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CHAPTER 1

Preliminaries

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1.1 Generalities

We assume basic knowledge in model theory, concerning formulae, types, theories, and models. Unless otherwise stated, a type means a complete type. Throughout we will denote by x, y, x^i, y^i tuples of variables, the subscript x_i, y_i will be used to denote a coordinate inside a tuple. Also, t will often denote a single variable. Capital letters A, B, C stand for sets whereas small latin letters a, b, c designate either singletons, finite or infinite tuples. For any tuple a (of elements or of variables), we denote by $|a|$ the length of a. For a set, $|C|$ is the cardinality of C. As usual in model theory, we denote by juxtaposition AB the union of the set A and the set B. We also identify juxtaposition of tuples ab as the concatenation of a and b. When dealing with independence relations or closure operators, we do not distinguish between tuples, enumerations, and sets.

Given a complete theory T in a language $\mathscr{L}$, a *monster model* $\mathbb{M}$ of T is a strongly κ-homogeneous and κ-saturated model of T, for some big enough κ. It is standard that $\mathbb{M}$ is κ^+-universal, in particular every model $\mathscr{M}$ of cardinality less than κ embeds in

$\mathbb{M}$. Furthermore, we will assume that any reduct of $\mathbb{M}$ is also a monster model[1] for its theory in the reduct language. As usual in that context, a *small* model of T (or a small set) is a model of T (or a subset of some model of T) of cardinal less than κ and, by κ-universality, we consider them as elementary substructures (subsets) of $\mathbb{M}$. A small cardinal is a cardinal $\lambda < \kappa$. We will sometimes forget about the "small" adjective and even about the cardinal κ, as it will always be implied that, in every single proof, every set we consider has cardinality smaller than κ.

Given a theory T we use the notations tp^T, $\equiv^T$, acl_T and dcl_T to precise that we work in the language of T, and when the language is clear, we just use tp, $\equiv$, acl, dcl. By strong κ-homogeneity if $a \equiv_C^T b$ then there is an automorphism σ of $\mathbb{M}$ over C (i.e. fixing C pointwise) such that $\sigma(a) = b$ (i.e. $\sigma(a_i) = b_i$ for $0 \leq i < |a|$, $|a|$ may be infinite). Such an automorphism is also called a C-automorphism. For two sets A, A', we denote by $A \equiv_C A'$ if for all enumeration $(a_\alpha)_{\alpha<|A|}$ of A there exists an automorphism σ of the monster over C, and an enumeration $(a'_\alpha)_{\alpha<|A'|}$ of A' such that $\sigma(a_\alpha) = a'_\alpha$ for all $\alpha < |A|$ (in particular $|A| = |A'|$); equivalently, there is a C-automorphism of the monster such that $\sigma(A) = A'$ setwise. The restriction of σ to the set A is called a T-*elementary bijection* (or T-elementary isomorphism) between A and A'. This must not be confused with the notion of elementary equivalent models over C: $\mathscr{M} \equiv_C \mathscr{N}$ if $C \subseteq \mathscr{M} \cap \mathscr{N}$ and for all $\mathscr{L}$-sentences θ with parameters in C, then $\mathscr{M} \models \theta$ if and only if $\mathscr{N} \models \theta$. If $\mathscr{M} \equiv_C \mathscr{N}$, in general there is no C-automorphism of the monster sending $\mathscr{M}$ on $\mathscr{N}$.

A theory T is *model-complete* if for all models $\mathscr{M}$ and $\mathscr{N}$ of T, if $\mathscr{M}$ is a substructure of $\mathscr{N}$ then $\mathscr{M}$ is an elementary substructure of $\mathscr{N}$. A model-complete theory need not be complete, see for instance ACF below. A *model-companion* of an $\mathscr{L}$-theory T is an $\mathscr{L}$-theory T^* such that:

- every model of T has an extension which is a model of T^*;
- every model of T^* has an extension which is a model of T;
- T is model-complete.

The model-companion of a theory need not exists, but if it does, it is unique (see for instance [Mar02, Exercise 3.4.13]).

An $\mathscr{L}$-structure $\mathscr{M}$ is *existentially closed* in some extension $\mathscr{N}$ if every existential formula with parameters in $\mathscr{M}$ that holds in $\mathscr{N}$ holds also in $\mathscr{M}$. An *existentially closed model of a theory* T is a model of T that is existentially closed in every extension which is a model of T. A theory is *inductive* if the union of any chain of models is still a model. Equivalently, it is $\forall\exists$-axiomatisable. Assume that T is inductive, then if the model-companion T^* exists, T^* is an axiomatization of the class of existentially closed models of T (see [Mar02, Exercise 3.4.13]). We say that a theory T has the *amalgamation property* if whenever $\mathscr{M}_0$, $\mathscr{M}_1$ and $\mathscr{M}_2$ are models of T such that there exists embedding

[1] By [Hod08, Chapter 10], choose $\mathbb{M}$ to be κ-*big*, then it strongly κ-homogeneous and κ-saturated, and any reduct is also κ-big. Note that in general, a reduct of a strongly κ-homogeneous structure need not be strongly κ-homogeneous, see [Hod08, 10.1, Exercice 11].

