-k}{m} \rfloor, \lceil \frac{c'-k}{m} \rceil]$, so we can rewrite $\psi(x,b)$ as $\bigvee_{i=1}^n x \in [c_i, c'_i]$. By reordering and combining intersecting intervals, we may assume that the intervals are disjoint and increasing, i.e., for all $i < n$, $c'_i \leq c_{i+1}$.

Now we show how from $\psi(x,b)$ we can get an L-definable formula equivalent to $[0,a]$, for a a positive nonstandard integer in $\mathcal{M}$. For each i, if $[c_i, c'_i]$ defines in $\mathcal{M}$ a finite set then it is L^--definable, and so $\psi(x,b) \wedge \neg x \in [c_i, c'_i]$ is also L-definable but not L^--definable (since $(\psi(x,b) \wedge \neg x \in [c_i, c'_i]) \vee x \in [c_i, c'_i]$ is again equivalent to $\psi(x,b)$). So we may assume that for all i, $[c_i, c'_i]$ is infinite. Note that as $\psi(x,b)$ is not L^--definable, it cannot be empty.

We want $\psi(x,b)$ to have a lower bound, i.e., $-\infty < c_1$. If $c_1 = -\infty$ but $c'_n \neq +\infty$, then we can just replace $\psi(x,b)$ with $\psi(-x,b)$. If both $c_1 = -\infty$ and $c'_n = +\infty$, we can replace $\psi(x,b)$ with $\neg\psi(x,b)$ and again remove all finite intervals. In both cases, $\psi(x,b)$ is still L-definable but not L^--definable, so it is still a nonempty disjunction of infinite disjoint intervals.

By replacing $\psi(x,b)$ with $\psi(x+c_1+1,b)$ we may assume that $c_1 = 0$, so the leftmost interval is $[0, c'_1]$. If $c'_1 \neq +\infty$ let $a' = c'_1$, otherwise let $a' \in \mathcal{M}$ be any positive nonstandard integer. Let $\theta(x,b')$ denote the formula $\psi(x,b) \wedge \psi(a'-x,b)$. Then $\theta(x,b')$ is L-definable and equivalent to the infinite interval $[0,a']$. The proof of the following claim is an obvious consequence of quantifier elimination for Presburger arithmetic and is left to the reader.

Claim 1. For every $c \geq 0$ there exist $a > c$ and b such that $\theta(x,b)$ is equivalent to the interval $[0,a]$.

In particular, as N is a small subset of $\mathcal{M}$, there exists $c \in \mathcal{M}$ bigger than all elements of N. By the claim, there exist $\tilde{a} > c$ and $\tilde{b}$ such that $\theta(x,\tilde{b})$ is equivalent to the interval $[0,\tilde{a}]$, and so $\theta(N,\tilde{b}) = \{s \in N : s \geq 0\}$.

Let $\chi(y,z)$ be the formula $\chi_1(y,z) \wedge \chi_2(y,z) \wedge \chi_3(y,z)$ where:

- $\chi_1(y,z)$ is the formula $\theta(0,z) \wedge \theta(y,z) \wedge \neg\theta(-1,z) \wedge \neg\theta(y+1,z) \wedge \neg\theta(2y,z)$.
- $\chi_2(y,z)$ is the formula $\forall w((w \neq 0 \wedge \theta(w,z)) \to \theta(w-1,z))$.

- $\chi_3(y,z)$ is the formula $\forall w((w \neq y \wedge \theta(w,z)) \to \theta(w+1,z))$.

So $\chi(y,z)$ is L-definable over $\emptyset$.

Claim 2. For every $a, b \in \mathcal{M}$, $\mathcal{M} \vDash \chi(a,b)$ if and only if $a > 0$ and $\theta(\mathcal{M}, b) = [0,a]$.

Proof of the claim. This can be formulated as a first order sentence in $L_<$ without parameters:

$$\mathcal{M} \vDash \forall y, z(\chi(y,z) \leftrightarrow (y > 0 \wedge \forall x(\theta(x,z) \leftrightarrow 0 \leq x \leq y))),$$

so it is enough to prove this for $\mathbb{Z}$. Let $a, b \in \mathbb{Z}$. If $a > 0$ and $\theta(\mathbb{Z}, b) = [0,a]$, then clearly $\mathbb{Z} \vDash \chi(a,b)$. Suppose $\mathbb{Z} \vDash \chi(a,b)$, and set $A := \theta(\mathbb{Z}, b)$. By χ_1, $0, a \in A$ and $-1, a+1, 2a \notin A$. Suppose towards contradiction that $a < 0$. Then from χ_2 it follows by induction that $(-\infty, a] \subseteq A$. But then $2a \in A$, a contradiction. So $a \geq 0$. If $a = 0$ then again $2a \in A$ is a contradiction. So $a > 0$. From χ_2 it follows by induction that $[0,a] \subseteq A$. Also, from $a+1 \notin A$ and χ_2 it follows by induction that $[a+1, \infty) \cap A = \emptyset$, and from $-1 \notin A$ and χ_3 it follows by induction that $(-\infty, -1] \cap A = \emptyset$. So $A = [0,a]$. ☐

Now, let $\delta(x)$ be the formula

$$\exists y, z(\chi(y,z) \wedge \theta(x,z)).$$

Then $\delta(x)$ is L-definable over $\emptyset$, and we claim that it defines $x \geq 0$ in $\mathcal{N}$: For $s \in N$, if $\mathcal{N} \vDash \delta(s)$ then there are $a, b \in N$ such that $\mathcal{N} \vDash \chi(a,b) \wedge \theta(s,b)$, so by Claim 2, $s \in [0,a]$ hence $s \geq 0$. On the other hand, suppose $s \geq 0$. By the choice of $\tilde{a}, \tilde{b}$, $\mathcal{M} \vDash \chi(\tilde{a}, \tilde{b}) \wedge \theta(s, \tilde{b})$, so $\mathcal{M} \vDash \delta(s)$, and by elementarity, $\mathcal{N} \vDash \delta(s)$. Therefore, $x \geq 0$ is definable over $\emptyset$ in $\mathcal{N}$. ☐

Remark 9.1.11. The part in the proof where we start with an L-formula $\phi(x,y)$ over $\emptyset$ with $|x| = 1$ and $b \in \mathcal{M}$ such that $\phi(x,b)$ is not L^--definable with parameters in $\mathcal{M}$, and show that there exists a formula $\theta(x,b')$ which is L-definable and equivalent to the infinite interval $[0,a']$, works the same for any structure $\mathcal{N}$ which is a proper expansion of $(N,+,0,1)$ and a reduct of $(N,+,0,1,<)$. $\mathcal{N}$ does not have to be a $\emptyset$-expansion of $(N,+,0,1)$ or a $\emptyset$-reduct of $(N,+,0,1,<)$, nor unstable, as long as such $\phi(x,y)$ and b exist (being a $\emptyset$-reduct is needed in the proof for $\phi(x,y)$ to also be $\emptyset$-definable in $L_<$). So in any structure $\mathcal{N}$ which is a proper expansion of $(N,+,0,1)$ and a reduct of $(N,+,0,1,<)$, and which has a definable one-dimensional set which is not definable in $(N,+,0,1)$, there exists a definable infinite interval, and hence it is unstable.

Combined with Fact 9.1.5, we recover Corollary 9.1.9 and Fact 9.1.7.

Proof of Corollary 9.1.9 from Theorem 9.1.10. Suppose for a contradiction that there exists a structure $\mathcal{N}$ with universe N, which is a $\emptyset$-proper $\emptyset$-expansion of $(N,+,0,1)$ and a $\emptyset$-proper $\emptyset$-reduct of $(N,+,0,1,<)$. So $\mathcal{N}$ is dp-minimal, and by Theorem 9.1.10, it must also be stable. By Lemma 9.1.2, relativization to $\mathbb{Z}$ gives us a structure $\mathcal{Z} \prec \mathcal{N}$ which is a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$ and a $\emptyset$-proper $\emptyset$-reduct of $(\mathbb{Z},+,0,1,<)$. As every element of $(\mathbb{Z},+,0,1)$ is $\emptyset$-definable, a reduct of $(\mathbb{Z},+,0,1)$ is in fact a $\emptyset$-reduct, and so a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$ is in fact a proper $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$, which is of course a proper expansion. So $\mathcal{Z}$ is a stable dp-minimal proper expansion of $(\mathbb{Z},+,0,1)$, a contradiction to Fact 9.1.5. ☐

Proof of Fact 9.1.7 from Corollary 9.1.9. Suppose for a contradiction that there exists a structure $\mathcal{Z}$ with universe $\mathbb{Z}$, which is a proper expansion of $(\mathbb{Z}, +, 0, 1)$ and a proper reduct of $(\mathbb{Z}, +, 0, 1, <)$. In $\mathcal{Z}$, $+$, 0, and 1 are definable, but not necessarily $\emptyset$-definable. We expand $\mathcal{Z}$ to a structure $\mathcal{Z}'$ by adding $+$, 0, and 1 to the language. So $\mathcal{Z}'$ is a proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$, and still a proper reduct of $(\mathbb{Z}, +, 0, 1, <)$. As every element of $(\mathbb{Z}, +, 0, 1, <)$ is $\emptyset$-definable, a reduct of $(\mathbb{Z}, +, 0, 1, <)$ is in fact a $\emptyset$-reduct. So $\mathcal{Z}'$ is a proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$, and a proper $\emptyset$-reduct of $(\mathbb{Z}, +, 0, 1, <)$. As a proper $\emptyset$-expansion/reduct is obviously a $\emptyset$-proper $\emptyset$-expansion/reduct, this contradicts Corollary 9.1.9. ☐

9.2 The main result: $(\mathbb{Z}, +, 0, |_p)$ is a minimal expansion of $(\mathbb{Z}, +, 0)$

In this section, we focus on a single valuation. Let p be any prime. Unless stated otherwise, we work in a monster model $\mathcal{M} = (M, +, 0, |_p)$ of T_p, and denote its value set by Γ. We may omit the subscript p when it is clear from the context. Recall that Γ is an elementary extension of $(\mathbb{N}, <, 0, S)$.

