Introduction

Foreword

This thesis is organised as follows. First, the Preliminaries consist of basics, facts and preparatory results, to be used in the main text. Part A is devoted to the generic expansion of a structure by a predicate for a submodel of a reduct. The results in Part A come from my second, third and fourth years of Ph.D., under the supervision of Thomas Blossier and Zoé Chatzidakis, and two preprints are available online. In Part B, we study the expansions of the group of integers $(\mathbb{Z}, +, 0)$ by p-adic valuations. This work resulted in a paper [AE19], co-authored with Eran Alouf, published in the Journal of Symbolic Logic. Part B was done during the first year of my Ph.D., under the supervision of Pierre Simon. Part A and Part B can be read independently.

We start by introducing Part A. § 1 and § 2 focus on the notions of existentially closed structures and generic expansions. The reader familiar with these notions can skip those and jump directly to § 3 and § 4 where we present our first results. § 5 and § 6 are concerned with recent history and state of the art on classification and NSOP_1 theories, and § 7 links the construction of § 3 to this notion. § 8 presents our results on ACFG, then starts the introduction of Part B. § 9 introduces the current situation regarding expansions of the group of integers. § 10 and § 11 present our results on this subject.

§ 1 Existentially closed structures

Model theory is the study of mathematical structures through the prism of its algebra of definable sets. This algebra is in general hard to grasp, hence model theorists have always been in search of structures in which a reasonable description of definable sets is possible. Tarski [Tar51] shows in the 1930's that the theory of algebraically closed fields ACF in the language of fields and the theory RCF of real closed fields in the language of ordered fields have quantifier elimination: the study of the algebra of definable sets is reduced to the study of the boolean algebra spanned by basic sets. One easily deduce Hilbert's Nullstellensatz and Chevalley's theorem on constructible sets from the former. The latter allowed Robinson to give an elementary proof of Hilbert's 17th problem. This was the starting point of the development of methods for proving quantifier elimination, and the second half of the twenty-first century witnessed numerous other quantifier elimination results. The theories DCF_0 of differentially closed fields of characteristic 0 [Rob58] [Rob59a] and $\mathrm{SCF}_{p,e}$ of separably closed fields of characteristic p and imperfection degree e [Ers67] [Del88] have quantifier elimination in suitable languages, these theories provided the adequate ambient structures in Hrushovski's celebrated proof of the Mordell-Lang conjecture [Bou+98].

A full quantifier elimination in a natural language is not always possible, this led Robinson to introduce the notion of *model-complete theory*, a weaker form of quantifier elimination, the elimination down to *existential* formulae. Wilkie [Wil96] proved that the theory of the real field with the exponential function $\mathbb{R}_{\exp}$ is model-complete, yielding o-minimality for $\mathbb{R}_{\exp}$, and answering partially a question asked by Tarski [Tar51]: the theory of $\mathbb{R}_{\exp}$ is decidable provided Schanuel's conjecture holds. Note that there is a comprehensible language in which $\mathbb{R}_{\exp}$ has quantifier elimination [DMM94], but it is rather complicated, and indicates that getting from model-completeness to quantifier elimination might be a hard step.

Intuitively, any form of quantifier elimination for a theory T witnesses when the language imposes a transfer principle, a "Nullstellensatz" between the models and some extensions (any superstructure for quantifier elimination, supermodels for model-completeness). Hence, forcing a tranfer principle for a structure would result in a well-behaved algebra of definable sets. A structure $\mathscr{M}$ is *existentially closed* in another structure $\mathscr{N}$ of the same language if every existential formula with parameters in $\mathscr{M}$ true in $\mathscr{N}$ is also true in $\mathscr{M}$. An algebraically (resp. separably) closed field is existentially closed in every field (resp. separable field)extension. A model of a theory T is *existentially closed* if it is existentially closed in every model of T extending it. If existentially closed models of a theory T exist and form an elementary class, their theory — the *model-companion* of T — is model-complete.

Pseudo algebraically closed (PAC) fields are pure fields which are existentially closed in every regular extension and, in general, there is no natural expansion of the language in which this theory is model-complete[8]. However, PAC fields have elementary invariants [CDM81], [FJ05]

[8] The theory of pseudo-finite fields, the theory of ω-free PAC fields are theories of PAC fields in which a natural expansion of the language allows model-completeness, see Section 1.5.2.

and have been studied on several occasions [Hru02], [Cha99], [Cha02], [CH04]. PAC fields also provide examples of complex phenomena that play an important role in recent developments of model theory, see [CR16], [KR17], [Cha19], [Ram18], and also § 6.

Existentially closed models of a theory have in general some randomness — or *generic* — aspect, resulting from their definition. Informally, we will call *generic*[9] a theory (or a model of such theory) that axiomatises the structures that are existentially closed in a reasonable class of extension.

In many familiar theories, existentially closed models does not form an elementary class: the theories of groups [ES70], of nilpotent groups [Sar74], of solvable groups [Sar76], of commutative rings [Che73], of skew fields (Sabbagh, 1970, unpublished). Existentially closed models of these theories all interpret the structure $(\mathbb{Z}, +, \cdot)$. However, existentially closed groups and skew fields have been studied in the seventies, leading to striking connections between model theory, group theory and recursion theory, see [Zie76], [SZ79], [Zie80], [HW75], [Bel74], [Bel74], [Bel78a], [Bel78b] and [Mac77, p. 5.2] for a survey. More recently, Haykazyan and Kirby [HK18] have studied another class of existentially closed structures which admits no model-companion, we discuss it in § 6.

§ 2 Expansions and genericity

The first study of an unfamiliar expansion of a familiar structure was initiated by Tarski, when he asked in [Tar51] whether the theory of $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$, the real field structure with a unary predicate for the real algebraic numbers is decidable. Later, Robinson answered positively the question by proving the model-completeness of this theory in a natural expansion of the language, and the same result for the pair $(\mathbb{C}, \overline{\mathbb{Q}})$ [Rob59b].

The structure $(\mathbb{C}, \overline{\mathbb{Q}})$ or more generally any proper pair (K, k) of algebraically closed fields is existentially closed in any extension (L, l) such that l and K are linearly disjoint over k, hence the expansion of ACF by a predicate for a proper algebraically closed subfield enjoys some genericity property, which is rather exceptionnal. In general, one studies the existentially closed models of an expansion, hence the terminology "generic expansions".

An important example of generic expansion is Winkler's construction [Win75]. Consider an $\mathscr{L}$-theory T and a language $\mathscr{L}' \supset \mathscr{L}$. One can see T as an $\mathscr{L}'$-theory which does not impose any structure on elements of $\mathscr{L}' \setminus \mathscr{L}$. Winkler proves that as an $\mathscr{L}'$-theory, T has a model-companion, provided T is model-complete and eliminates $\exists^\infty$. The particular case in which the expansion is by a unary predicate –the *generic predicate*– has been studied by Chatzidakis and Pillay [CP98]. These generic expansions have connections with neostability theory, we investigate this direction in § 6.