$f_1 : \mathscr{M}_0 \to \mathscr{M}_1$ and $f_2 : \mathscr{M}_0 \to \mathscr{M}_2$ then there exists a model $\mathscr{N}$ of T and embeddings $g_1 : \mathscr{M}_1 \to \mathscr{N}$ and $g_2 : \mathscr{M}_2 \to \mathscr{N}$ such that the following diagram commutes.

$$\begin{tikzcd} \mathscr{M}_0 \arrow[r, "f_1"] \arrow[d, "f_2"] & \mathscr{M}_1 \arrow[d, "g_1"] \\ \mathscr{M}_2 \arrow[r, "g_2"] & \mathscr{N} \end{tikzcd}$$

If it exists, the model-companion of a theory which has the amalgamation property is called the *model-completion*.

Let $\mathbb{M}$ be a monster model of T and A, B small subsets of $\mathbb{M}$. Let p be a (complete) type over B. We say that p is *finitely satisfiable in* A if for all formula $\phi(x, b)$ in p, there exists a tuple a from A with $|a| = |x|$ and $\mathbb{M} \models \phi(a, b)$. We say that p is A*-invariant* if for all A-automorphism, $\sigma p = p$. If p is finitely satisfiable in A, then it is A-invariant (see [Sim15, Example 2.17]). A *global type* is a type over $\mathbb{M}$, a *global extension* q of p is a global type such that $q \upharpoonright B = p$, where $q \upharpoonright B$ is the type consisting of formulae in q with parameters in B. If a type p over B is finitely satisfiable in A then it has a global extension which is finitely satisfiable in A (see [Sim15, Example 2.17]). As every type over a model $\mathscr{M}$ is clearly finitely satisfiable in $\mathscr{M}$, it follows that every type over a model $\mathscr{M}$ has a global extension which is finitely satisfiable in $\mathscr{M}$, hence also $\mathscr{M}$-invariant. If p is a type over B and q is an extension of p which is finitely satisfiable in B, then q is called a *coheir* of p.

Let λ be a small cardinal and C a small set. A sequence $(b_i)_{i<\lambda}$ is C*-indiscernible* if for all $n < \omega$ and $\alpha_1 < \cdots < \alpha_n < \lambda$ we have $b_{\alpha_1}, \cdots, b_{\alpha_n} \equiv_C b_1, \cdots, b_n$. For all $\alpha < \lambda$, the sequence $(b_i)_{\alpha \le i < \kappa}$ is $Cb_{<\alpha}$-indiscernible.

Given an $\mathscr{L}$-theory T, we recall the construction of T^{eq}, see for example [TZ12, Chapter 8]. To each model $\mathscr{M}$ of T is associated a structure $\mathscr{M}^{eq}$ consisting of the following: one sort (the *home* sort) for the structure $\mathscr{M}$ and for each definable equivalence class $E(x, y)$ in $\mathscr{M}$ without parameters (we will also say 0-*definable*) an associated *imaginary* sort S_E in the language of $\mathscr{M}^{eq}$, and a projection $\pi_E : \mathscr{M}^{|x|} \to S_E$ such that $\pi_E(x) = \pi_E(y)$ if and only if $E(x, y)$. Let T^{eq} be the theory of $\mathscr{M}^{eq}$. Basic facts about T^{eq} are first that it doesn't depend on the model $\mathscr{M}$ of T. Every automorphism of $\mathscr{M}$ extends to an automorphism of $\mathscr{M}^{eq}$ and every automorphism of $\mathscr{M}^{eq}$ restricted to $\mathscr{M}$ is an automorphism of $\mathscr{M}$. It follows that elements in the home sort $\mathscr{M}$ have the same type (over $\emptyset$) in $\mathscr{M}$ if and only if they have the same type in $\mathscr{M}^{eq}$. Also T^{eq} *eliminates imaginaries*, that is, for a monster model $\mathscr{N}$ of T^{eq} and formula $\phi(x, b)$ with parameters in $\mathscr{N}$, there exists a finite tuple d that is fixed (pointwise) by the same automorphisms which fix $\phi(\mathscr{N}, b)$ setwise. Such a tuple is called a *canonical parameter* for $\phi(x, b)$. A theory T has elimination of imaginaries if and only if in T^{eq} every element of the home sort is interdefinable with an element of an imaginary sort. A theory T has *weak elimination of imaginaries* if in T^{eq}, for every element e in an imaginary sort, there is a tuple d from the home sort such that e is definable over d and d is algebraic over e. We denote by acl_T^{eq} and dcl_T^{eq} the algebraic closure and definable closure in the sens of the theory T^{eq}.