9.2.1 Preparatory lemmas

For $a \in M$, $\gamma \in \Gamma$, we denote by $B(a, \gamma)$ the definable set $\{x : v(x - a) \geq \gamma\}$ and call it the *ball of radius* γ around a. If $\gamma = \infty$ then $B(a, \gamma)$ is just $\{a\}$, and we call such balls *trivial*. Unless stated otherwise, balls are assumed to be nontrivial. Of course, $a \in B(a, \gamma)$, and if $b \in B(a, \gamma)$ then $B(b, \gamma) = B(a, \gamma)$. Also, by Lemma 8.1.2 (2), if $\delta \neq \gamma$ then $B(a, \delta) \neq B(a, \gamma)$. So the radius of a ball is well defined. We denote the radius of a ball B by $rad(B)$.

We call a *swiss cheese* any non-empty set F that can be written as $F = B_0 \backslash \bigcup_{i=1}^{n} B_i$, where $\{B_i\}_{i=0}^{n}$ are balls. Note that this representation is not unique. As the intersection of any two balls is either empty or equals one of them, we may always assume that $\{B_i\}_{i=1}^{n}$ are nonempty, pairwise disjoint and contained in B_0.

Remark 9.2.1. Rephrasing Lemma 8.1.9 *(2)*, if $B_0, B_1, \ldots, B_n$ are balls such that for all $i \geq 1$, $rad(B_i) \geq rad(B_0) + \underline{n}$, then $B_0 \backslash \bigcup_{i=1}^{n} B_i \neq \emptyset$. In particular, this holds if $|rad(B_i) - rad(B_0)| \notin \mathbb{N}$.

Proposition 9.2.2. *Let $\emptyset \neq F = B_0 \backslash \bigcup_{i=1}^{n} B_i$ be a swiss cheese. Then there exists a unique ball B_0' such that $F \subseteq B_0'$ and B_0' is minimal with respect to this property. This B_0' satisfies $B_0' \subseteq B_0$, $|rad(B_0') - rad(B_0)| \in \mathbb{N}$, and it is also the unique ball $B \subseteq B_0$ such that there are at least two distinct balls B_1'' and B_2'', satisfying $rad(B_j'') = rad(B_0') + 1$ and $B_j'' \cap F \neq \emptyset$ for $j = 1, 2$.*

Proof. Let $I_1 = \{1 \leq i \leq n : |rad(B_i) - rad(B_0)| \in \mathbb{N}\}$, $I_2 = \{1, \ldots, n\} \backslash I_1$. By applying Lemma 8.1.8 *(2)* to $B_0 \backslash \bigcup_{i \in I_1} B_i \neq \emptyset$, we see that $B_0 \backslash \bigcup_{i \in I_1} B_i = \bigsqcup_{j=1}^{l} B_j''$, where $l \geq 1$ and for all j, $B_j'' \subseteq B_0$ and $rad(B_j'') = \max\{rad(B_i) : i \in I_1\}$. So $F = \bigsqcup_{j=1}^{l} (B_j'' \backslash \bigcup_{i \in I_2} B_i)$. By Remark 9.2.1, for each j, $B_j'' \backslash \bigcup_{i \in I_2} B_i \neq \emptyset$. If C is a ball such that $F \subseteq C$, then for each j, $B_j'' \backslash \bigcup_{i \in I_2} B_i \subseteq C$, and we claim that in fact $B_j'' \subseteq C$. Indeed, by Axiom 8, $B_j'' = \bigsqcup_{t=1}^{p} B_{j,t}''$ with $rad(B_{j,t}'') = rad(B_j'') + 1$, and again by Remark 9.2.1, for each t, $B_{j,t}'' \backslash \bigcup_{i \in I_2} B_i \neq \emptyset$. So $C \cap B_{j,t}'' \neq \emptyset$ but $C \nsubseteq B_{j,t}''$ (as also for $s \neq t$, $C \cap B_{j,s}'' \neq \emptyset$), therefore $B_{j,t}'' \subseteq C$. This holds for all t, hence $B_j'' \subseteq C$. In particular, $B_1'' \subseteq C$. As $|rad(B_1'') - rad(B_0)| \in \mathbb{N}$, there are only finitely many balls B such that $B_1'' \subseteq B \subseteq B_0$, so we may choose B_0' to be a minimal one (with respect to inclusion) among those that also satisfy $F \subseteq B$ (exists, since B_0 satisfies this). By this choice, $B_0' \subseteq B_0$ and $|rad(B_0') - rad(B_0)| \in \mathbb{N}$. If B is another ball such that $F \subseteq B$, then $F \subseteq B \cap B_0'$, and $B \cap B_0' \neq \emptyset$ is also a ball. Also, as we have shown, $B_1'' \subseteq B$, so $B''_1 \subseteq B \cap B'_0 \subseteq B_0$. Hence by the choice of B'_0, $B'_0 = B \cap B'_0 \subseteq B$. This shows that B'_0 is the unique minimal ball containing F. Finally, let D be a ball and assume $F \subseteq D$. By Axiom 8 write $D = \bigsqcup_{t=1}^{p} D''_t$ with $rad(D''_t) = rad(D) + 1$. Then D is minimal if and only if for all t, $F \nsubseteq D''_t$, iff there are $t \neq s$ such that $F \cap D''_t \neq \emptyset$ and $F \cap D''_s \neq \emptyset$. ☐

Let F be a swiss cheese. By Proposition 9.2.2 we may write $F = B_0 \backslash \bigcup_{i=1}^{n} B_i$ where B_0 is the unique minimal ball containing F. We may also assume that $\{B_i\}_{i=1}^{n}$ are nonempty, pairwise disjoint and contained in B_0. Unless stated otherwise, all representations are assumed to satisfy these conditions. We call B_0 the *outer ball* of F, and define the *radius* of F to be $rad(F) := rad(B_0)$. We also call $\{B_i\}_{i=1}^{n}$ the *holes* of F. Note that this representation is still not unique (unless there are no holes at all), as each hole may always be split into p smaller holes, and sometimes there are sets of p holes which may each be combined into a single hole. There is a canonical representation for F, namely, the one with the minimal number of holes. But we will not use it. Rather, when dealing with holes without mentioning a specific representation, either the intended representation is clear from the context (e.g., when using Remark 9.2.3 (2) or (3) to split a swiss cheese with a given representation), or we may choose any representation and stick with it.

We say that B_i is a *proper hole* of F if $|rad(B_i) - rad(B_0)| \notin \mathbb{N}$. We call F a *proper cheese* if all of its holes are proper. Note that by Remark 9.2.1, being a proper cheese does not depend on the representation of the holes.

Remark 9.2.3.

(1) If $B_0, B_1, \ldots, B_n$ are balls such that for all $i \geq 1$, $B_i \subseteq B_0$ and $|rad(B_i) - rad(B_0)| \notin \mathbb{N}$, then B_0 is the outer ball of the swiss cheese $F = B_0 \backslash \bigcup_{i=1}^{n} B_i$, which is therefore proper.

(2) Let F be a swiss cheese, and let $k \geq 1$. Then F may be written as a disjoint union $F = \bigsqcup_{i=1}^{l} F_i$, where $1 \leq l \leq p^k$, and for each i, F_i is a swiss cheese such that $rad(F_i) \geq rad(F) + k$ and $|rad(F_i) - rad(F)| \in \mathbb{N}$. Each hole of F_i is already a hole of F, and each hole of F is a hole of at most one of the $\{F_i\}_i$.
If F is proper, then $l = p^k$ and each F_i is a proper cheese of radius $rad(F_i) = rad(F) + k$. In this case, each hole of F is a hole of exactly one of the $\{F_i\}_i$.

(3) Let $F = B_0 \backslash \bigcup_{i=1}^{n} B_i$ be a swiss cheese, let $I_1 = \{1 \leq i \leq n : |rad(B_i) - rad(B_0)| \in \mathbb{N}\}$, and let $k_0 = \max\{rad(B_i) - rad(B_0) : i \in I_1\} \in \mathbb{N}$. Then for each $k \geq k_0$, F may be written as a disjoint union $F = \bigsqcup_{i=1}^{l} F_i$, where $1 \leq l \leq p^k$, and for each i, F_i is a *proper* swiss cheese of radius $rad(F_i) = rad(F) + k$. Each hole of F_i is already a proper hole of F, and each proper hole of F is a hole of exactly one of the $\{F_i\}_i$.

(4) Let F', F'' be two swiss cheeses of radiuses γ', γ'' respectively, and let $\gamma = \max\{\gamma', \gamma''\}$. Then $F' \cap F''$ is either empty, or also a swiss cheese of radius $rad(F' \cap F'') \geq \gamma$ such that $|rad(F' \cap F'') - \gamma| \in \mathbb{N}$.

(5) If both F', F'' are proper and $\gamma' = \gamma''$, and if $F' \cap F''$ is not empty, then F', F'' have the same outer ball, and $F' \cap F''$ is also a proper cheese of the same outer ball.