A recent breakthrough in the area of generic expansions is the *interpolative fusion* construction [KTW18] by Kruckman, Tran and Walsberg. Given arbitrary many model-complete theories, they describe a general setting for the model-companion of the union of these theories to exists. It appears that many generic structures are bi-interpretable with an interpolative fusion of simpler structures.

[9] The term generic has classically been used for models of a theory in which infinite forcing and model-theoretic satisfaction coincide (see for instance [Mac77] or [Che76]). If the theory is model-complete, then the notion of existentially closed model and generic model coincide. The term generic for a structure or a theory has nowadays an unclear meaning closer to ours.

Consider the following two expansions of ACF: T_1 is the expansion by a generic predicate, T_2 the expansion by a predicate for a proper algebraically closed subfield. These two theories are generic expansions of ACF by a predicate for a reduct (trivial in both cases), the first one is the theory of and infinite set in the trivial language, the second one is the full theory in the full language. The first result of this thesis, Theorem A, present a general setting for expanding a theory by a predicate for a reduct.

§ 3 Generic expansion by a pregeometric reduct

Let T be a theory in a language $\mathscr{L}$. Let $\mathscr{L}_0 \subseteq \mathscr{L}$ and T_0 a reduct of T to the language $\mathscr{L}_0$. Let acl_0 be the algebraic closure in the sense of $\mathscr{L}_0$. Let $\mathscr{L}_S = \mathscr{L} \cup \{S\}$, for S a new unary predicate symbol, and T_S be the $\mathscr{L}_S$-theory whose models $(\mathscr{M}, \mathscr{M}_0)$ consist in a model $\mathscr{M}$ of T in which S is a predicate for a model $\mathscr{M}_0$ of T_0 which is a substructure of $\mathscr{M}$. We present a setting in which we get partial results toward an axiomatisation of generic models of T_S. Assume the following.

(H_1) T is model complete;

(H_2) T_0 is model complete and $\mathrm{acl}_0(A) \models T_0$, for all infinite set A;

(H_3) T_0 is pregeometric (i.e. acl_0 satisfies exchange);

(H_4) for all $\mathscr{L}$-formula $\phi(x, y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for all $\mathscr{M} \models T$ and tuple b from $\mathscr{M}$,

$$\mathscr{M} \models \theta_\phi(b) \iff \text{there exists } \mathscr{N} \succ \mathscr{M} \text{ and } a \in \mathscr{N} \text{ such that } \phi(a, b) \text{ and } a \text{ is an independent tuple over } \mathscr{M}, \text{ in the sense of the pregeometry } \mathrm{acl}_0.$$

We denote by $\mathop{\smile\!\!\!\!\!\mid}^0$ the independence relation in the sense of the pregeometry acl_0. We call an extension $(\mathscr{N}, \mathscr{N}_0)$ of $(\mathscr{M}, \mathscr{M}_0)$ *strong* if $\mathscr{N}_0 \mathop{\smile\!\!\!\!\!\mid}^0_{\mathscr{M}_0} \mathscr{M}$.

Theorem A. *There exists a unique theory TS containing T_S such that*

- *every model of T_S has a strong extension which is a model of TS;*
- *if $(\mathscr{M}, \mathscr{M}_0) \models TS$ and $(\mathscr{N}, \mathscr{N}_0) \models T_S$ is a strong extension of $(\mathscr{M}, \mathscr{M}_0)$ then $(\mathscr{M}, \mathscr{M}_0)$ is existentially closed in $(\mathscr{N}, \mathscr{N}_0)$.*

If acl_0 *defines a* modular *pregeometry, TS is the model-companion of T_S and in TS the algebraic closure is given by the algebraic closure in T.*

As usual in the proof of this kind of result, the axiomatisation gives an outline of the proof, it is given in Theorem 2.1.5. For a given tuple b in a model $\mathscr{M}$ of T, the formula $\theta_\phi(b)$ witnesses whenever the formula $\phi(x, b)$ have a realisation a in an elementary extension $\mathscr{N}$ of $\mathscr{M}$ such that for any partition $A_1 \cup A_2$ of the coordinates of a, there is an $\mathscr{L}_0$-substructure $\mathscr{N}_0$ of $\mathscr{N}$ model of T_0 which separates A_1 from A_2, by which we mean $A_1 \subseteq \mathscr{N}_0$ and $A_2 \cap \mathscr{N}_0 = \emptyset$. An existentially closed model of T_S should be able to realise all these possible attributions of coordinate for any such $\mathscr{L}$-formula. The main point here is that this is expressible in a first-order way, provided that formulae $\theta_\phi(y)$ exist.

Hypothesis (H_1) is obviously necessary, and (H_3) provides a general setup to express basic notions in a first-order way, and allows us to give a geometric treatment of the proof, by which we mean using $\mathop{\smile\!\!\!\!\!|}^0$ as in forking calculus, in order to prepare an adaptation of the proof to wider contexts. In (H_2), the fact that acl_0-closed infinite sets are models of T_0 can certainly be weakened to a condition close to "every acl_0-closed set embeds in a model of T_0", although this would increase the technicalities of the proof; and provide no more applications of the theorem than the ones we give in this thesis.

Hypothesis (H_4) is, in practice, a difficult condition to obtain. It can be thought of as a generalisation of elimination of $\exists^\infty$. If $T_0 = T$ and T is pregeometric, it is actually equivalent to elimination of $\exists^\infty$ (see Fact 1.3.10), hence for a geometric theory with $T = T_0$, one only need to check (H_2). The resulting theory is a generic pair of models of T. If T_0 is the theory of an infinite set in the emty language, condition (H_4) is again equivalent to elimination of $\exists^\infty$, (H_2) and (H_3) are trivial and Theorem A gives nothing more than the generic predicate construction of [CP98]. Condition (H_4) can also be seen as a "definability of the dimension" condition, although in a strong sense, as the condition involves independence in the sense of the reduct T_0 and not in the sense of T. An equivalent statement of (H_4) in terms of existence of bounds is given in Section 2.2, as well as a weak converse: assuming (H_1, H_2, H_3), if TS exists, then for each $\mathscr{L}$-formula $\phi(x, y)$ and $k \leq |x|$, there exists $\theta^k_\phi(y)$ such that for all $b \in \mathscr{M} \models T$, $\mathscr{M} \models \theta^k_\phi(b)$ if and only if there exists $\mathscr{N} \succ \mathscr{M}$ and $a \in \mathscr{N}$ such that $\phi(a, b)$ and $a_k \mathop{\smile\!\!\!\!\!|}^0 \mathscr{M}(a_i)_{i\neq k}$. In particular T eliminates $\exists^\infty$. In general, (H_4) is not equivalent to elimination of $\exists^\infty$, this is discussed in § 4.