1.2 Independence relations

Let $\mathbb{M}$ be a monster model for a theory T.

Definition 1.2.1. For any a, b (finite or infinite) tuples from $\mathbb{M}$ and C small set in $\mathbb{M}$ we recall the following various ternary relations (see for instance [CK17]).

(1) *Algebraic independence.* $a \mathop{\smile\!\!\!\!\!\vert}^{a}_{C} b$ if and only if $\mathrm{acl}(Ca) \cap \mathrm{acl}(Cb) = \mathrm{acl}(C)$

(2) *Imaginary algebraic independence.* $a \mathop{\smile\!\!\!\!\!\vert}^{aeq}_{C} b$ if and only if $\mathrm{acl}^{eq}(Ca) \cap \mathrm{acl}^{eq}(Cb) = \mathrm{acl}^{eq}(C)$. This relation is defined over subsets of $\mathbb{M}^{eq}$. We write $\mathop{\smile\!\!\!\!\!\vert}^{aeq}\restriction \mathbb{M}$ to specify the restriction to elements of the sort $\mathbb{M}$.

(3) *Kim-dividing independence.* $a \mathop{\smile\!\!\!\!\!\vert}^{Kd}_{C} b$ if and only if $tp(a/Cb)$ does not Kim-divides over C if and only if for all global C-invariant extension p of $tp(b/C)$ and sequences $(b_i)_{i<\omega}$ such that $b_i \models p \restriction Cb_{<i}$ for all $i < \omega$, there exists a' such that $a'b_i \equiv_C ab$ for all $i < \omega$;

(4) *Kim-(forking)independence.* $a \mathop{\smile\!\!\!\!\!\vert}^{K}_{C} b$ if and only if $tp(a/Cb)$ does not Kim-fork over C if and only if for any $b' \supseteq b$ there exists $a' \equiv_{Cb} a$ such that $a' \mathop{\smile\!\!\!\!\!\vert}^{Kd}_{C} b'$;

(5) *Dividing independence.* $a \mathop{\smile\!\!\!\!\!\vert}^{d}_{C} b$ if and only if $tp(a/Cb)$ does not divide over C if and only if for any C-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 = b$ there exists a' such that $a'b_i \equiv_C ab$ for all $i < \omega$;

(6) *Forking independence.* $a \mathop{\smile\!\!\!\!\!\vert}^{f}_{C} b$ if and only if $tp(a/Cb)$ does not fork over C if and only if for any $b' \supseteq b$ there exists $a' \equiv_{Cb} a$ such that $a' \mathop{\smile\!\!\!\!\!\vert}^{d}_{C} b'$;

(7) *Coheir independence.* $a \mathop{\smile\!\!\!\!\!\vert}^{u}_{C} b$ if and only if $tp(a/Cb)$ is finitely satisfiable in C.

As usual, we extend these notions to sets by the following: $A \mathop{\smile\!\!\!\!\!\vert}_{C} B$ if and only if for all enumeration a of A and b of B, then $a \mathop{\smile\!\!\!\!\!\vert}_{C} b$. The following is a list of properties for a ternary relation $\mathop{\smile\!\!\!\!\!\vert}$ defined over small subsets of $\mathbb{M}$, sometimes relatively to another ternary relation $\mathop{\smile\!\!\!\!\!\vert}'$, also defined over small subsets of $\mathbb{M}$.

- Invariance. If $ABC \equiv A'B'C'$ then $A \mathop{\smile\!\!\!\!\!\vert}_{C} B$ if and only if $A' \mathop{\smile\!\!\!\!\!\vert}_{C'} B'$.
- Finite Character. If $a \mathop{\smile\!\!\!\!\!\vert}_{C} B$ for all finite $a \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\vert}_{C} B$.
- Symmetry. If $A \mathop{\smile\!\!\!\!\!\vert}_{C} B$ then $B \mathop{\smile\!\!\!\!\!\vert}_{C} A$.
- Closure $A \mathop{\smile\!\!\!\!\!\vert}_{C} B$ if and only if $A \mathop{\smile\!\!\!\!\!\vert}_{\mathrm{acl}(C)} \mathrm{acl}(BC)$.
- Monotonicity. If $A \mathop{\smile\!\!\!\!\!\vert}_{C} BD$ then $A \mathop{\smile\!\!\!\!\!\vert}_{C} B$.
- Base Monotonicity. If $A \mathop{\smile\!\!\!\!\!\vert}_{C} BD$ then $A \mathop{\smile\!\!\!\!\!\vert}_{CD} B$.
- Transitivity. If $A \mathop{\smile\!\!\!\!\!\vert}_{CB} D$ and $B \mathop{\smile\!\!\!\!\!\vert}_{C} D$ then $AB \mathop{\smile\!\!\!\!\!\vert}_{C} D$.