Lemma 9.2.4. *Let F, F' be two swiss cheeses of radiuses $\gamma \leq \gamma'$ respectively. If $F \cap F' \neq \emptyset$, then $F \cup F'$ is also a swiss cheese, of radius exactly γ. The set of holes of $F \cup F'$ is a subset of the union of the set of holes of F and the set of holes of F'.*

Proof. Write $F = B_0 \backslash \bigcup_{i=1}^{n} B_i$, $F' = B'_0 \backslash \bigcup_{j=1}^{m} B'_j$. If $F \cap F' \neq \emptyset$ then $B_0 \cap B'_0 \neq \emptyset$, hence $B_0 \supseteq B'_0$. Therefore,

$$F' \backslash F = F' \backslash \left(B_0 \backslash \bigcup_{i=1}^{n} B_i \right) = F' \backslash B_0 \cup \left(F' \cap \bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} F' \cap B_i.$$

For each i: if $B'_0 \cap B_i = \emptyset$ then $F' \cap B_i = \emptyset$. Otherwise, as $B_0 \supseteq B'_0$, we also get $B_i \subseteq B'_0$ ($B_i \supseteq B'_0$ is impossible, as it implies $F \cap F' = \emptyset$), and in this case, $F' \cap B_i = B_i \setminus \bigcup_{j=1}^{m} (B_i \cap B'_j)$. Together, we get

$$F \cup F' = F \cup (F' \setminus F) = B_0 \setminus \left(\bigcup_{i \in I_1} B_i \cup \bigcup_{i \in I_2} \bigcup_{j=1}^{m} (B_i \cap B'_j) \right)$$

where I_1 is the set of i such that $B'_0 \cap B_i = \emptyset$ and I_2 is the set of i such that $B_i \subseteq B'_0$. This is a swiss cheese, and as $F \subseteq F \cup F' \subseteq B_0$ and $rad(F) = rad(B_0) = \gamma$, also $rad(F \cup F') = \gamma$ and B_0 is its outer ball. For each i such that $B_i \subseteq B'_0$ and each j, either $B_i \cap B'_j = \emptyset$ (in which case $B_i \cap B'_j$ does not appear as a hole of $F \cup F'$), or $B_i \cap B'_j = B_i$ or $B_i \cap B'_j = B'_j$, so the last part holds. □

Sometimes we want disjoint swiss cheeses to also have disjoint outer balls, but unfortunately, that is not always possible. An example for this is a union of two swiss cheeses, $F_1 \cup F_2$, with $F_2 \subseteq B$ where B is one of the holes of F_1. If $|rad(B) - rad(F_1)| \in \mathbb{N}$, we may rewrite F_1 as a union of swiss cheeses of radius $rad(B)$, and, together with F_2, we have a union of swiss cheeses with disjoint outer balls. But if $|rad(B) - rad(F_1)| \notin \mathbb{N}$, we cannot do such a thing.

Definition 9.2.5. A *pseudo swiss cheese* is a definable set P such that there is a swiss cheese F with outer ball B such that $F \subseteq P \subseteq B$. By the following remark, we may call B the *outer ball* of P, and define the *radius* of P to be $rad(P) := rad(B)$. We also call P *pseudo proper cheese* if there is a proper cheese F with outer ball B such that $F \subseteq P \subseteq B$.

Remark 9.2.6. (1) In the previous definition, B is uniquely determined by P. Indeed, suppose F_1,F_2 are two swiss cheeses with outer balls B_1,B_2 respectively, such that $F_1 \subseteq P \subseteq B_1$ and $F_2 \subseteq P \subseteq B_2$. Then $rad(B_1) = rad(F_1) \geq rad(B_2)$ and $rad(B_2) = rad(F_2) \geq rad(B_1)$, so $rad(B_1) = rad(B_2)$. Also, $P \subseteq B_1 \cap B_2 \neq \emptyset$, so we must have $B_1 = B_2$.

(2) For every $k \geq 1$, every proper pseudo swiss cheese of radius γ can be written as a union of exactly p^k proper pseudo cheeses with disjoint outer balls of radius exactly $\gamma + k$.

(3) Note that the analogue to Remark 9.2.3 (2) is not true for pseudo swiss cheeses. For example, let B be a ball of radius γ, let $\{B_i\}_{i=0}^{p-1}$ be all the balls of radius $\gamma + 1$ contained in B, let $\{B_{i,j}\}_{j=0}^{p-1}$ be all the balls of radius $\gamma + 2$ contained in B_i, and let $C \subseteq B_{0,1}$ be a ball of radius $\delta > \gamma$ such that $|\delta - \gamma| \notin \mathbb{N}$. Then $P = C \sqcup \bigsqcup_{i=0}^{p-1} B_{i,0}$ is a pseudo swiss cheese of radius γ, but cannot be written as $\leq p$ pseudo swiss cheeses of radius $\geq \gamma + 1$, because $P \cap B_0$ is not a pseudo swiss cheese. Also, note that the intersection of two pseudo swiss cheeses is not necessarily a single pseudo swiss cheese. For example, take $P \cap B_0$ from above.

Lemma 9.2.7.

(1) Let P_1,P_2 be two pseudo swiss cheeses with outer balls B_1,B_2 respectively, such that $rad(B_1) \geq rad(B_2)$. If $B_1 \cap B_2 \neq \emptyset$ then $P_1 \cup P_2$ is also a pseudo swiss cheese, with outer ball B_2. If P_2 is proper, then $P_1 \cup P_2$ is also proper.

(2) Any finite union of pseudo swiss cheeses may be written as a union of pseudo swiss cheeses having disjoint outer balls. Also, any finite union of pseudo proper cheeses may be written as a union of pseudo proper cheeses having disjoint outer balls.

Proof. We prove *(1)*. $B_1 \cap B_2 \neq \emptyset$ and $rad(B_1) \geq rad(B_2)$, so $B_1 \subseteq B_2$ and therefore also $P_1 \subseteq B_2$. Let F_2 be a swiss cheese with outer ball B_2 such that $F_2 \subseteq P_2 \subseteq B_2$. Then

$F_2 \subseteq P_1 \cup P_2 \subseteq B_2$. If P_2 is proper, then we may take F_2 to be proper, and so $P_1 \cup P_2$ is also proper.

We prove *(2)*. Let $A = \bigcup_{i=1}^n P_i$ such that for each i, P_i is a pseudo swiss cheese with outer ball B_i. Let $\{B'_j\}_{j=1}^m$ be the set of all the maximal balls (with respect to inclusion) among $\{B_i\}_{i=1}^n$. Then $\{B'_j\}_{j=1}^m$ are pairwise disjoint. For each $1 \leq j \leq m$, let $I_j = \{i : B_i \cap B'_j \neq \emptyset\}$ and $P'_j = \bigcup_{i \in I_j} P_i$. So $\{1, \dots, n\} = \bigsqcup_{j=1}^m I_j$ and therefore $A = \bigcup_{j=1}^m P'_j$. By *(1)*, P'_j is a pseudo swiss cheese with outer ball B'_j. If for each i, P_i is proper, then by *(1)*, for each j, P'_j is also proper. □

Remark 9.2.8. The valuation v_p induces a topology on $\mathcal{M}$, generated by the balls. By Lemma 8.1.9 *(3)*, if $\gcd(m, p) = 1$, then the sets defined by $D_m(x - r)$ are dense in $\mathcal{M}$.

Lemma 9.2.9. *Let P be a pseudo swiss cheese with outer ball B and radius α, and assume $0 \in B$. Let G be a dense subgroup of $\mathcal{M}$, and let $A = P \cap G$. Then there exists $N \in \mathbb{N}$ and $a_1, \dots, a_N \in B \cap G$ such that $\bigcup_{i=1}^N (A + a_i) = B \cap G$.*

Proof. Observe that B is a subgroup of $\mathcal{M}$ since $0 \in B$. Let F be a swiss cheese with outer ball B such that $F \subseteq P \subseteq B$. By Remark 9.2.3 (3), for some finite k we may find a proper cheese $F' \subseteq F$ of radius $\alpha + k$. Let s be the number of holes in F'. By Remark 9.2.3 (2), we may write F' as a union of exactly p^s proper cheeses of radius $\alpha + k + s$. As $p^s > s$, at least one of these proper cheeses must have no holes, i.e., must be a ball, say D. Let $x \in D$ and $D_0 = D - x$. Then D_0 is a subgroup of B of index $N := p^{k+s}$. Let $x_1, \dots, x_N$ be representatives of the cosets, so $B = \bigcup_{i=1}^N x_i + D_0$. For each i, let $a_i \in x_i + D_0 \cap G$. As $a_i \in B \cap G$ and $A \subseteq B \cap G$, we have $(A + a_i) \subseteq B \cap G$, and therefore $\bigcup_{i=1}^N (A + a_i) \subseteq B \cap G$. On the other hand, as $A \supseteq D \cap G$, we also have $\bigcup_{i=1}^N (A + a_i) \supseteq B \cap G$. □

Lemma 9.2.10. *Let $A = G \cap \bigsqcup_{i=1}^n F_i$ where G is a dense subgroup of $\mathcal{M}$ and $\{F_i\}_{i=1}^n$ are disjoint proper cheeses with nonstandard radiuses. Then there are $N, m \in \mathbb{N}$ and $c_1, \dots, c_N \in G$ such that $\bigcap_{i=1}^N (A - c_i) = G \cap \bigsqcup_{i=1}^m P_i$ with P_i* pseudo *proper cheeses with disjoint outer balls, all of the same nonstandard radius, and $0 \in P_1$.*

Proof. It is of course enough to prove the lemma without the requirement $0 \in P_1$, as we may then arrange that by shifting by some $c \in G \cap P_1$.

Preparation step. By Remark 9.2.3 *(2)*, if F is a proper cheese of infinite radius γ then, for all $k \geq 0$, F can be written as a disjoint union of proper cheeses of radius $\gamma + k$. So there exists $\gamma_1, \dots, \gamma_n$, in distinct archimedean classes of Γ, such that we can write

$$\bigsqcup_{i=1}^n F_i = \bigsqcup_{i=1}^m \bigsqcup_{j=1}^{s_i} F_j^i,$$

where $s_1, \dots, s_m \geq 1$ and for all $1 \leq i \leq m$ and $1 \leq j \leq s_i$, $rad(F_j^i) = \gamma_i$ and F_j^i has a swiss cheese representation in which the radiuses of all the holes are in

$$R := \{\alpha \in \Gamma : \text{for all } 1 \leq k \leq m, \text{ if } |\alpha - \gamma_k| \in \mathbb{N} \text{ then } \alpha \leq \gamma_k\}.$$

We call this representation of A a *good representation of A with respect to* $\{\gamma_i\}_{i=1}^m$.

If $m = 1$, we already have what we want, so we may assume that $m > 1$. For each i, j, let B_j^i be the outer ball of F_j^i. There are two cases:

Case 1: For every $1 < l \leq m$ and every $1 \leq u \leq s_l$ there is some $1 \leq v \leq s_1$ such that $B_v^1 \cap B_u^l \neq \emptyset$.