In the sense of [KTW18], the theory TS of Theorem A is the interpolative fusion of T with the theory of generic pairs of models of T_0.

We turn now to applications of Theorem A. Let $\mathbb{F}_{q_1}, \cdots, \mathbb{F}_{q_n}$ be finite fields, and T a theory in a language

$$\mathscr{L} = \left\{+, 0, (\lambda_\alpha)_{\alpha\in\mathbb{F}_{q_1}}, \dots, (\lambda_\alpha)_{\alpha\in\mathbb{F}_{q_n}}, \cdots\right\},$$

such that every model of T carries, for all $1 \leq i \leq n$, a structure of infinite $\mathbb{F}_{q_i}$-vector space in the language $\{+, 0, (\lambda_\alpha)_{\alpha\in\mathbb{F}_{q_i}}\}$, where λ_α is the function interpreted as the multiplication by α.

Theorem B. *Let $V_1, \cdots, V_n$ be unary predicates and $T_{V_1\dots V_n}$ the theory of models of T where V_i is a predicate for a vector subspace over $\mathbb{F}_{q_i}$. If T is model-complete and eliminates $\exists^\infty$, then $T_{V_1\dots V_n}$ admits a model-companion.*

The framework described above encompasses the hypotheses (H_1, H_2, H_3), and elimination of $\exists^\infty$ gives, in that particular case, the condition (H_4), because of the uniform finiteness of the pregeometry and a classical lemma from [CP98]. Furthermore, applying once Theorem A results in a theory that also eliminates $\exists^\infty$, hence we may iterate and add as many generic vector subspaces as we want. The pregeometry associated to an $\mathbb{F}_{q_i}$-vector space is modular, hence the resulting theory is the model-companion.

§ 4 Generic expansions of fields

Generic additive subgroups in positive characteristic. Let p be a prime number. The additive group of a field of characteristic p is an $\mathbb{F}_p$-vector space, hence Theorem B applies. Let T be one of the following theory

- ACF_p the theory of algebraically closed fields of characteristic p in the language of fields;
- $\mathrm{SCF}_{p,e}$ the theory of separably closed fields of characteristic p and imperfection degree $e \leq \infty$, in the language of fields with predicates for p-independence;

- Psf_c the theory of pseudo-finite fields of characteristic p in the language of fields expanded by constants for coefficients of irreducible polynomials (Section 1.5.2);
- ACFA_p the theory of algebraically closed fields of characteristic p with a generic automorphism in the language of difference fields.

Then the expansion of T by finitely many generic additive subgroups exists. The expansion of ACF_p by a generic additive subgroup will be denoted by ACFG, and it only exists in positive characteristic, as we will see later. Chapters 5, 6 and 7 of this thesis are devoted to a study of ACFG, those results are described in § 8. We also get the following generic expansion of perfect PAC fields in positive characteristic.

Theorem C. *Let* PAC_G *be the theory whose models are perfect* PAC *fields of characteristic* p *with a predicate* G *for an additive subgroup. Then there exists a theory* PACG *such that*

(1) every model (F, G) *of* PAC_G *extends to a model* (K, G) *of* PACG *such that* K *is a regular extension of* F*;*

(2) every model (K, G) *of* PACG *is existentially closed in every extension* (F, G) *such that* F *is a regular extension of* K*.*

It is possible to iterate this construction.

All the previous results concerning generic expansions of fields of positive characteristic are also true when replacing G by an $\mathbb{F}_q$-vector space V, for some finite subfield $\mathbb{F}_q$ of the ambiant field.

Generic additive subgroups in characteristic zero. The previous results have no analogue in characteristic 0. Let T be *any* inductive theory of a field of characteristic 0 in a language $\mathscr{L}$ containing the language of rings. Let G be a new predicate and T_G be the $\mathscr{L} \cup \{G\}$-theory of models of T in which G is a predicate for an additive subgroup of the field, this is an inductive theory. An simple argument (Proposition 3.2.7) shows that if (K, G) is an existentially closed model of T_G, then $\{a \in K \mid aG \subseteq G\} = \mathbb{Z}$. In particular, the theory T_G does not admit a model-companion. Similarly, if one imposes G to be divisible, the stabiliser of the group is $\mathbb{Q}$. Furthermore, consider the case in which T is the theory of $\mathbb{R}$ or of $\mathbb{C}$, then hypotheses (H_1, H_2, H_3) hold, so by the contrapositive of Theorem A, condition (H_4) does not hold, even though T eliminate $\exists^\infty$.

Generic multiplicative subgroups in all characteristic. The generic expansion by an additive subgroup fails in characteristic zero, however, we have the following.

Theorem D. *Let* p *be a prime number or zero. The expansion of* ACF_p *by a generic multiplicative subgroup exists.*

Hypotheses (H_1, H_2, H_3) are easy to check. We prove hypothesis (H_4) only for formulae that define quasi-affine varieties, which is sufficient for proving that the model-companion exists. Hypothesis (H_4) follows from a definability result in abstract Kummer theory. Let $W \subset K^n \setminus \{(0, \cdots, 0)\}$ be an affine irreducible algebraic variety in an algebraically closed field K of characteristic $p \geq 0$. We say that W is *free* if it is not contained in any translate of a proper algebraic subgroup of the torus $\mathbb{G}_m^n(K)$. Bays, Gavrilovitch and Hils show in [BGH13] that W is free if and only if every element in $\mathbb{G}_m^n(K)$ is the product of $2n$ elements from W, which is a definable condition[10].

[10]Note that Minh Chieu Tran [Tra17] also obtained the definability of the freeness of an affine irreducible variety (which he calls *multiplicatively largeness*), using Zilber's indecomposability theorem. As a matter of fact, the result of Bays, Gavrilovitch and Hils [BGH13] is more general, they prove it when replacing $\mathbb{G}_m^n(K)$ by any semiabelian variety.

§ 5 Shelah's Classification

An important part of model theory consists in defining various notions of "tameness" in order to classify and understand mathematical structures. A leading idea, initiated by Shelah, is that the "wildness" of a structure can be detected in the combinatorial complexity of bipartite graphs associated with definable sets. Thus, tameness is associated with the absence of some combinatorial configuration in the bipartite graph of any formula. For instance, the so-called stable structures are those that avoid defining an infinite half graph. One of the most striking facts in stability theory is that its combinatorial definition is equivalent to the existence of a well-behaved notion of independence in every model, based on Shelah's forking. During the last two decades, model theorists have tried to apply stability theoretic methods to unstable theories, this is called neostability theory. For instance, simplicity is a generalisation of stability. Simple theories are also defined by a combinatorial condition and are also characterized by the good behaviour of forking independence, this is known as Kim-Pillay theorem [KP97]. The theories of infinite random graphs, of bounded PAC fields, ACFA, are examples of simple theories. However, the theories of ω-free PAC fields [Cha99], of generic $K_{m,n}$-free bipartite graph [CK17] (for $n, m \geq 2$) are not simple, they are NSOP_1 theories, a generalisation of simplicity.