- Existence. For any C and A we have $A \mathop{\smile\!\!\!\!\!\!|}_C C$.
- Full Existence. For all A, B and C there exists $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B$.
- Extension. If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then for all D there exists $A' \equiv_{CB} A$ and $A' \mathop{\smile\!\!\!\!\!\!|}_C BD$.
- Local Character. For all finite tuple a and infinite B there exists $B_0 \subset B$ with $|B_0| \leq \aleph_0$ and $a \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$.
- Strong Finite Character over E. If $a \not\mathop{\smile\!\!\!\!\!\!|}_E b$, then there is a formula $\Lambda(x, b, e) \in tp(a/Eb)$ such that for all a', if $a' \models \Lambda(x, b, e)$ then $a' \not\mathop{\smile\!\!\!\!\!\!|}_E b$.
- $\mathop{\smile\!\!\!\!\!\!|}'$-amalgamation over E. If there exists tuples c_1, c_2 and sets A, B such that
  - $c_1 \equiv_E c_2$
  - $A \mathop{\smile\!\!\!\!\!\!|}'_E B$
  - $c_1 \mathop{\smile\!\!\!\!\!\!|}_E A$ and $c_2 \mathop{\smile\!\!\!\!\!\!|}_C B$

  then there exists $c \mathop{\smile\!\!\!\!\!\!|}_E A, B$ such that $c \equiv_A c_1$, $c \equiv_B c_2$, $A \mathop{\smile\!\!\!\!\!\!|}^a_{Ec} B$, $c \mathop{\smile\!\!\!\!\!\!|}^a_{EA} B$ and $c \mathop{\smile\!\!\!\!\!\!|}^a_{EB} A$.
- Stationnarity over E. If $c_1 \equiv_E c_2$ and $c_1 \mathop{\smile\!\!\!\!\!\!|}_E A$, $c_2 \mathop{\smile\!\!\!\!\!\!|}_E A$ then $c_1 \equiv_{EA} c_2$.
- Witnessing. Let a, b be tuples, $\mathscr{M}$ a model and assume that $a \not\mathop{\smile\!\!\!\!\!\!|}_{\mathscr{M}} b$. Then there exists a formula $\Lambda(x, b) \in tp(a/\mathscr{M}b)$ such that for any global extension $q(x)$ of $tp(b/\mathscr{M})$ finitely satisfiable in $\mathscr{M}$ and for any $(b_i)_{i<\omega}$ such that for all $i < \omega$ we have $b_i \models q \upharpoonright \mathscr{M}b_{<i}$, the set $\{\Lambda(x, b_i) \mid i < \omega\}$ is inconsistent.

If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, the set C is called the *base set.*

Definition 1.2.2. Let $\mathop{\smile\!\!\!\!\!\!|}$, $\mathop{\smile\!\!\!\!\!\!|}^0$ be two ternary relations. We say that $\mathop{\smile\!\!\!\!\!\!|}$ *is stronger than* $\mathop{\smile\!\!\!\!\!\!|}^0$ (or $\mathop{\smile\!\!\!\!\!\!|}^0$ *is weaker than* $\mathop{\smile\!\!\!\!\!\!|}$) if for all a, b, C we have $a \mathop{\smile\!\!\!\!\!\!|}_C b \implies a \mathop{\smile\!\!\!\!\!\!|}^0_C b$. We denote it by $\mathop{\smile\!\!\!\!\!\!|} \to \mathop{\smile\!\!\!\!\!\!|}^0$.

Assume that $\mathop{\smile\!\!\!\!\!\!|} \to \mathop{\smile\!\!\!\!\!\!|}^0$, then if $\mathop{\smile\!\!\!\!\!\!|}$ satisfies Full Existence or Local Character, so does $\mathop{\smile\!\!\!\!\!\!|}^0$. Similarly, if a relation satisfies $\mathop{\smile\!\!\!\!\!\!|}^0$-amalgamation then it also satisfies $\mathop{\smile\!\!\!\!\!\!|}$-amalgamation.