This means that $\{B^1_j\}_{j=1}^{s_1}$ is the set of all the maximal balls with respect to inclusion among $\{B^i_j : 1 \leq i \leq m\,,\, 1 \leq j \leq s_i\}$. It follows that $\{B^1_j\}_{j=1}^{s_1}$ are outer balls of pseudo proper cheese containing all the F^i_j. Indeed, by the proof of Lemma 9.2.7 *(2)*, we may write

$$\bigsqcup_{i=1}^{m}\bigsqcup_{j=1}^{s_i} F^i_j = \bigsqcup_{j=1}^{s_1} P_j,$$

where for each j, P_j is a pseudo proper cheese such that $F^1_j \subseteq P_j \subseteq B^1_j$. So these are pseudo proper cheeses with disjoint outer balls, all of the same radius γ_1. So in this case we are done.

Case 2: There are $1 < l \leq m$ and $1 \leq v \leq s_l$ such that for every $1 \leq j \leq s_1$, $B^1_j \cap B^l_v = \emptyset$.

Let $a \in F^1_1 \cap G$ and $b \in F^l_v \cap G$ and set $A' = (A - a) \cap (A - b)$. Then $0 \in A' \neq \emptyset$. We show that A' has a good representation with respect to a subset of $\{\gamma_i\}_{i=1}^m$, of the form

$$A' = G \cap \bigsqcup_{i=1}^{m'}\bigsqcup_{j=1}^{s'_i} \tilde{F}^i_j$$

such that either there are no more proper cheeses of radius γ_1, or the number s'_1 of proper cheeses of radius γ_1 is strictly less than s_1. By reiterating this process, it will terminate either to the case in which every proper cheese is of the same radius or to *Case 1*, which proves the Lemma.

Write $A' = G \cap (\bigsqcup_{i=1}^m \bigsqcup_{j=1}^{s_i} \bigsqcup_{q=1}^m \bigsqcup_{r=1}^{s_i} (F^i_j - a) \cap (F^q_r - b))$. By the good representation, for each i, j we write $F^i_j = B^i_j \backslash \bigsqcup_t B^i_{j,t}$ with $rad(B^i_{j,t}) \in R$.

For every i and j, k, if $B^i_j - a \neq B^i_k - b$, then $(F^i_j - a) \cap (F^i_k - b) = \emptyset$, and if $B^i_j - a = B^i_k - b$, then $(F^i_j - a) \cap (F^i_k - b)$ is a proper cheese of radius $\gamma_i \geq \gamma_1$ such that all its holes can be written with radiuses in R.

For every $i < i'$ and j, k, if $(B^i_j - a) \cap (B^{i'}_k - b) = \emptyset$, then also $(F^i_j - a) \cap (F^{i'}_k - b) = \emptyset$. Otherwise, $(B^i_j - a) \supseteq (B^{i'}_k - b)$ and

$$(F^i_j - a) \cap (F^{i'}_k - b) = ((B^{i'}_k - b) \backslash \bigsqcup_{t'} (B^{i'}_{k,t'} - b)) \backslash \bigsqcup_{t} (B^i_{j,t} - a).$$

For each t such that $(B^i_{j,t} - a) \cap (B^{i'}_k - b) \neq \emptyset$ there are three cases:

(1) $rad(B^{i'}_k - b) > rad(B^i_{j,t} - a)$. Then $(B^{i'}_k - b)$ is included in the hole $(B^i_{j,t} - a)$ hence $(F^i_j - a) \cap (F^{i'}_k - b) = \emptyset$.

(2) $rad(B^{i'}_k - b) \leq rad(B^i_{j,t} - a)$ and $rad(B^i_{j,t} - a)$ is at finite distance from $\gamma_{i'}$. As $rad(B^i_{j,t} - a) = rad(B^i_{j,t}) \in R$, we get

$$rad(B^{i'}_k - b) = rad(B^{i'}_k) = \gamma_{i'} \geq rad(B^i_{j,t} - a).$$

So $rad(B^{i'}_k - b) = rad(B^i_{j,t} - a)$, and so $(B^{i'}_k - b) = (B^i_{j,t} - a)$ and therefore $(F^i_j - a) \cap (F^{i'}_k - b) = \emptyset$.

(3) $rad(B^{i'}_k - b) \leq rad(B^i_{j,t} - a)$ and $rad(B^i_{j,t} - a)$ is not at finite distance from $\gamma_{i'}$. Then $B^i_{j,t} - a$ is a proper hole of $(F^i_j - a) \cap (F^{i'}_k - b)$.

Therefore $(F^i_j - a) \cap (F^{i'}_k - b)$ is either empty or a proper cheese of radius $\gamma_{i'} > \gamma_i \geq \gamma_1$ such that all its holes can be written with radiuses in R.

So A' has a good representation that is the intersection of G with a (nonempty) disjoint union of proper cheeses, with radiuses among $\{\gamma_i\}_{i=1}^m$, such that all their holes have radiuses in R. Now either $s_1 = 1$, hence F_1^1 is the only cheese of radius γ_1 in the good representation of A and hence in the good representation of A' there are no more proper cheese of radius γ_1. Otherwise we have a good representation with respect to a subset of $\{\gamma_i\}_{i=1}^m$ of the form

$$A' = G \cap \bigsqcup_{i=1}^{m'} \bigsqcup_{j=1}^{s'_i} \tilde{F}_j^i$$

where $s'_1, \ldots, s'_{m'} \geq 1$, and s'_1 is the number of cheese of radius γ_1. For every $1 \leq l \leq s'_1$, there must be j, k such that $\tilde{F}_l^1 = (F_j^1 - a) \cap (F_k^1 - b)$. As $(F_j^1 - a) \cap (F_k^1 - b) \neq \emptyset \iff B_j^1 - a = B_k^1 - b$, for every j there is at most one k such that $(F_j^1 - a) \cap (F_k^1 - b) \neq \emptyset$, therefore $s'_1 \leq s_1$. Suppose towards contradiction that $s'_1 = s_1$. Then for every j there is exactly one k such that $(F_j^1 - a) \cap (F_k^1 - b) \neq \emptyset$, in particular, for $j = 1$ there is exactly one l such that $(F_1^1 - a) \cap (F_l^1 - b) \neq \emptyset$, and so also $B_1^1 - a = B_l^1 - b$. By the choice of a, b, we have $0 \in (B_1^1 - a) \cap (B_v^l - b) = (B_l^1 - b) \cap (B_v^l - b)$, so $b \in B_l^1 \cap B_v^l \neq \emptyset$, a contradiction. Therefore $s'_1 < s_1$. □

Lemma 9.2.11. *Let* $A = G \cap \bigsqcup_{i=1}^n P_i$ *where* G *is a dense subgroup of* $\mathcal{M}$ *and* $\{P_i\}_{i=1}^n$ *are pseudo proper cheeses with disjoint outer balls, all of the same nonstandard radius* α*, such that* $0 \in P_1$*. Then there exists* $N \in \mathbb{N}$ *and* $c_1, \ldots, c_N \in G$ *such that* $\bigcap_{i=1}^N (A - c_i) = G \cap P$ *for some pseudo proper cheese* P *of nonstandard radius such that* $0 \in P$*.*

Proof. It is of course enough to prove the lemma without the requirement $0 \in P$. We proceed by induction on n. For $n = 1$ we have nothing to prove. Suppose that the lemma holds for all $n' < n$. For each $1 \leq i \leq n$ let B_i be the outer ball of P_i, and let F_i be a proper cheese with outer ball B_i such that $F_i \subseteq P_i \subseteq B_i$. Let S be the set of all the balls of radius α, and let $S' = \{B_i : 1 \leq i \leq n\}$. Observe that $(S, +)$ is an infinite group with neutral element B_1 (since $0 \in P_1 \subseteq B_1$), and in particular, $S' \subsetneq S$. Let $C := \bigcup S' = \bigsqcup_{i=1}^n B_i$.

Claim. If for every $1 \leq i \leq n$ there is $a \in B_i$ such that $S' - a = S'$, then S' is a subgroup of S.

Proof of the claim. If $B, B' \in S$ then $rad(B) = rad(B')$, hence $(B - a) \cap B' \neq \emptyset \Rightarrow B - a = B'$. Also, for all $B'' \in S$ and $a, a' \in B''$, $a - a' \in B_1$ and therefore $B - a' = (B - a) + (a - a') = B - a$. From this and the hypothesis of the claim it follows that for each $1 \leq i \leq n$, $S' - B_i := \{B - B_i : B \in S'\} = S'$, which implies that S' is a subgroup of S. □

There are two cases:

Case 1: S' is a subgroup of S. Then $(C, +)$ is a subgroup of $(M, +)$, and S' is the quotient group C/B_1. As $(C, +)$ is definable, by Lemma 8.2.4 it must be of the form $C = B(0, \beta)$ (as $B_1 \not\subseteq mM$ for every $m > 1$ with $\gcd(m, p) = 1$). In fact, since $|S'| = n$, it must be that $\beta = \alpha - k$, where k satisfies $n = p^k$. In particular, β is nonstandard. For each i, let H_i be (any choice for) the set of holes of F_i, and let $H = \bigcup_i H_i$. Then we can rewrite $\bigsqcup_{i=1}^n F_i$ as $F = B(0, \beta) \setminus \bigcup H$, which is a single proper cheese, with outer ball $B(0, \beta)$. Let $P = \bigsqcup_{i=1}^n P_i$. Then $F \subseteq P \subseteq B(0, \beta)$, so P is a pseudo proper cheese, and we are done.

Case 2: S' is not a subgroup of S. Then by the claim, there is some $1 \leq i_0 \leq n$ such that for all $a \in B_{i_0}$, $S' - a \neq S'$ (in fact $1 < i_0$). Let $a \in G \cap P_{i_0} \subseteq B_{i_0}$ (which exists because G is dense), and let $A' = A \cap (A - a)$. Then $0 \in A' \neq \emptyset$.