§ 6 A recent history of NSOP_1 theories

NSOP_1 theories, for "not strong order property 1", were defined by Džamonja and Shelah in [DS04] (together with NSOP_2) as an extension of the $(\mathrm{NSOP}_n)_{n\geq 3}$ hierarchy. In [SU08], Shelah and Usvyatsov proved that T^*_{feq} (the model completion of the theory of infinitely many independent parametrized equivalence relations) is NSOP_1 and not simple. For the past three years, NSOP_1 theories have been intensively studied through two different approaches (not mutually exclusive): the abstract one, in which combinatorics and pure model theory are involved; and the applied one, which consists in the study of particular examples.

The first breakthrough concerning the abstract study of NSOP_1 theories was made by Chernikov and Ramsey in [CR16]. They proved a Kim-Pillay style result [CR16] which states that a theory is NSOP_1 provided there exists an independence relation satisfying some specific properties. This result turned out to be a very useful tool to prove that a theory is NSOP_1. The ω-free PAC fields case is a good example. A PAC field is simple if [CP98] and only if [Cha99] it is bounded. Nonetheless, in her work [Cha02] on ω-free PAC fields (which are unbounded), Chatzidakis defined a weak notion of independence and showed that it satisfied some nice properties, in particular, the so-called *independence theorem*. It turned out that almost all the properties of the criterion [CR16] were proved at that time. Chernikov and Ramsey used this weak independence to deduce that the theory of ω-free PAC fields is NSOP_1. They also showed that Granger's example of generic bilinear form over an infinite dimensional vector space over an algebraically closed field is NSOP_1 (see [Gra99] or [CR16, Example 6.1]), as well as the combinatorial example of a generalised parametrized structure (see [CR16, Example 6.3]). The second breakthrough in the abstract study of NSOP_1 theories was the development of Kim-independence by Kaplan and Ramsey in [KR17]. They introduced analogues of forking and dividing –Kim-forking and Kim-dividing– which behave nicely in NSOP_1 theories. Kim-dividing is defined as dividing with respect to some particular indiscernible sequences, namely sequences in a global invariant type. Numerous properties of forking in simple theories appears for Kim-forking in NSOP_1 theories. For instance, a theory is NSOP_1 if and only if Kim-independence is symmetric. Kaplan and Ramsey also completed the Kim-Pillay style criterion in [CR16] to get a characterisation of Kim-independence[11] in terms of properties of a ternary relation, similarly to the Kim-Pillay

[11] Actually they proved that if $\mathop{\smile\!\!\!\!\!|}\,$ satisfies the conditions of [CR16, Proposition 5.3] then $\mathop{\smile\!\!\!\!\!|}\,$ strengthens Kim-independence. The *Witnessing* condition ensures that Kim-dividing independence strengthens $\mathop{\smile\!\!\!\!\!|}\,$.

classical result. Using this tool, they identified Kim-independence in various NSOP_1 theories. Chatzidakis' weak independence in ω-free PAC fields turned out to be Kim-independence. In Granger's example, the independence relation that he studied which satisfied the Chernikov and Ramsey's criterion for NSOP_1 is strictly stronger than Kim-independence.

Concerning the applied approach, Conant and Kruckman's generic incidence structures [CK17], Barbina and Casanovas' Steiner triple system [BC18] are new examples of NSOP_1 theories. As we will see in § 7, ACFG and almost all examples in § 4 are new NSOP_1 theories. Most of these examples are generic constructions, and they share many common features. Simple theories have commonly been considered as stable ones with some "random noise". A strongly supporting fact for this is the construction of the generic predicate [CP98]. Adding a generic predicate preserves simplicity, by which we mean that if T is simple, then the theory of the generic predicate on T is also simple. However, if more complex genericity is involved, simplicity may not be preserved, even starting with a very tame theory. We will see in § 7 that ACFG is not simple, even though it is the generic expansion of a strongly minimal theory. Kruckman and Ramsey [KR18] prove that Winkler's generic expansion by an arbitrary language [Win75], discussed in § 2, preserves NSOP_1. They also show that another construction from Winkler [Win75], the expansion of a theory by generic Skolem functions, preserves NSOP_1, and deduce the following interesting fact: any NSOP_1 theory that eliminates $\exists^\infty$ can be expanded to an NSOP_1 theory that has built-in Skolem functions. Concerning PAC fields, the general intuition is that their model-theoretic features can be deduced from the model-theoretic features of their absolute Galois group. Recent results from Chatzidakis and Ramsey strongly support this idea: a PAC field is NSOP_n if and only if its absolute Galois group is NSOP_n ([Cha19] for $n \geq 3$, [Ram18] for $n = 1, 2$).

Recent work from Haykazyan and Kirby [HK18], highlights a new source of NSOP_1 theories, in the sense of positive logic. They study the class of existentially closed exponential fields (an exponential field is a field with a group homomorphism from the additive group to the multiplicative group of the field). As for the class of existentially closed skew fields or fields with an additive subgroup in characteristic 0, this class is not elementary. However, an idea which goes back to Shelah [She75], consists in dealing with those non-elementary classes by considering only *existential* formulae, this is called positive logic, and was developed in different ways by Ben-Yaacov [Ben03a] and Pillay [Pil00]. Suitable notions of stability [Pil00] [Ben03b] and simplicity [Ben03b] were further developed. Haykazyan and Kirby [HK18] adapted the result of Chernikov and Ramsey [CR16] to prove that the class of existentially closed exponential fields is NSOP_1 in the sense of positive logic, using the existence of a well-behaved independence relation. It is likely that the theory developed by Haykazyan and Kirby can be used to show that the class of algebraically closed fields of characteristic 0 with a generic additive subgroup is NSOP_1 in the sense of positive logic.

§ 7 Preservation of NSOP_1

Our next result gives a condition which makes the expansion by a generic reduct § 3 an NSOP_1-preserving construction. As in § 3, let T be an $\mathscr{L}$-theory, $\mathscr{L}_0 \subseteq \mathscr{L}$ and $T_0 \restriction \mathscr{L}_0$. Assume that hypotheses (H_1) to (H_4) are satisfied. We denote by acl_T the algebraic closure in the sense of $\mathscr{L}$, acl_0 the algebraic closure in the sense of $\mathscr{L}_0$ and $\mathop{\smile\!\!\!\!\!\!|}^0$ the independence relation associated with the pregeometry acl_0.