Fact 1.2.3. *The following are standard facts more or less obvious from the definition.*

(1) $\mathop{\smile\!\!\!\!\!\!|}^a$ *satisfies* Invariance, Monotonicity, Transitivity, Existence, Extension *and* Full Existence*;*

(2) $\mathop{\smile\!\!\!\!\!\!|}^d$ *satisfies* Invariance, Monotonicity, Base Monotonicity, Transitivity*;*

(3) $\mathop{\smile\!\!\!\!\!\!|}^f$ *satisfies* Invariance, Monotonicity, Base Monotonicity, Transitivity *and* Extension*;*

(4) $\mathop{\smile\!\!\!\!\!\!|}^u$ *satisfies* Invariance, Monotonicity, Base Monotonicity, Transitivity, Extension, Existence *over models,* Full Existence *over models;*

(5) $\mathop{\smile\!\!\!\!\!\!|}^d \to \mathop{\smile\!\!\!\!\!\!|}^{aeq} \restriction \mathbb{M}$*;*

(6) $\mathop{\smile\!\!\!\!\!\!|}^u \to \mathop{\smile\!\!\!\!\!\!|}^f \to \mathop{\smile\!\!\!\!\!\!|}^d \to \mathop{\smile\!\!\!\!\!\!|}^{aeq} \restriction \mathbb{M} \to \mathop{\smile\!\!\!\!\!\!|}^a$*;*

(7) $\mathop{\smile\!\!\!\!\!\!|}^f \to \mathop{\smile\!\!\!\!\!\!|}^K$ *and* $\mathop{\smile\!\!\!\!\!\!|}^d \to \mathop{\smile\!\!\!\!\!\!|}^{Kd}$*.*

Proof. *(1)* is [Adl09a, Proposition 1.5]. *(2)* and *(3)* are [Adl09b, Proposition 1.3]. *(4)* is [CK12, Remark 2.16], Base Monotonicity is trivial. For *(5)*, it is clear that if $a \mathop{\smile\!\!\!\!\!\!|}^d_C b$ in $\mathbb{M}$, then $a \mathop{\smile\!\!\!\!\!\!|}^d_C b$ in $\mathbb{M}^{eq}$, and by [Adl09a, Remark 5.4] it follows that $\mathrm{acl}^{eq}(Ca) \cap \mathrm{acl}^{eq}(Cb) = \mathrm{acl}^{eq}(C)$ hence $a \mathop{\smile\!\!\!\!\!\!|}^{aeq}_C b$. *(6)* follows from [CK12, Example 2.22], and the previous results. *(7)* is by definition. □

Lemma 1.2.4. *Let* $\mathop{\smile\!\!\!\!\!\!|}$ *be a relation satisfying* Symmetry, Monotonicity, Existence *and* Strong Finite Character *over* C*.*

$$\text{If } a \mathop{\smile\!\!\!\!\!\!|}^u_C b \text{ then } a \mathop{\smile\!\!\!\!\!\!|}_C b.$$

In particular, as $\mathop{\smile\!\!\!\!\!\!|}^u$ *satisfies* Full Existence *over models, so does* $\mathop{\smile\!\!\!\!\!\!|}$*.*

Proof. Indeed, assume $a \mathop{\not\smile\!\!\!\!\!\!|}_C b$ then by Strong Finite Character there is a formula $\phi(x,b) \in tp(a/Cb)$ such that if $a' \models \phi(x,b)$ then $a' \mathop{\not\smile\!\!\!\!\!\!|}_C b$. As $tp(a/Cb)$ is finitely satisfiable in C there is $c \in C$ such that $c \models \phi(x,b)$, so $c \mathop{\not\smile\!\!\!\!\!\!|}_C b$, so by Symmetry and Monotonicity $b \mathop{\not\smile\!\!\!\!\!\!|}_C C$ which contradicts Existence. □

Lemma 1.2.5. *If* $\mathop{\smile\!\!\!\!\!\!|}$ *satisfies* Invariance *and* Extension*, then* $A \mathop{\smile\!\!\!\!\!\!|}_C B$ *implies* $A \mathop{\smile\!\!\!\!\!\!|}_C \mathrm{acl}(CB)$*. If* $\mathop{\smile\!\!\!\!\!\!|}$ *satisfies* Invariance, Extension *and* Base Monotonicity*, then* $\mathop{\smile\!\!\!\!\!\!|}$ *satisfies* Closure*.*

Proof. Assume that $A \mathop{\smile\!\!\!\!\!\!|}_C B$. By Extension, let A' be such that $A' \equiv_{BC} A$ and $A' \mathop{\smile\!\!\!\!\!\!|}_C \mathrm{acl}(BC)$. There is an automorphism σ over BC sending A' to A hence by Invariance, $A \mathop{\smile\!\!\!\!\!\!|}_C \sigma(\mathrm{acl}(BC))$. Now, as sets, $\sigma(\mathrm{acl}(BC)) = \mathrm{acl}(BC)$ so $A \mathop{\smile\!\!\!\!\!\!|}_C \mathrm{acl}(BC)$. The last assertion is trivial, as $\mathrm{acl}(C) \subseteq \mathrm{acl}(BC)$. □