Write $A' = G \cap (\bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{n} P_i \cap (P_j - a))$. Then

$$G \cap \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{n} F_i \cap (F_j - a) \subseteq A' \subseteq G \cap \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{n} B_i \cap (B_j - a).$$

For all $1 \leq i, j \leq n$, $rad(B_i) = rad(B_j) = \alpha$ and therefore, as in Lemma 9.2.10, $B_i \cap (B_j - a) \neq \emptyset \iff B_i = B_j - a \iff F_i \cap (F_j - a) \neq \emptyset$, and in this case, $F_i \cap (F_j - a)$ is a proper cheese with outer ball B_i. We also have that $F_i \cap (F_j - a) \subseteq P_i \cap (P_j - a) \subseteq B_i \cap (B_j - a)$, so $P_i \cap (P_j - a) \neq \emptyset \iff B_i \cap (B_j - a) \neq \emptyset$, and in this case, $P_i \cap (P_j - a)$ is a pseudo proper cheese with outer ball B_i. Therefore, $G \cap (\bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{n} B_i \cap (B_j - a)) = G \cap (\bigsqcup_{i=1}^{n'} B'_i)$, $G \cap (\bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{n} F_i \cap (F_j - a)) = G \cap (\bigsqcup_{i=1}^{n'} F'_i)$, and $A' = G \cap (\bigsqcup_{i=1}^{n'} P'_i)$, where for each i, $B'_i \in S'$, F'_i is a proper swiss cheese with outer ball B'_i, and P'_i is a pseudo proper cheese such that $F'_i \subseteq P'_i \subseteq B'_i$.

Moreover, for every i there is at most one j such that $B_i \cap (B_j - a) \neq \emptyset$, therefore $n' \leq n$. But by the choice of a, $S' - a \neq S'$, so there is an $1 \leq i \leq n$ such that $B_i \neq B_j - a$ for all $1 \leq j \leq n$. Therefore $n' < n$, and by the induction hypothesis we are done. ☐

9.2.2 Proof of the theorem

The proof of Theorem 9.2.12 is similar to the proof of Theorem 9.1.10 but more involved and relies on Subsection 9.2.1.

Theorem 9.2.12. *Let* $(N, +, 0, 1, |_p)$ *be an elementary extension of* $(\mathbb{Z}, +, 0, 1, |_p)$. *Then* $(N, +, 0, 1, |_p)$ *is* $\emptyset$*-minimal among the* unstable $\emptyset$*-proper* $\emptyset$*-expansions of* $(N, +, 0, 1)$.

Proof. Let $\mathcal{N}$ be any unstable structure with universe N, which is a $\emptyset$-proper $\emptyset$-expansion of $(N, +, 0, 1)$ and a $\emptyset$-reduct of $(N, +, 0, 1, |_p)$. We show that $\mathcal{N}$ is $\emptyset$-interdefinable with $(N, +, 0, 1, |_p)$. It is enough to show that $x|_p y$ is definable over $\emptyset$ in $\mathcal{N}$. Let L be the language of $\mathcal{N}$ and $L^- = \{+, 0, 1\}$. As in the proof of Theorem 9.1.10, we may assume that all languages contain $\{-\} \cup \{D_n : n \geq 1\}$, and (by being a $\emptyset$-reduct and $\emptyset$-expansion) that $L^- \subseteq L \subseteq L_p^E$.

Let $\mathcal{M}$ be a monster model for T_p, so $\mathcal{M}|_L$ is a monster for $Th(\mathcal{N})$. As $(N, +, 0, 1)$ is stable but $\mathcal{N}$ is not, by Lemma 1.4.3 there exist an L-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ and $b \in \mathcal{M}$ such that $\phi(x, b)$ is not L^--definable with parameters in $\mathcal{M}$. By Theorem 8.2.1 (quantifier elimination) and Remark 8.2.3, $\phi(x, b)$ is equivalent to a formula of the form

$$\bigvee_i \left(D_m(x - r_i) \wedge kx \in F_i \wedge \bigwedge_j k'x \neq a_{i,j} \right) \vee \bigvee_{i'} x = c_{i'}$$

where $m, k, k', r_i \in \mathbb{Z}$, $\gcd(m, p) = \gcd(k, p) = 1$, $k' = p^l k$ for some $l \geq 0$, $a_{i,j}, c_{i'} \in \mathcal{M}$ and each F_i is a swiss cheese in $\mathcal{M}$.

The first step of the proof is to show the existence of an L-definable formula which is equivalent to a formula of the form $D_m(x) \wedge x \in B(0, \gamma)$, i.e. $D_m(x) \wedge v(x) \geq \gamma$, for some nonstandard $\gamma \in \Gamma$ and integer m such that $\gcd(m, p) = 1$. Let $\phi'(x, b)$ be the formula

$$\bigvee_i (D_m(x - r_i) \wedge kx \in F_i).$$

The symmetric difference $\phi(x,b)\triangle\phi'(x,b)$ is finite, hence L^--definable, and therefore $\phi'(x,b)$ is also L-definable but not L^--definable. So we may replace $\phi(x,b)$ by $\phi'(x,b)$. For each i, the formula $D_m(x-r_i)$ is equivalent to $D_{km}(kx-kr_i)$, so $\phi(x,b)$ is equivalent to the formula

$$\bigvee_i (D_{km}(kx-kr_i)\wedge kx\in F_i).$$

Let $\phi'(x,b)$ be the formula $D_k(x)\wedge\phi(\frac{x}{k},b)$. Then $\phi'(x,b)$ is L-definable and equivalent to the formula

$$\bigvee_i (D_{m'}(x-r'_i)\wedge x\in F_i)$$

where $m'=km$ and $r'_i=kr_i$. This substitution is reversible as $\phi(x,b)$ is equivalent to $\phi'(kx,b)$, therefore also $\phi'(x,b)$ is not L^--definable. So again we may replace $\phi(x,b)$ by $\phi'(x,b)$.

We want each F_i to have a nonstandard radiuses. For each i, choose a representation for F_i as a swiss cheese $F_i=B_{i,0}\setminus\bigcup_{j=1}^{n_i}B_{i,j}$, where $B_{i,j}=B(a_{i,j},\gamma_{i,j})$. Let $J_i=\{1\le j\le n_i : \gamma_{i,j}\notin\mathbb{N}\}$, i.e., the set of indices of the infinite holes, and let

$$B'_{i,0}=\begin{cases}B(0,0) & \gamma_{i,0}\in\mathbb{N}\\ B_{i,0} & \gamma_{i,0}\notin\mathbb{N}\end{cases}\quad\text{and}\quad B''_{i,0}=\begin{cases}B_{i,0} & \gamma_{i,0}\in\mathbb{N}\\ B(0,0) & \gamma_{i,0}\notin\mathbb{N}\end{cases}$$

(note that $B(0,0)=M$). Let $F'_i=B'_{i,0}\setminus\bigcup_{j\in J_i}B_{i,j}$, and let $F''_i=B''_{i,0}\setminus\bigcup_{j\notin J_i}B_{i,j}$. Then $F_i=F'_i\cap F''_i$, and so $\phi(x,b)$ is equivalent to

$$\bigvee_i (D_{m'}(x-r'_i)\wedge x\in F''_i\wedge x\in F'_i).$$

Each hole of F'_i has nonstandard radius, and its outer ball either has an infinite radius or has radius 0. On the other hand, both the outer ball and all the holes of F''_i have finite radiuses. In general, if $B(a,\gamma)$ has finite radius, then the formula $x\in B(a,\gamma)$ is equivalent to $D_{p^\gamma}(x-a)$. So $x\in F''_i$ is equivalent to a boolean combination of such formulas, and therefore, by Lemma 8.1.3 (1) (choosing the same m'' for all the i's and rearranging the disjunction), $\phi(x,b)$ is equivalent to a formula of the form

$$\bigvee_i (D_{m''}(x-r'_i)\wedge x\in F'_i)$$

where each hole of F'_i has a nonstandard radius, and its outer ball either has an nonstandard radius or has radius 0. Note that now it may be that $p|m''$. By grouping together disjuncts with the same r'_i, we can rewrite this as

$$\bigvee_i (D_{m''}(x-r'_i)\wedge\bigvee_j x\in F'_{i,j})$$

where for $i_1\neq i_2$, $r'_{i_1}\not\equiv r'_{i_2}\bmod m''$. As this formula is equivalent to $\phi(x,b)$, which is not L^--definable with parameters in $\mathcal{M}$, there must be an i_0 such that $D_{m''}(x-r'_{i_0})\wedge\bigvee_j x\in F'_{i_0,j}$ is not L^--definable with parameters in $\mathcal{M}$. This latter formula, which we denote by $\phi_{i_0}(x,b)$, is equivalent to $\phi(x,b)\wedge D_{m''}(x-r'_{i_0})$, and so is L-definable. So we may replace $\phi(x,b)$ by $\phi_{i_0}(x,b)$. For simplicity of notation we rewrite this as

$$D_m(x-r)\wedge\bigvee_i x\in F_i.$$

By Lemma 9.2.4 we may assume that $\{F_i\}_i$ are pairwise disjoint, and still have that for each i, all the holes of F_i have infinite radiuses and its outer ball either has an infinite radius or has radius 0. By Remark 9.2.1 two proper cheeses having the same outer ball must intersect.

Applying this to all the F_i's having radius 0 (which are all proper, as all the holes are of infinite radius), we see that there can be at most one i such that F_i has radius 0.

We want all proper cheeses to have infinite radius. If there is i_0 such that the proper cheese F_{i_0} has radius 0, let $\phi'(x, b)$ be the formula $D_m(x - r) \wedge \neg\phi(x, b)$. Then $\phi'(x, b)$ is L-definable and, as $\phi(x, b)$ is equivalent to $D_m(x - r) \wedge \neg\phi'(x, b)$, it is also not L^--definable. The formula $\phi'(x, b)$ is equivalent to

$$D_m(x - r) \wedge \bigwedge_i x \in F_i^c.$$

We may write $F_{i_0} = B(0, 0) \setminus \bigcup_{j=1}^n B_j$, where for each j, $rad(B_j)$ is infinite. So $F_{i_0}^c = \bigcup_{j=1}^n B_j$, and $\phi'(x, b)$ is equivalent to

$$D_m(x - r) \wedge \bigvee_{j=1}^n (x \in B_j \wedge \bigwedge_{i \neq i_0} x \in F_i^c).$$

For each $i \neq i_0$, F_i^c is a finite union of swiss cheeses (specifically, a union of a single swiss cheese of radius 0 and a finite number of balls). Therefore, by Remark 9.2.3 *(4)*, for each j, $B_j \cap \bigcap_{i \neq i_0} F_i^c$ is a finite union of swiss cheeses, each of radius at least $rad(B_j)$, so infinite. So $\phi'(x, b)$ is equivalent to a formula of the form

$$D_m(x - r) \wedge \bigvee_i x \in F_i'$$

where each F_i' is a swiss cheese of infinite radius. Again by Lemma 9.2.4, we may assume in addition that $\{F_i'\}_i$ are pairwise disjoint. As $\phi'(x, b)$ is not L^--definable, the disjunction cannot be empty. So we may replace $\phi(x, b)$ by $\phi'(x, b)$ and rename F_i' as F_i.