Theorem E. *Assume that* acl_0 *defines a modular pregeometry,* T *is* NSOP_1 *and* $\mathop{\smile\!\!\!\!\!\!|}^T$ *is the Kim-independence relation in* T*. Let* $\mathscr{M} \models T$ *and* A, B, C *algebraically closed containing* $\mathscr{M}$*, in a monster model of* T*. Assume the following.*

$$(A) \qquad \textit{For all such } A,B,C,\mathscr{M}, \textit{ if } C \mathop{\smile\!\!\!\!\!|}^{T}_{\mathscr{M}} A,B \textit{ and } A \mathop{\smile\!\!\!\!\!|}^{T}_{\mathscr{M}} B \textit{ then}$$

$$(\mathrm{acl}_T(AC), \mathrm{acl}_T(BC)) \mathop{\smile\!\!\!\!\!|}^{0}_{A,B} \mathrm{acl}_T(AB).$$

Then TS is NSOP_1 *and the Kim-independence relation in TS is given by the relation $\mathop{\smile\!\!\!\!\!|}^{w}$, defined by*

$$A \mathop{\smile\!\!\!\!\!|}^{T}_{\mathscr{M}} B \textit{ and } S(\mathrm{acl}_0(\mathrm{acl}_T(A\mathscr{M}), \mathrm{acl}_T(B\mathscr{M}))) = \mathrm{acl}_0(S(\mathrm{acl}_T(A\mathscr{M})), S(\mathrm{acl}_T(B\mathscr{M}))).$$

Theorem E is a consequence of more general considerations. Starting from an independence relation $\mathop{\smile\!\!\!\!\!|}^{T}$ in models of T, we define two independence relations for models of TS, a *strong* independence $\mathop{\smile\!\!\!\!\!|}^{st}$ and a *weak* independence $\mathop{\smile\!\!\!\!\!|}^{w}$. Both relations $\mathop{\smile\!\!\!\!\!|}^{w}$ and $\mathop{\smile\!\!\!\!\!|}^{st}$ extend the relation $\mathop{\smile\!\!\!\!\!|}^{T}$. In Chapter 4, we analyse properties of $\mathop{\smile\!\!\!\!\!|}^{T}$ that are transferred to $\mathop{\smile\!\!\!\!\!|}^{w}$ and $\mathop{\smile\!\!\!\!\!|}^{st}$. If T is NSOP_1 and $\mathop{\smile\!\!\!\!\!|}^{T}$ is Kim-independence, then all properties satisfied by $\mathop{\smile\!\!\!\!\!|}^{T}$ that characterize Kim-independence and NSOP_1 theories are transferred to $\mathop{\smile\!\!\!\!\!|}^{w}$, apart from the independence theorem, which requires hypothesis *(A)*. This gives Theorem E. We give a fine analysis of the conservation of properties from $\mathop{\smile\!\!\!\!\!|}^{T}$ to $\mathop{\smile\!\!\!\!\!|}^{w}$ and $\mathop{\smile\!\!\!\!\!|}^{st}$. For instance, if $\mathop{\smile\!\!\!\!\!|}^{T}$ is stationnary over algebraically closed sets, so is $\mathop{\smile\!\!\!\!\!|}^{st}$. If $\mathop{\smile\!\!\!\!\!|}^{T}$ satisfies $\mathop{\smile\!\!\!\!\!|}'$-*amalgamation* (a version of the independence theorem in which the parameters can be choosen independent in the sense of another independence relation $\mathop{\smile\!\!\!\!\!|}'$, see Section 1.2) then $\mathop{\smile\!\!\!\!\!|}^{w}$ also satisfies $\mathop{\smile\!\!\!\!\!|}'$-amalgamation (Theorem 4.1.5). This is used to show that when it is possible, iterating the expansion by a generic reduct also preserves NSOP_1 (Corollary 4.2.4).

Kaplan and Ramsey [KR17] also give a necessary and sufficient geometric condition for an NSOP_1 theory to be simple: Kim-independence has to satisfy the so-called *Base monotonicity* property (if $a \mathop{\smile\!\!\!\!\!|}_C bd$ then $a \mathop{\smile\!\!\!\!\!|}_{Cb} d$, see Section 1.2). This translates in our context as a useful criterion to show when TS is not simple, this depends on the entanglement of acl_0 and acl_T (Corollary 4.2.3). Also, condition *(A)* expresses how $\mathop{\smile\!\!\!\!\!|}^{T}$ controls $\mathop{\smile\!\!\!\!\!|}^{0}$, the independence in the sense of the pregeometric reduct T_0. The "proximity" between T and T_0 plays an interesting role concerning the preservation of neostability notions for the expansion TS, as we can summarize in the following table.

Configuration $T_0 \subseteq T$	Generic expansion TS
$T_0 = T$	Preserves stability
$T_0 \subseteq T$	Preserves NSOP_1
$T_0 = \emptyset$	Preserves simplicity

If T is a theory of fields of positive characteristic and T_0 is the additive reduct of T, then condition *(A)* follows from a simpler assumption, *(B)* below. For a field A we denote by $\overline{A}$ the field theoretic algebraic closure of A.

Theorem F. *Let T be a model-complete theory of an* NSOP_1 *field of positive characteristic that eliminates $\exists^\infty$. Let A, B be acl_T-closed and $E \models T$ contained in A and B. Let $\mathop{\smile\!\!\!\!\!|}^{T}$ be Kim-independence in T and assume that :*

$$(B) \qquad \textit{for all such } A, B \textit{ and } E, \textit{ if } A \mathop{\smile\!\!\!\!\!|}^{T}_{E} B \textit{ then } \mathrm{acl}_T(AB) \subseteq \overline{AB}.$$

Then the expansion of T by generic additive subgroups $TG_1 \ldots G_n$ is NSOP_1*. Kim-independence in $TG_1 \ldots G_n$ is given by*

$$A \mathop{\smile\!\!\!\!\!|}^{T}_{E} B \textit{ and for all } i \leq n \ G_i(A+B) = G_i(A) + G_i(B),$$

for A, B and E as above. Furthermore, $TG_1 \cdots G_n$ is not simple.

In particular, all examples in positive characteristic given in § 4 (except the one in Theorem C) are new NSOP_1 and not simple theories. Concerning the theory in Theorem C, not all perfect PAC fields are NSOP_1, however they always satisfy hypothesis *(B)* (because they are *algebraically bounded* [CH04]). It follows that the expansion in a suitable language of the theory of perfect Frobenius fields, or perfect ω-free PAC fields by generic additive subgroups, is NSOP_1. Note that in Theorem F, each G_i can be replaced by an $\mathbb{F}_{q_i}$-vector space, for a subfield $\mathbb{F}_{q_i}$ of any model of T.

The proof of Theorem F consists in deducing *(A)* from *(B)*. It involves a description of Kim-independence in any theory of fields by Kaplan and Ramsey [KR17], following the work of Chatzidakis [Cha99]. The theorem follows from stability flavoured arguments in the separable closure of the field and some Galois theory.

Finally, for any p prime or zero, the expansion of ACF_p by a generic multiplicative subgroup is also NSOP_1 and not simple. We use Theorem E, condition *(A)* follows from a coheir argument in the stable theory ACF_p.

§ 8 The theory ACFG

Chapters 5, 6 and 7 are dedicated to the study of the theory ACFG, the expansion of ACF_p by a generic additive subgroup, for a fixed prime p.