Remark 1.2.6. If $\mathop{\smile\!\!\!\!\!\!|}$ satisfies Invariance, Symmetry, Transitivity and Full Existence, then $\mathop{\smile\!\!\!\!\!\!|}$ satisfies Extension. Also if $\mathop{\smile\!\!\!\!\!\!|}$ satisfies Existence, Monotonicity and Extension then it satisfies Full Existence. Hence for relations $\mathop{\smile\!\!\!\!\!\!|}$ satisfying Invariance, Monotonicity, Existence, Transitivity, Symmetry, the properties Full Existence and Extension are equivalent. Unfortunately, when dealing with non-symmetrical independence relations, we need to differentiate both properties. In Chapter 7, we see an example of a relation which is not symmetric but satisfies Invariance, Monotonicity, Existence, Transitivity, Full Existence: forking independence in ACFG. We show that it also satisfies Extension by non-trivial arguments.

Remark 1.2.7. Most of the properties above are familiar to anyone who knows forking in stable or simple theories.

The property STRONG FINITE CHARACTER is always satisfied by forking independence relation: take the formula ϕ to be a forking formula. This property is needed to use [CR16, Proposition 5.3] and prove that under the right assumptions on T, any completion of TS is NSOP$_1$.

If $\mathscr{M}$ is a model of the ambient theory, our formulation of $\mathop{\smile\!\!\!\!\!\mid}$-AMALGAMATION over $\mathscr{M}$ is what is called *The algebraically reasonable independence theorem* in [KR18], which holds for Kim-forking in any NSOP$_1$ theory (see [KR18, Theorem 2.21]). In simple theories, the forking independence relation also satisfies this property. The conclusion $A \mathop{\smile\!\!\!\!\!\mid}^a_{Ec} B$, $c \mathop{\smile\!\!\!\!\!\mid}^a_{EA} B$ and $c \mathop{\smile\!\!\!\!\!\mid}^a_{EB} A$ is always true in the simple case by SYMMETRY, BASE MONOTONICITY and TRANSITIVITY of the forking independence relation (and Fact 1.2.3 *(5)*). In many examples, one can prove the independence theorem under weaker assumptions, for instance assuming $\mathop{\smile\!\!\!\!\!\mid}'$ to be $\mathop{\smile\!\!\!\!\!\mid}^a$, or the base set to be acl-closed. Actually, there is no known example of an NSOP$_1$ theory in which $\mathop{\smile\!\!\!\!\!\mid}^a$-AMALGAMATION is *not* satisfied.

1.3 Pregeometry

This section introduces basic notions about pregeometries, as can be found in e.g. [TZ12, Appendix C]. We denote by $\mathscr{P}(S)$ the powerset of a set S.

Definition 1.3.1. A *pregeometry* (S, cl) is a set S and a *closure operator* $\mathrm{cl} : \mathscr{P}(S) \to \mathscr{P}(S)$ satisfying the following conditions, for all $A \subseteq S$ and a, b elements of S:

- (*Reflexivity*) $A \subseteq \mathrm{cl}(A)$;
- (*Finite Character*) $\mathrm{cl}(A) = \bigcup_{A_0 \subseteq B,\ A_0 \text{ finite}} \mathrm{cl}(A_0)$;
- (*Transitivity*) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;
- (*Exchange*) If $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$ then $b \in \mathrm{cl}(Aa)$.

A tuple $(b_i)_{i<\kappa}$ is *independent over* A if $b_i \notin \mathrm{cl}(A(b_j)_{j \neq i})$ for all $i < \kappa$. Similarly a set B is independent over A if for all enumeration b of B, b is independent over A. If $A \subseteq B$, and $B = \mathrm{cl}(B)$, a *basis* of B over A is a subset B' of B which is independent over A and such that $\mathrm{cl}(AB') = B$.

Fact 1.3.2. *Let (S, cl) be a pregeometry, $A \subset B \subset S$, and $B = \mathrm{cl}(B)$. Then every independent tuple in B over A can be completed into a basis of B over A, in particular B admits a basis over A. Every basis of B over A have the same cardinality, we call it the* dimension of B over A, *denoted by* $\dim_{\mathrm{cl}}(B/A)$ *(or* $\dim_{cl}(B)$ *if* $A = \emptyset$*).*

In any pregeometry, there is a notion of independence.

$$A \mathop{\smile\!\!\!\!\!\mid}^{cl}_C B \iff \text{for all basis } A_0 \text{ of } \mathrm{cl}(A) \text{ over } C \text{ and } B_0 \text{ of } \mathrm{cl}(B) \text{ over } C,$$
$$A_0B_0 \text{ is a basis of } \mathrm{cl}(AB) \text{ over } C$$

When there is a pregeometry in a wider context, we will say that a tuple a is $\mathop{\smile\!\!\!\!\!\!\;|}^{cl}$-independent over B to precise that this is relatively to the pregeometry (S, cl).