We may assume that for each i, $D_m(x - r) \wedge x \in F_i$ defines a nonempty set, as otherwise we may just drop the i'th disjunct. Write $m = p^k m'$ with $\gcd(m', p) = 1$. Then $D_m(x - r)$ is equivalent to $D_{m'}(x - r_1) \wedge (v_p(x - r_2) \geq k)$, where $r_1 = r \bmod m'$ and $r_2 = r \bmod p^k$. So $\phi(x, b)$ is equivalent to

$$D_{m'}(x - r_1) \wedge \bigvee_i (v_p(x - r_2 \geq k) \wedge x \in F_i).$$

The formula $v_p(x - r_2) \geq k$ defines the ball $B(r_2, k)$, of finite radius k, and for each i, the outer ball of F_i has an infinite radius. As $D_m(x - r) \wedge x \in F_i$ defines a nonempty set, so too does $v_p(x - r_2) \geq k \wedge x \in F_i$, and hence the outer ball of F_i is contained in $B(r_2, k)$. Therefore $v_p(x - r_2) \geq k \wedge x \in F_i$ is equivalent to just $x \in F_i$, and so $\phi(x, b)$ is equivalent to

$$D_{m'}(x - r_1) \wedge \bigvee_i x \in F_i.$$

By Remark 9.2.3 (3) we may assume that each F_i is a proper cheese. We replace $\phi(x, b)$ by $\phi(x + r_1, b)$, and rename m' as m and each $F_i - r_1$ as F_i. Altogether, $\phi(x, b)$ is equivalent to a formula of the form

$$D_m(x) \wedge \bigvee_i x \in F_i$$

where $\gcd(m, p) = 1$, and $\{F_i\}_i$ are disjoint proper cheeses having infinite radiuses. As $\phi(x, b)$ is not L^--definable, the disjunction cannot be empty.

By Remark 9.2.8, $D_m(x)$ defines a dense subgroup of $\mathcal{M}$. By successively applying Lemmas 9.2.10, 9.2.11 and 9.2.9, we get an L-definable formula of the form

$$D_m(x) \wedge x \in B(0, \gamma) \tag{$\star$}$$

with γ nonstandard and $\gcd(m, p) = 1$. We will now assume that $\phi(x, b)$ is of this form.

To finish, we need the following:

Claim. Let $\psi(x,z)$ be any L_p-formula with $|x|=1$.

(1) Suppose there exists $a \in \mathcal{M}$ with $v(a)$ nonstandard, for which there exists b such that $\psi(x,b)$ is equivalent to $v(x) \geq v(a)$. Then for any c such that $v(c)$ is nonstandard there is $b' \in \mathcal{M}$ such that $tp(b'/\emptyset) = tp(b/\emptyset)$ (in L_p) and $\psi(x,b')$ is equivalent to $v(x) \geq v(c)$.

(2) Let $\theta(z)$ be an L_p-formula. Then there exists $K \in \mathbb{N}$ such that for any $a \in \mathcal{M}$ with $v(a) \geq K$, if there exists b such that $\theta(b)$ holds and $\psi(x,b)$ is equivalent to $v(x) \geq v(a)$, then for any c such that $v(c) \geq K$ there is $b' \in \mathcal{M}$ such that $\theta(b')$ and $\psi(x,b')$ is equivalent to $v(x) \geq v(c)$. That is, let $\alpha(w)$ be the formula defined by

$$\exists z(\theta(z) \wedge \forall x(\psi(x,z) \leftrightarrow v(x) \geq v(w)))$$

and let $\chi(w)$ be the formula defined by

$$\alpha(w) \rightarrow \forall w'(v(w') \geq K \rightarrow \alpha(w')).$$

Then $\chi(w)$ is satisfied by any a such that $v(a) \geq K$.

Proof of the claim. Proof of (1). We show that we can find $a' \in \mathcal{M}$ such that $tp(a'/\emptyset) = tp(a/\emptyset)$ and $v(a') = v(c)$. Indeed, let $\Sigma(x)$ be the partial type $tp(a/\emptyset) \cup \{v(x) = v(c)\}$. We show that it is consistent. Let $F \subseteq \Sigma(x)$ be a finite subset. As $v(a)$ is nonstandard, we may assume that F is of the form

$$\{x \neq j \, : \, -n \leq j \leq n\} \cup \{D_{m_k}(x - r_k) \, : \, 1 \leq k \leq s\} \cup \{v(x) = v(c)\}.$$

Let $m = \prod_k m_k$, and write $m = p^l m'$ with $\gcd(m',p) = 1$. By **??**, there exists $\tilde{a} \in \mathcal{M}$ satisfying the formula $D_{m'}(x-a) \wedge (v(x) = v(c))$. So $v(\tilde{a}) = v(c)$ is nonstandard. As $v(a)$ is also nonstandard, $\tilde{a}$ also satisfies $D_{p^l}(x-a)$, so it satisfies $D_m(x-a)$, and therefore it satisfies $\{D_{m_k}(x-r_k) \, : \, 1 \leq k \leq s\}$. Also, as $v(\tilde{a})$ is nonstandard, $\tilde{a} \notin \mathbb{Z}$. Together we have that $\tilde{a}$ satisfies F.
So $\Sigma(x)$ is consistent. Let $a' \in \mathcal{M}$ be a realization of $\Sigma(x)$. As $tp(a'/\emptyset) = tp(a/\emptyset)$, there is an automorphism of L_p-structures $\sigma \in Aut(\mathcal{M}/\emptyset)$ such that $\sigma(a) = a'$. Let $b' = \sigma(b)$. So $tp(b'/\emptyset) = tp(b/\emptyset)$ and $\psi(x,b')$ is equivalent to $v(x) \geq v(a')$. As $v(a') = v(c)$, we have what we want.
Proof of (2). Let $\xi(w,w')$ be the formula defined by $\alpha(w) \rightarrow \alpha(w')$. By (1), $\xi(a,c)$ holds for any a, c such that $v(a)$ and $v(c)$ are nonstandard, so the result follows by compactness. □

Now, let $\theta(z)$ be the formula expressing that $(\phi(x,z),+)$ is a subgroup. By Lemma 8.2.4 there are $n_1, \ldots, n_k$, having $\gcd(n_i, p) = 1$ for each i, such that for all $c \in \mathcal{M}$ for which $\theta(c)$ holds, $\phi(x,c)$ is equivalent to a formula of the form $D_{n_i}(x) \wedge v(x) \geq v(d)$ for some i and some $d \in \mathcal{M}$. As $(N,+,0,|_p)$ is an elementary substructure, if $c \in N$ then there exists such $d \in N$. Let $n = \prod_i n_i$, and let $\psi(x,z)$ be the formula $\phi(nx,z)$. Then for all $c \in \mathcal{M}$ for which $\theta(c)$ holds, $\psi(x,c)$ is equivalent to $v(x) \geq v(d)$, for the same d corresponding to $\phi(x,c)$ (as $v(n) = 0$).

Let $K \in \mathbb{N}$ be as given by the claim for $\psi(x,z)$ and $\theta(z)$, and let $\alpha(w)$ and $\chi(w)$ be as in the claim. We have that $\psi(x,b)$ is equivalent to $v(x) \geq \gamma$. In particular, the formula $\rho(z)$ defined by

$$\theta(z) \wedge \exists w(v(w) \geq K \wedge \forall x(\psi(x,z) \leftrightarrow v(x) \geq v(w)))$$

is satisfied by b. Since $\rho(z)$ contains no parameters, there exists $c \in N$ such that $(N,+,0,|_p) \vDash \rho(c)$. So $\theta(c)$ holds and there exists $d \in N$ such that $v(d) \geq K$ and $\psi(x,c)$ is equivalent to $v(x) \geq v(d)$. So $(N,+,0,|_p) \vDash \alpha(d)$. As $v(d) \geq K$, by the claim we have $\mathcal{M} \vDash \chi(d)$. Since $\chi(w)$ contains no parameters, also $(N,+,0,|_p) \vDash \chi(d)$. Hence, as v_p is surjective, for every $\gamma \in \Gamma(N)$ such that $\gamma \geq K$ there exists $c_\gamma \in N$ such that $\theta(c_\gamma)$ holds and $\psi(x,c_\gamma)$ is equivalent to $v(x) \geq \gamma$.