Almost simple. In ACFG, there is no independence relation satisfying the Kim-Pillay criterion for simplicity. However, we have the following.

Theorem G.
- *In* ACFG, *Kim-independence satisfies all the properties of the Kim-Pillay characterisation of simple theories except* Base Monotonicity.
- *In* ACFG, *there is an independence relation which satisfies all the properties of the Kim-Pillay characterisation of simple theories except* Local Character.

The independence in the second item is the strong independence, mentioned in § 7.

Models of ACFG. Let (K, G) be a model of ACFG, G is the generic subgroup of the field K. The group G enjoys some interesting algebraic properties, for instance, it is dense and codense for the Zariski topology on K. Also, every element in K is the product of two elements of the group, which implies that G is stably embedded in K.

Let $\overline{\mathbb{F}_p}$ be the field theoretic algebraic closure of the prime field $\mathbb{F}_p$. Using that $\overline{\mathbb{F}_p}$ is locally finite and quantifier elimination in ACF_p, we construct by union of chain a subgroup G of $\overline{\mathbb{F}_p}$ such that $(\overline{\mathbb{F}_p}, G)$ is a model of ACFG. The space $\mathrm{Sg}(\overline{\mathbb{F}_p})$ of additive subgroups of $\overline{\mathbb{F}_p}$ is endowed with the Chabauty topology (Section 1.6). We show that almost all (in the sense of Baire) additive subgroups G of $\overline{\mathbb{F}_p}$ are generic.

Theorem H. *The set of additive subgroups G of $\overline{\mathbb{F}_p}$ such that $(\overline{\mathbb{F}_p}, G) \models$ ACFG is a dense G_δ of $\mathrm{Sg}(\overline{\mathbb{F}_p})$, for the Chabauty topology on $\mathrm{Sg}(\overline{\mathbb{F}_p})$.*

This result is proved in the same way as the analogous result from Hrushovski about models of ACFA in $\overline{\mathbb{F}_p}$ [Hru04].

Imaginaries in ACFG. Let (K, G) be a model of ACFG. There are no canonical parameters for elements of the quotient group K/G. A natural question to ask is the following: if one adds to (K, G) canonical parameters for the quotient K/G, does one add canonical parameters for every definable equivalence class in (K, G)? In order to answer this question, denote by $(K, K/G)$ the 2-sorted structure consisting in one sort for the field K and one sort for the group K/G, together with the canonical projection $\pi : K \to K/G$.

Theorem I. *For any model (K, G) of* ACFG*, the structure $(K, K/G)$ has weak elimination of imaginaries.*

Weak elimination of imaginaries is optimal for $(K, K/G)$. Indeed, $(K/G, +)$ carries the structure of a pure $\mathbb{F}_p$-vector space, hence finite imaginaries from this sort cannot be eliminated. Furthermore, canonical parameters for finite imaginaries of K/G are not enough to code all finite imaginaries in $(K, K/G)$, see Example 6.3.6. The proof of Theorem I follows the same classical pattern as the proof of elimination of imaginaries in [Hru02], [CH99] or [CP98], and is based on the independence theorem. In our case, however, Kim-independence will play the role that forking independence plays in those classical proofs. Kim-independence in (K, G) is given by $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_C B$ and $G(A+B) = G(A)+G(B)$, for $C = A\cap B$ and A, B, C algebraically closed. In $(K, K/G)$, the condition $G(A + B) = G(A) + G(B)$ translates to $\pi(A) \cap \pi(B) = \pi(C)$, a "stable modular" independence coming from the pure $\mathbb{F}_p$-vector space structure on the sort K/G. Hence in $(K, K/G)$, Kim-independence is given by the conjunction of the independence in the sense of ACF in the sort K and the "stable modular" independence in the sort K/G, which is easier to apprehend that Kim-independence in (K, G). Also, conditions involving elements of K/G are translated in terms of special representatives (*minimal* and *maximal*) in K, in order to deduce a version of the independence theorem and adapt the classical proofs. The proof of Theorem I should go through as it is by replacing the theory ACF_p by any stable theory of fields with elimination of imaginaries.

The study of the theory of $(K, K/G)$ suggests a "dual" generic expansion. Starting from a theory T and a reduct T_0 of T in the language $\mathscr{L}_0$. Consider the two sorted theory T' whose models consists in a model $\mathscr{M}$ of T in the first sort, a model $\mathscr{M}_0$ of T_0 in the second sort and a surjective $\mathscr{L}_0$-homomorphism $\pi : \mathscr{M} \to \mathscr{M}_0$. Then under hypotheses (H_1) to (H_4) one constructs the model-companion of T', in which imaginaries should be easier to deal with.

Forking and Thorn-forking in ACFG. Forking is not symmetric in ACFG, otherwise, the theory would be simple. Theorem G gives the loose intuition that Kim-independence and forking independence differs only by the property Base Monotonicity. Indeed, we prove that forking independence is obtained by "forcing" the property base monotonicity on Kim-independence. We show that forking equals dividing for types and that it also equals thorn forking.

Theorem J. *Let $\mathop{\smile\!\!\!\!\!\!\mid}^f$, $\mathop{\smile\!\!\!\!\!\!\mid}^d$, $\mathop{\smile\!\!\!\!\!\!\mid}^{þ}$, $\mathop{\smile\!\!\!\!\!\!\mid}^K$ be respectively the forking, dividing, thorn-forking*[12]*, and Kim independence relation in* ACFG*. Then*

$$a \mathop{\smile\!\!\!\!\!\!\mid}^f_C b \iff a \mathop{\smile\!\!\!\!\!\!\mid}^d_C b \iff a \mathop{\smile\!\!\!\!\!\!\mid}^{þ}_C b \iff \text{for all } C \subseteq D \subseteq Cb,\ a \mathop{\smile\!\!\!\!\!\!\mid}^K_D b.$$

In NSOP_1 theories, forking independence and Kim independence are different notions, no good description of forking independence exists, but most of the known examples of NSOP_1 and not simple theories share the same description of forking as ACFG [Cha02], [CK17], [KR18]. A similar pattern for proving that forking equals dividing for types seems to emerge from different examples. It involves a "mixed transitivity" result between the strong independence and Kim-independence, from which one deduces that dividing independence satisfies the property *Extension*, hence equals forking. A discussion on the features shared by the main examples of NSOP_1 theories is given in Section 7.4, as well as the main differences between ACFG and

[12] Here we mean the restriction of thorn forking to the home sort, as thorn forking in general is defined in T^{eq}.

the other examples (see also Figure 7.2). We also advertise some nice phenomena that appear when one forces the *Base Monotonicity* property on a given independence relation (Section 7.1).