Fact 1.3.3. *The relation $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}$ satisfies the following properties.*

- Finite Character. *If for all finite tuple a from A we have $a \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_C B$ then $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_C B$.*
- Symmetry. *If $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_C B$ then $B \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_C A$.*
- Closure *$A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_C B$ if and only if $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_{\mathrm{cl}(C)} \mathrm{cl}(BC)$.*
- Monotonicity. *If $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_C BD$ then $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_C B$.*
- Base Monotonicity. *If $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_C BD$ then $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_{CD} B$.*
- Transitivity. *If $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_{CB} D$ and $B \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_C D$ then $AB \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_C D$.*
- Existence. *For all A, C, $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{cl}}_C C$.*

As there are no theory lying around (yet), properties like Invariance and Full Existence doesn't make sense here.

Definition 1.3.4. A pregeometry (S, cl) is *modular* if for all $A = \mathrm{cl}(A), B = \mathrm{cl}(B)$, $\dim(AB) + \dim(A \cap B) = \dim(A) + \dim(B)$.

Fact 1.3.5. *Let (S, cl) be a pregeometry. The following are equivalent.*

(1) (S, cl) is modular.

(2) for all $A, B \subseteq S$ if $c \in \mathrm{cl}(AB)$ then there exists $a \in \mathrm{cl}(A)$ and $b \in \mathrm{cl}(B)$ such that $c \in \mathrm{cl}(a, b)$.

(3) for all A, B, C: $A \mathop{\smile\!\!\!\!\!\!\;|}^{cl}_C B$ if and only if $\mathrm{cl}(AC) \cap \mathrm{cl}(BC) = \mathrm{cl}(C)$.

(4) for all A, B, C such that $A = \mathrm{cl}(A)$, $B = \mathrm{cl}(B)$ and $C = \mathrm{cl}(C)$, if $C \subseteq B$ then $\mathrm{cl}(AB) \cap C = \mathrm{cl}(\mathrm{cl}(A \cap C), C)$.

Throughout, we will refer to any of these properties using "by modularity".

Example 1.3.6 (Algebraically closed fields)**.** Let K be an algebraically closed field and cl the closure operator defined for $A \subseteq K$ by $\mathrm{cl}(A) = \overline{\mathbb{F}(A)}$ where $\overline{\mathbb{F}(A)}$ is the algebraic closure (in K) of the subfield of K generated by A and the prime field $\mathbb{F}$. Then (K, cl) defines a pregeometry. The dimension is the transcendence degree, it is not a modular pregeometry (see [Bou06b, A.V.110, §3]).

Example 1.3.7 (Vector spaces)**.** Let V be a vector space over some field k and defined the closure operator $\langle A \rangle$ to be the span of $A \subseteq V$. Then (V, cl) defines a modular pregeometry. The dimension is the dimension as a k-vector space.

In a model theoretical context, a closure operator likely to define a pregeometry in a model is the model-theoretic algebraic closure, as it always satisfies *Reflexivity*, *Finite Character* and *Transitivity*.

Definition 1.3.8. A theory T is *pregeometric* if $(\mathscr{M}, \mathrm{acl}^{\mathscr{M}}(\cdot))$ defines a pregeometry, for all model $\mathscr{M}$ of T. We denote by $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ the associated independence relation. We say that T *eliminates* $\exists^\infty$ if for all formula $\phi(x,y)$ there is an integer $n \in \mathbb{N}$ such that for all $|y|$-tuple b in any model $\mathscr{M}$ of T, $\phi(\mathscr{M}, b)$ is either infinite or of cardinality less than n. A pregeometric theory that eliminates $\exists^\infty$ is called *geometric*.

Note that if a theory is geometric, it does not mean that the algebraic closure defines a geometry, see [TZ12, Appendix C] for a definition of a geometry.

Fact 1.3.9 ([Gag05]). *Let T be a pregeometric theory and $\mathbb{M}$ a monster model for T. For all B small subset of $\mathbb{M}$ and finite tuple x there exists a partial type $p_B(x)$ such that a realizes p_B if and only if a is $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$-independent over B. Furthermore for any type q in and expansion of $\mathscr{M}$, and $B \subseteq D$, if $q \cup p_B$ is consistent, then so is $q \cup p_D$. The relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ satisfies* Invariance, Finite Character, Symmetry, Closure, Monotonicity, Base Monotonicity, Transitivity, Existence, Full Existence *and* Extension.