Let $\delta(x,y)$ be the formula

$$\bigwedge_{k=1}^{K-1}(D_{p^k}(x) \to D_{p^k}(y)) \wedge \forall z(\theta(z) \to (\psi(x,z) \to \psi(y,z))).$$

Then $\delta(x,y)$ is L-definable over $\emptyset$, and we claim that it defines $v(x) \leq v(y)$ in $\mathcal{N}$: Let $a_1, a_2 \in N$, and suppose $v(a_1) \leq v(a_2)$. Then of course $\bigwedge_{k=1}^{K-1}(D_{p^k}(a_1) \to D_{p^k}(a_2))$. Let $c \in N$ such that $\theta(c)$. Then there exists $d \in N$ such that $\psi(x,c)$ is equivalent to $v(x) \geq v(d)$, and therefore also $\psi(a_1,c) \to \psi(a_2,c)$. So we have $\delta(a_1,a_2)$. On the other hand, suppose $\delta(a_1,a_2)$. If $v(a_1) \leq K-1$, then by $\bigwedge_{k=1}^{K-1}(D_{p^k}(a_1) \to D_{p^k}(a_2))$ we get $v(a_1) \leq v(a_2)$. Otherwise, we have that $\gamma := v(a_1) \geq K$ and hence $\psi(a_1, c_\gamma)$. From $\forall z(\theta(z) \to (\psi(a_1,z) \to \psi(a_2,z)))$, as $\theta(c_\gamma)$ holds, we get in particular $\psi(a_1,c_\gamma) \to \psi(a_2,c_\gamma)$, and therefore we get $\psi(a_2,c_\gamma)$, which means $v(a_2) \geq \gamma = v(a_1)$. Therefore, $v(x) \leq v(y)$ is definable over $\emptyset$ in $\mathcal{N}$. ☐

Combined with Fact 9.1.5 and Theorem 8.3.2, we obtain:

Theorem 9.2.13. *Let $(N,+,0,1,|_p)$ be an elementary extension of $(\mathbb{Z},+,0,1,|_p)$. Then $(N,+,0,1,|_p)$ is $\emptyset$-minimal among the $\emptyset$-proper $\emptyset$-expansions of $(N,+,0,1)$.*

Proof. Identical to the proof of Corollary 9.1.9 from Theorem 9.1.10. ☐

In particular:

Corollary 9.2.14. *$(\mathbb{Z},+,0,1,|_p)$ is minimal among the proper expansions of $(\mathbb{Z},+,0,1)$.*

Proof. Identical to the proof of Fact 9.1.7 from Corollary 9.1.9. ☐

9.3 Intermediate structures in elementary extensions: some counterexamples

In this section, we show that Fact 9.1.5, Fact 9.1.7 and Corollary 9.2.14 are no longer true if we replace $\mathbb{Z}$ by an elementarily equivalent structure. In the case of Corollary 9.2.14, there are both stable and unstable counterexamples. For Fact 9.1.7 there are unstable counterexamples, but we do not know whether there are stable ones.

For each of the above, we give a family of counterexamples.

Remark 9.3.1. Let $L \subseteq L^+$ be two first-order languages, let $\phi(x,y)$ be an L^+-formula, and let P be a new relation symbol. Let $\mathcal{N}$ be an L^+-structure, let $a, b \in \mathcal{N}$ be such that $tp(a/\emptyset) = tp(b/\emptyset)$ (in L^+), and let $A = \phi(\mathcal{N}, a)$, $B = \phi(\mathcal{N}, b)$. Let $\mathcal{N}_1$,$\mathcal{N}_2$ be two reducts of $\mathcal{N}$, both in the language $L \cup \{P\}$, such that $\mathcal{N}_1|_L = \mathcal{N}_2|_L = \mathcal{N}|_L$, $P(\mathcal{N}_1) = A$, $P(\mathcal{N}_2) = B$. Then $\mathcal{N}_1 \equiv \mathcal{N}_2$.

Proposition 9.3.2. *Let $(N,+,0,1,|_p)$ be a nontrivial elementary extension of $(\mathbb{Z},+,0,1,|_p)$, let γ be a nonstandard element from Γ. Let $B = \{a \in N : v_p(a) \geq \gamma\}$. Then $(N,+,0,1,B)$ is a stable proper expansion of $(N,+,0,1)$ of dp-rank 1. In particular, it is a proper reduct of $(N,+,0,1,|_p)$.*

Proof. It is clear that $(N,+,0,1,B)$ is a proper expansion of $(N,+,0,1)$, and, as a reduct of $(N,+,0,1,|_p)$, by Theorem 8.3.2 it is of dp-rank 1. It remains to show stability. This follows from a theorem of Wagner, see Remark 9.3.3, but we also give a direct proof. First, we show that $Th(N,+,0,1,B)$ does not depend on N or b, as long as $v_p(b)$ is infinite, so it is enough to prove stability for just one particular choice of $(N,+,0,1,|_p)$ and b. Let $(N_2,+,0,1,|_p) \equiv (N,+,0,1,|_p)$, let $c \in N_2$ be such that $\delta := v_p(c)$ is nonstandard, and let $C = \{a \in N_2 : c|_p a\} =$

$\{a \in N_2 : v_p(a) \geq \delta\}$. Let $(M, +, 0, 1, |_p)$ be a monster model, and let $B' = \{a \in M : b|_p a\}$, $C' = \{a \in M : c|_p a\}$. So $B = B' \cap N$, $C = C' \cap N_2$, and $(N, +, 0, 1, B) \prec (M, +, 0, 1, B')$, $(N_2, +, 0, 1, C) \prec (M, +, 0, 1, C')$. By Claim 9.2.2 (1), there exists $d \in M$ such that $tp(d/\emptyset) = tp(b/\emptyset)$ (in $\{+, 0, 1, |_p\}$) and $v_p(d) = v_p(c)$. Let $D' = \{a \in M : d|_p a\}$. Then $D' = C'$, and by Remark 9.3.1, $(M, +, 0, 1, D') \equiv (M, +, 0, 1, B')$. So $(N_2, +, 0, 1, C) \equiv (M, +, 0, 1, C') = (M, +, 0, 1, D') \equiv (M, +, 0, 1, B') \equiv (N, +, 0, 1, B)$.

Now, consider the valued ring $(\mathbb{Z}, +, \cdot, 0, 1, |_p)$, and let $\mathcal{M}_1 = (M, +, \cdot, 0, 1, |_p)$ be a monster model for its theory. Consider the partial type $\Sigma(x) = \{p^n|_p x : n \in \mathbb{N}\} \cup \{\forall y(x|_p y \leftrightarrow \exists z(y = x \cdot z))\}$. Then for each $n_0 \in \mathbb{N}$, p^{n_0} satisfies $\{p^n|_p x : n \leq n_0\} \cup \{\forall y(x|_p y \leftrightarrow \exists z(y = x \cdot z))\}$, so Σ is consistent. Let $b \vDash \Sigma$. Let $\mathcal{M}_2 = (M, +, 0, 1, \{\tilde{r}\}_{r \in M})$, where for each $r \in M$, $\tilde{r} : M \to M$ is the function $\tilde{r}(a) := r \cdot a$. So $\mathcal{M}_2$ is an $\mathcal{M}_1$-module in the language of $\mathcal{M}_1$-modules (expanded by the constant 1), and therefore it is stable (see e.g. [Poi00, Theorem 13.14]). Let $B = \{a \in M : b|_p a\}$, and let $\mathcal{M}_3 = (M, +, 0, 1, B)$. As $b \vDash \Sigma$, $B = \{a \in M : \exists z(a = b \cdot z)\} = \{a \in M : \exists z(a = \tilde{b}(z))\}$, so B is definable in $\mathcal{M}_2$. Hence $\mathcal{M}_3$ is a reduct of $\mathcal{M}_2$, and therefore it is stable. □

Remark 9.3.3. In [Wag97, Example 0.3.1 and Theorem 4.2.8], Wagner defines an abelian structure to be an abelian group together with some predicates for subgroups of powers of this group. Every module is an abelian structure. Wagner proves that, as with modules, in an abelian structure every definable set is equal to a boolean combination of cosets of $\operatorname{acl}(\emptyset)$-definable subgroups. As a consequence, every abelian structure is stable. Under the assumptions of Proposition 9.3.2, B is a subgroup of N, so $(N, +, 0, 1, B)$ is an abelian structure. This immediately proves its stability.

Let $(N, +, 0, 1, |_p)$ be a nontrivial elementary extension of $(\mathbb{Z}, +, 0, 1, |_p)$. For $\gamma \in \Gamma$ we define

$$C_\gamma = \left\{(a, b) \in N^2 : v_p(a) \leq \gamma \wedge v_p(b) \leq \gamma \wedge v_p(a) \leq v_p(b)\right\}.$$

Proposition 9.3.4. *Let $(N, +, 0, 1, |_p) \succ (\mathbb{Z}, +, 0, 1, |_p)$ be a nontrivial elementary extension, let $\gamma \in \Gamma$ be nonstandard. Then $(N, +, 0, 1, C_\gamma)$ is an unstable proper expansion of $(N, +, 0, 1)$ and a proper reduct of $(N, +, 0, 1, |_p)$.*

Proof. Let R be the relation symbol corresponding to C. It is clear that $(N, +, 0, 1, C_\gamma)$ is an unstable proper expansion of $(N, +, 0, 1)$. We show that $|_p$ is not definable with parameters in $(N, +, 0, 1, C)$. First, exactly as in Proposition 9.3.2, $Th(N, +, 0, 1, |_p, C)$ does not depend on N or c, as long as γ is nonstandard. That is, if $(N_2, +, 0, 1, |_p) \equiv (N, +, 0, 1, |_p)$, $d \in N_2$ is such that $\delta := v_p(d)$ is nonstandard, then $(N, +, 0, 1, |_p, C_\delta) \equiv (N, +, 0, 1, |_p, C_\gamma)$. So it is enough to prove this for just one particular choice of $(N, +, 0, 1, |_p)$ and γ..

For each $m \in \mathbb{N}$, let

$$\begin{aligned} C_m &= \{(a, b) \in \mathbb{Z}^2 : \neg D_{p^{m+1}}(a) \wedge \neg D_{p^{m+1}}(b) \wedge \bigwedge_{i=1}^{m} (D_{p^i}(a) \to D_{p^i}(b))\} \\ &= \{(a, b) \in \mathbb{Z}^2 : a|_p p^m \wedge b|_p p^m \wedge a|_p b\} \end{aligned}$$

and let $\mathcal{Z}_m = (\mathbb{Z}, +, 0, 1, |_p, C_m)$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and let $\mathcal{N} = \prod_{\mathcal{U}} \mathcal{Z}_m = (N, +, 0, 1, |_p, C)$ be the ultraproduct of $\{\mathcal{Z}_m\}_m$ with respect to $\mathcal{U}$. Let $\psi(z)$ be the formula $\forall x, y(R(x, y) \leftrightarrow x|_p z \wedge y|_p z \wedge x|_p y)$. For each $k \in \mathbb{N}$, for every $m \geq k$, $\mathcal{Z}_m \vDash (\exists z \psi(z)) \wedge \forall z(\psi(z) \to p^k|_p z)$, and therefore also $\mathcal{N} \vDash (\exists z \psi(x)) \wedge \forall z(\psi(z) \to p^k|_p z)$. Hence there exists $c \in N$ such that $\gamma := v_p(c)$ is infinite and $C = C_\gamma$.