Our treatment of thorn-forking in ACFG uses the geometric description of the thorn-forking independence given by Adler [Adl09a]. More generally, Part A and especially Chapter 7 make use of the geometric treatment of independence relations taking roots in Kim-Pillay theorem but mainly developed by Adler [Adl08a], [Adl09a], [Adl09b], then followed by [CK12], [CR16], [CK17] among others.

We now turn to part B.

§ 9 Expansions of the group of integers

From a model-theoretic point of view, the structure $(\mathbb{Z}, +, \cdot)$ is not understandable, this follows from Godel's celebrated work on Peano arithmetic. Starting from this fact, it is natural to study tame reducts of $(\mathbb{Z}, +, \cdot)$. The study of the structures $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, +, 0, <)$ dates back from 1929 by Presburger, the theory of $(\mathbb{Z}, +, 0, <)$ is still referred to as *Presburger arithmetic*[13]. Both structures admit quantifier elimination in the expansion of the language by the constant 1 and predicates for the subgroups $n\mathbb{Z}$, for all $n \in \mathbb{N}$. The theory of $(\mathbb{Z}, +, 0)$ is superstable of U-rank 1. Presburger arithmetic, however, is unstable but NIP and moreover dp-minimal.

The study of tame expansions of $(\mathbb{Z}, +, 0)$ is a recent subject. Until not long ago, no examples of such structures were studied, other than $(\mathbb{Z}, +, 0, <)$. The first examples were given independently by Palacín and Sklinos [PS18] and by Poizat [Poi14]. Specifically, they both proved, using different methods, that for any integer $q \geq 2$ the structure $(\mathbb{Z}, +, 0, \prod_q)$ is superstable of U-rank ω, where $\prod_q = \{q^n \mid n \in \mathbb{N}\}$. Palacín and Sklinos also showed the same result for other examples, such as $(\mathbb{Z}, +, 0, \mathrm{Fac})$, where $\mathrm{Fac} = \{n! \mid n \in \mathbb{N}\}$. Conant [Con17b] and Lambotte and Point [LP17] independently generalized these results. For a subset $A \subseteq \mathbb{Z}$ with either an upper bound or a lower bound, they give some sparsity conditions on A which are sufficient for the structure $(\mathbb{Z}, +, 0, A)$ to be superstable of U-rank ω. Conant also gives sparsity conditions which are necessary for the structure $(\mathbb{Z}, +, 0, A)$ to be stable.

A different kind of example was given recently by Kaplan and Shelah in [KS17]. They proved that for $\mathrm{Pr} = \{p \in \mathbb{Z} \mid |p| \text{ is prime}\}$, the structure $(\mathbb{Z}, +, 0, \mathrm{Pr})$ has the independence property (and even the n-independence property for all n) hence it is unstable. On the other hand, assuming Dickson's Conjecture [14], it is supersimple of U-rank 1.

In contrast to the above, $(\mathbb{Z}, +, 0, <)$ remained the only known unstable dp-minimal expansion of $(\mathbb{Z}, +, 0)$. In [Asc+13, Question 5.32], Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko ask the following question: is every dp-minimal expansion of $(\mathbb{Z}, +, 0)$ a reduct of $(\mathbb{Z}, +, 0, <)$ $\quad (\star)$. In [Asc+16] the same authors prove that $(\mathbb{Z}, +, 0, <)$ has no proper dp-minimal expansions. They use a strong result from automaton theory of Michaux and Villemaire [MV96], that can be stated as follows: every proper expansion of $(\mathbb{Z}, +, 0, <)$ defines a new subset of $\mathbb{Z}$. This was later strengthened by Dolich and Goodrick [DG17], they obtain that $(\mathbb{Z}, +, 0, <)$ has no proper strong expansions. Together with a result of Conant [Con18]

[13]Note that for some logicians, Presburger arithmetic is the theory of $(\mathbb{Z}, +, 0)$ and indeed, in his 1929 paper, Presburger studied mainly the theory of $(\mathbb{Z}, +, 0)$. However, in the same paper, he explains that his results extend to the theory of $(\mathbb{Z}, +, 0, <)$, see [Haa18] for a comprehensive survey on Presburger arithmetic.

[14]A strong number-theoretic conjecture about the distribution of prime numbers in arithmetic progressions, which generalizes Dirichlet's theorem on prime numbers.

which we describe below, any other unstable dp-minimal expansion of $(\mathbb{Z}, +, 0)$, if exists, is not a reduct, nor an expansion of $(\mathbb{Z}, +, 0, <)$, thus question $(\star)$ becomes:

$$\text{Is } (\mathbb{Z}, +, 0, <) \text{ the only dp-minimal non-trivial expansion of } (\mathbb{Z}, +, 0)? \qquad (\star\star)$$

§ 10 New dp-minimal expansions of the integers

We introduce a new family of dp-minimal expansions of $(\mathbb{Z}, +, 0)$, thus giving a negative answer to question $(\star\star)$ above. More generally, for every $n \in \mathbb{N} \cup \{\omega\}$ we introduce a family of expansions of $(\mathbb{Z}, +, 0)$ having dp-rank n. After proving this we were informed that a similar result has been proved independently by François Guignot [Gui16], and by Nathanaël Mariaule [Mar17, Corollary 2.11].

Let P be a (finite or infinite) set of prime numbers. For each $p \in P$ consider the preorder on integers given by

$$a \mid_p b \iff v_p(a) \leq v_p(b)$$

where v_p is the p-adic valuation on $\mathbb{Z}$.

Theorem K. *The structure $(\mathbb{Z}, +, 0, (\cdot \mid_p \cdot)_{p \in P})$ has quantifier elimination in the language expanded by the constant* 1 *and predicates for the subgroups $n\mathbb{Z}$, for all $n \in \mathbb{N}$. Furthermore, its dp-rank equals $|P|$, the cardinality of P.*

Each p-adic valuation on $\mathbb{Z}$ imposes a tree-like structure on the integers, and a tree topology similar to the one we describe in Section 1.6. Each integer is represented as a branch on which each node represents its coordinate in the p-adic representation. This tree structure is preserved in elementary extensions and allows a graphical treatment of the arguments. The proof of quantifier elimination is technical but does not use more complicated arithmetic result than the Chinese remainder theorem, which translates here as the topological density of each subgroup $n\mathbb{Z}$ in $\mathbb{Z}$, for n coprime with p.

The computing of the dp-rank of $(\mathbb{Z}, +, 0, (\cdot \mid_p \cdot)_{p \in P})$ is in two steps. First we prove dp-minimality for the case $P = \{p\}$, this involves quantifier elimination and the dp-minimality of the structure $(\mathbb{Q}_p, +, 0, \mid_p)$ [DGL11]. Then we deduce from quantifier elimination that the dp-rank of $(\mathbb{Z}, +, 0, (\cdot \mid_p \cdot)_{p \in P})$ cannot grow more than the sum of the dp-ranks of each reduct $(\mathbb{Z}, +, 0, \mid_p)$, and we conclude by exhibiting an ict-pattern of length $|P|$.