Suppose for a contradiction that $|_p$ is definable in $(N, +, 0, 1, C)$. Then there is a formula $\phi(x, y, z)$ in the language of $(N, +, 0, 1, C)$ with $|x| = |y| = 1$, and there is $d \in N$, such that $\mathcal{N} \models \forall x, y(x|_p y \leftrightarrow \phi(x, y, d))$. Let $(d_m)_{m\in\mathbb{N}}$ be a representative for d mod $\mathcal{U}$. Then $\{m \in \mathbb{N} : \mathcal{Z}_m \models \forall x, y(x|_p y \leftrightarrow \phi(x, y, d_m))\} \in \mathcal{U}$. In particular, this set is not empty, so there exists $m \in \mathbb{N}$ such that $\mathcal{Z}_m \models \forall x, y(x|_p y \leftrightarrow \phi(x, y, d_m))$. Hence $|_p$ is definable in $(\mathbb{Z}, +, 0, 1, C_m)$. But C_m is definable in $(\mathbb{Z}, +, 0, 1)$, a contradiction. □

Proposition 9.3.5. *Let $(N, +, 0, 1, <) \succ (\mathbb{Z}, +, 0, 1, <)$ be a non-trivial elementary extension, let $b \in N$ be a positive infinite element, and let $B = [0, b]$. Then $(N, +, 0, 1, B)$ is an unstable proper expansion of $(N, +, 0, 1)$ and a proper reduct of $(N, +, 0, 1, <)$.*

Proof. Let P be the relation symbol corresponding to B. It is clear that $(N, +, 0, 1, B)$ is a proper expansion of $(N, +, 0, 1)$. The formula $P(y-x)$ defines the ordering on B, so this structure is unstable. It remains to show that $<$ is not definable with parameters in $(N, +, 0, 1, B)$. First, we show that it is enough to prove this for a single choice of N and b (though in this case, the theory does depend on $tp(b/\emptyset)$). Let $(N_2, +, 0, 1, <) \equiv (N, +, 0, 1, <)$, let $c \in N_2$ be a positive infinite element, and let $C = \{a \in N : 0 \leq a \leq c\} = [0, c]$. Suppose that $<$ is not definable with parameters in $(N_2, +, 0, 1, C)$. Let $(M, +, 0, 1, <)$ be a monster model, and let $B' = \{a \in M : 0 \leq a \leq b\}$, $C' = \{a \in M : 0 \leq a \leq c\}$. So $B = B' \cap N$, $C = C' \cap N_2$. By Lemma 9.1.2 (with $A = \{c\}$), $(N_2, +, 0, 1, C) \prec (M, +, 0, 1, C')$ and $<$ is not definable with parameters in $(M, +, 0, 1, C')$. Similarly, $(N, +, 0, 1, B) \prec (M, +, 0, 1, B')$, and $<$ is definable with parameters in $(N, +, 0, 1, B)$ if and only if it is definable with parameters in $(M, +, 0, 1, B')$. As c is a positive infinite element, $tp(c/\emptyset)$ in $\{+, 0, 1, <\}$ is unbounded from above in $(M, +, 0, 1, <)$. Let $d \in M$ such that $d > b$ and $tp(d/\emptyset) = tp(c/\emptyset)$. Let $D' = \{a \in M : 0 \leq a \leq d\}$. By Remark 9.3.1 (with $L = L^+ = \{+, 0, 1, <\}$), $(M, +, 0, 1, <, C') \equiv (M, +, 0, 1, <, D')$, so in particular, $<$ is not definable in $(M, +, 0, 1, D')$. As $d > b$, $[0, b] = [0, d] \cap [-d + b, b]$, and so the formula $P(x) \wedge P(-x + b)$ defines B' in $(M, +, 0, 1, D')$. So $(M, +, 0, 1, B')$ is a reduct of $(M, +, 0, 1, D')$, and hence $<$ is not definable in $(M, +, 0, 1, B')$.

Now, for each $m \in \mathbb{N}$, let $B_m = [0, m]$, and let $\mathcal{Z}_m = (\mathbb{Z}, +, 0, 1, <, B_m)$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and let $\mathcal{N} = \prod_{\mathcal{U}} \mathcal{Z}_m = (N, +, 0, 1, <, B)$ be the ultraproduct of $\{\mathcal{Z}_m\}_m$ with respect to $\mathcal{U}$. For each $k \in \mathbb{N}$, for every $m \geq k$,

$$\mathcal{Z}_m \models \exists! x((\forall y(P(y) \leftrightarrow 0 \leq y \leq x)) \wedge x \geq \underline{k})$$

and therefore also $\mathcal{N} \models \exists! x((\forall y(P(y) \leftrightarrow 0 \leq y \leq x)) \wedge x \geq \underline{k})$. Hence there exists a positive infinite element $b \in N$ such that $B = [0, b]$.

Suppose for a contradiction that $<$ is definable in $(N, +, 0, 1, B)$. Then there is a formula $\phi(x, y, z)$ in the language of $(N, +, 0, 1, B)$ with $|x| = |y| = 1$, and there is $c \in N$, such that $\mathcal{N} \models \forall x, y(x < y \leftrightarrow \phi(x, y, c))$. Let $(c_m)_{m\in\mathbb{N}}$ be a representative for c mod $\mathcal{U}$. Then $\{m \in \mathbb{N} : \mathcal{Z}_m \models \forall x, y(x < y \leftrightarrow \phi(x, y, c_m))\} \in \mathcal{U}$. In particular, this set is not empty, so there exists $m \in \mathbb{N}$ such that $\mathcal{Z}_m \models \forall x, y(x < y \leftrightarrow \phi(x, y, c_m))$. Hence $<$ is definable in $(\mathbb{Z}, +, 0, 1, B_m)$, a contradiction. □

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Index des citations

Dans les remerciements, dans l'ordre de lecture. Tuna Altınel. Stephan Remmler, *Alles hat ein Ende, nur die Wurst hat zwei.* Anthony Kiedis *Can't stop.* Jean Giono *Un de Baumugnes.* Lucien Jacques *Lucien Jacques et la grande guerre.* Venus Khoury-Ghata *Qui parle au nom du jasmin ?* (2 fois). Sylvain Tesson *Sur les chemins noirs.* Sylvain Tesson, *Dans les forêts de Sibérie.* Sylvain Tesson *Sur les chemins noirs* (2 fois). Inconnu, *Ah, que nos pères étaient heureux.* Luis Mariano, *Airetun Txikitun.* Juan Carlos Perez, *Lau Teilatu.*

Aux pages XIII, 91, et 125, c'est Edmond Rostand, *Cyrano de Bergerac.* À la page 33, c'est Hermann Melville, *Moby Dick*, traduit de l'anglais par Lucien Jacques.

English Abstract

This thesis is concerned with the expansions of some algebraic structures and their fit in Shelah's classification landscape.

The first part deals with the expansion of a theory by a random –or *generic*– predicate for a substructure model of a reduct of the theory. We describe a setup allowing such an expansion to exist, which is suitable for several algebraic structures. In particular, we obtain the existence of additive generic subgroups of some theories of fields and multiplicative generic subgroups of algebraically closed fields in all characteristic. We also study the preservation of certain neostability notions, for instance, the NSOP_1 property is preserved but the simplicity is not in general. Thus, this construction produces new examples of NSOP_1 not simple theories, and we study in depth a particular example: the expansion of an algebraically closed field of positive characteristic by a generic additive subgroup.

The second part studies expansions of the groups of integers by p-adic valuations. We prove quantifier elimination in a natural language and compute the dp-rank of these expansions: it equals the number of distinct p-adic valuations considered. Thus, the expansion of the integers by one p-adic valuation is a new dp-minimal expansion of the group of integers. Finally, we prove that the latter expansion does not admit intermediate structures: any definable set in the expansion is either definable in the group structure or is able to "reconstruct" the valuation using only the group operation.

Keywords: Generic expansions; fields with generic subgroups; NSOP_1 theories; forking; Kim-forking; p-adic valuations on integers; finite dp-rank.

Expansions et néostabilité en théorie des modèles

Résumé. Cette thèse est consacrée à l'étude d'expansions de certaines structures algébriques et leur place dans la classification modèle-théorique des structures, initiée par Shelah.

La première partie aborde de manière abstraite l'expansion d'une théorie par un prédicat aléatoire –ou *générique*– pour une sous-structure modèle d'un réduit de la théorie. Nous élaborons un critère pour l'existence d'une telle expansion, qui est vérifié pour certaines théories de structures algébriques. En particulier, nous montrons l'existence de sous-groupes additifs génériques pour certaines théories de corps, ainsi que de sous-groupes multiplicatifs génériques pour les corps algébriquement clos en toute caractéristique. Nous étudions aussi la conservation de diverses notions de néostabilité, en particulier nous montrons que cette expansion préserve la propriété NSOP_1, mais en général ne préserve pas la simplicité. Nous produisons par cette construction de nouveaux exemples de structures NSOP_1 non simples, et faisons une étude toute particulière de l'une d'entre elles : l'expansion d'un corps algébriquement clos de caractéristique positive par un sous-groupe additif générique.

La deuxième partie étudie les expansions du groupe des entiers par des valuations p-adiques. Nous montrons l'élimination des quantificateurs dans un langage naturel et calculons le dp-rang d'une telle expansion : il est égal au nombre de valuations considérées. L'expansion du groupe des entiers par une seule valuation p-adique est donc une nouvelle expansion dp-minimale du groupe des entiers. Enfin, nous montrons que cette dernière n'admet pas de structures intermédiaires : tout ensemble définissable dans l'expansion est soit définissable dans le groupe des entiers, soit capable de “reconstruire” la valuation en utilisant seulement la structure additive.

Mots-clés : Sous-groupes génériques de corps ; théories NSOP_1 ; Kim-déviation ; valuations p-adiques sur les entiers ; dp-rang fini.

Image de couverture : H. Fantin-Latour, Ariane abandonnée, 1899 - Huile sur toile.

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