Quantifier elimination in $(\mathbb{Z}, +, 0, \mid_p)$ implies that every definable set is a boolean combination of $\emptyset$-definable sets and balls. C-minimality is a notion introduced by Macpherson and Steinhorn [MS96] to give an analogue of o-minimality in the context of valued or tree-like structures which admits quantifier elimination. Morally in a C-minimal tree-like structure, every definable set is a boolean combination of balls. This suggests a definition of *quasi*-C-minimality, analogously to quasi-o-minimality [BPW00], which would hopefully imply dp-minimality. Similarly for the even more general notion of VC-minimality [Adl08b].

§ 11 Minimality phenomena

Given a class $\mathcal{C}$ of structures with the same underlying universe, and $\mathscr{M} \in \mathcal{C}$, we say that $\mathscr{M}$ is *minimal* in $\mathcal{C}$ if there is no proper reduct of $\mathscr{M}$ in $\mathcal{C}$. Similarly, $\mathscr{M}$ is *maximal* in $\mathcal{C}$ if there are no proper expansion of $\mathscr{M}$ in $\mathcal{C}$.

A first example of this phenomenon was given by Pillay and Steinhorn [PS87]: $(\mathbb{N}, <)$ has no proper *o*-minimal expansions, in other words, $(\mathbb{N}, <)$ is maximal in the class of *o*-minimal structures with underlying universe $\mathbb{N}$. Another example was given by Marker, motivated by the following question from Zilber: can an algebraically closed field have a proper strongly minimal expansion? Although this question was answered positively by Hrushovski [Hru92], Marker proved [Mar90] that $(\mathbb{C}, +, \cdot, 0, 1)$ has no proper expansion *at all* which is a proper reduct of $(\mathbb{C}, +, \cdot, 0, 1, \mathbb{R})$ i.e., $(\mathbb{C}, +, \cdot, 0, 1, \mathbb{R})$ is minimal among the proper expansions of $(\mathbb{C}, +, \cdot, 0, 1)$.

The study of expansions of the group $(\mathbb{Z}, +, 0)$ recently produced numerous minimality/maximality flavoured results. As we saw above, $(\mathbb{Z}, +, 0, <)$ is a maximal dp-minimal structure on $\mathbb{Z}$ [Asc+16]. Based on a result by Palacín and Sklinos [PS18], Conant and Pillay [CP18] proved the following: $(\mathbb{Z}, +, 0)$ has no proper stable expansions of finite dp-rank. In other words, $(\mathbb{Z}, +, 0)$ is maximal among the stable structures of finite dp-rank on $\mathbb{Z}$. As $(\mathbb{Z}, +, 0, <)$ is dp-minimal, an immediate consequence of the above is that there is no stable structures which is both a proper expansion of $(\mathbb{Z}, +, 0)$ and a proper reduct of $(\mathbb{Z}, +, 0, <)$. In [Con18] Conant strengthened this result by proving that there are no structures *at all* which are both proper expansions of $(\mathbb{Z}, +, 0)$ and proper reducts of $(\mathbb{Z}, +, 0, <)$, hence Conant proved that $(\mathbb{Z}, +, 0, <)$ is a minimal proper expansion of $(\mathbb{Z}, +, 0)$. We proved the corresponding result for our expansion of $(\mathbb{Z}, +, 0)$.

Theorem L. *$(\mathbb{Z}, +, 0, |_p)$ is a minimal proper expansions of $(\mathbb{Z}, +, 0)$.*

Conant's proof [Con18] does not use that $(\mathbb{Z}, +, 0)$ has no proper stable expansion which is a reduct of $(\mathbb{Z}, +, 0, <)$. His proof involves detailed analysis of definable sets in arbitrary dimension. Conant asked whether his theorem could be proved using model-theoretic methods which incorporate the result [CP18]. This is the strategy we adopt to prove Theorem L, hence the content of Theorem L is really that there is no *unstable* structure which is a reduct of $(\mathbb{Z}, +, 0, |_p)$ and a proper expansion of $(\mathbb{Z}, +, 0)$. We also give a shorter proof of the result of Conant, by the same method we used for the proof of Theorem L, which we describe now. By a classical result of Shelah, the unstability of a theory can be witnessed by a formula for which one of the tuples of variables have length one. If $\mathcal{Z}$ is an unstable (hence proper) expansion of $(\mathbb{Z}, +, 0)$ and a reduct of $(\mathbb{Z}, +, 0, |_p)$, then Shelah's result implies that, at the cost of working in an elementary extension $\mathcal{Z}'$ of $\mathcal{Z}$, there is a formula in the language of $\mathcal{Z}$ which defines a new subset of the domain in $\mathcal{Z}'$, hence the problem is reduced to an analysis of unidimensional definable subsets of $\mathcal{Z}'$, allowed by quantifier elimination in $(\mathbb{Z}, +, 0, |_p)$. The rest of the proof consists in defining the relation $x |_p y$ by applying transformations in the language $\{+, 0\}$ to the new formula. It uses an analysis of uniformly definable subgroups of the domain in elementary extensions of $(\mathbb{Z}, +, 0, |_p)$. We prove that any such new formula can be transformed to uniformly define (and only define) a chain of balls centered in 0 (hence subgroups) of strictly increasing consecutive radiuses. When considering this formula back in $\mathbb{Z}$, it uniformly defines cofinitely many subgroups of the form $p^k\mathbb{Z}$, and only sets of this form. This yields the result, as $a |_p b$ if and only if for all $k \in \mathbb{N}$, $a \in p^k\mathbb{Z} \to b \in p^k\mathbb{Z}$.

Theorem L and Conant's analogous result does not hold in elementary extensions. We give counter examples in Section 9.3. However, for stronger notions of expansions and reducts, the minimality results goes through to elementary extensions, see Corollary 9.1.9 and Theorem 9.2.12.

In regard of question $(\star\star)$ above, one would formulate the following trichotomy: a dp-minimal expansion of $(\mathbb{Z}, +, 0)$ is either stable (and hence interdefinable with $(\mathbb{Z}, +, 0)$ itself), either $(\mathbb{Z}, +, 0, <)$ or it defines a valuation. This conjecture is inspired by the following result for fields due to Will Johnson [Joh15]: if K is a dp-minimal field, then it is either algebraically closed, real-closed or admits a definable henselian valuation. However, the conjecture for $(\mathbb{Z}, +, 0)$ is false. Indeed, Tran and Walsberg present in [TW17] a new family of dp-minimal expansions of

$(\mathbb{Z}, +, 0)$ obtained by adding cyclical orderings. It would be interesting to know whether other dp-minimal expansions of $(\mathbb{Z}, +, 0)$ exist, or if the expansion of $(\mathbb{Z}, +, 0)$ by a cyclical ordering satisfies the same minimality property as $(\mathbb{Z}, +, 0, <)$ and $(\mathbb{Z}, +, 0, |_p)$ do.