Proof. The first two assertions are in [Gag05], the fact that q can be choosen in an expansion of $\mathscr{M}$ follows easily by inspection of the proof. A consequence of the first part is that $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}_C b$ is type-definable for every basis a of $\mathrm{acl}(Ca)$ over C and b basis of $\mathrm{acl}(Cb)$ over C. As any automorphism fixes (setwise) the algebraic closure, it follows that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ satisfies Invariance. We prove that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ satisfies Extension, the rest follows from Remark 1.2.6, and Fact 1.3.3. Assume that for some finite a, $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}_C B$ and D is arbitrary. Let a' be a basis of $\mathrm{acl}(CBa)$ over CB. As a' realizes $tp(a'/CB) \cup p_{CB}(x)$, the type $tp(a'/CB) \cup p_{CBD}$ is consistent, let a'' be a realisation. A CB-automorphism sending a' to a'' sends a to some $\tilde{a}$ such that $\tilde{a} \subseteq \mathrm{acl}(a''CB)$. As $a'' \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}_C BD$, by Closure, Symmetry and Monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$, we have $\tilde{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}_C BD$. □

Let T be a pregeometric theory with monster $\mathbb{M}$, b a tuple from $\mathbb{M}$ and $\phi(x,b)$ a formula. By $\dim(\phi(x,b))$ we mean the maximum dimension of $\mathrm{acl}(cb)$ over b, for realisations c of $\phi(x,b)$.

Fact 1.3.10. *Let T be a geometric theory and $\mathbb{M}$ a monster model for T. Then for all formula $\phi(x,y)$ there exists a formula $\theta_\phi(y)$ such that $\theta_\phi(b)$ holds if and only if there exists a realisation a of $\phi(x,b)$ which is an $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$-independent tuple over* $\mathrm{acl}_T(b)$.

Proof. From [Gag05, Fact 2.4], for each $k \leq |x|$ there exists a formula $\theta_k(y)$ such that $\theta_k(b)$ if and only if $\dim(\phi(\mathbb{M}, b)) = k$. The formula $\theta_{|x|}(y)$ holds if and only if there is a realisation a of $\phi(x,b)$ such that $\dim(\mathrm{acl}(ab)/b) = |x|$, hence it is $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$-independent over $\mathrm{acl}_T(b)$. □

Note that a reduct of a pregeometric theory is pregeometric, and the reduct of a geometric theory is also a geometric theory (see [Hil08, Fait 2.15]).

1.4 Classification Theory

Let $\mathcal{T}$ be a tree (such as $2^{<\omega}, \omega^{<\omega}, \kappa^{<\lambda}$). We denote by $\lhd$ the natural partial order on $\mathcal{T}$. For $\nu, \eta \in \kappa^{<\lambda}$ we denote by $\nu^\frown\eta$ the concatenation of the two.

1.4.1 Stable and simple theories

Definition 1.4.1. Let T be a complete theory, $\mathbb{M}$ a monster model of T. Let $\phi(x, y)$ be a formula.

- We say that $\phi(x, y)$ has the *order property* (or is *unstable*) if there are two indiscernible sequences $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ in $\mathbb{M}$ such that $\mathbb{M} \models \phi(a_i, b_j)$ if and only if $i < j$. A theory T is *stable* if no formula in any monster model $\mathbb{M}$ is unstable.
- We say that $\phi(x, y)$ has the *tree property* (TP) if there is a sequences $(a_\mu)_{\mu\in\omega^{<\omega}}$ in $\mathbb{M}$ and $k \in \mathbb{N}$ such that

  (1) $\{\phi(x, a_{\mu^\frown i}) \mid i < \omega\}$ is k-inconsistent, for all $\mu \in \omega^{<\omega}$;

  (2) $\{\phi(x, a_{s\restriction n}) \mid n \in \mathbb{N}\}$ is consistent, for any $s \in \omega^\omega$.

  A theory T is *simple* if no formula in any monster model $\mathbb{M}$ has the tree property.

Fact 1.4.2 ([She90, Theorem 2.13])**.** *A theory T is stable if and only if all formulas $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ are stable.*

Lemma 1.4.3. *Let $\mathscr{L}$ be any language and let T be an unstable $\mathscr{L}$-theory with monster model $\mathbb{M}$. Let $\mathscr{L}^- \subseteq \mathscr{L}$ be such that $T\restriction_{\mathscr{L}^-}$ is stable. Then there exists an $\mathscr{L}$-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ and a tuple b from $\mathbb{M}$ such that $\phi(x, b)$ is not $\mathscr{L}^-$-definable with parameters in $\mathbb{M}$.*

Proof. By Fact 1.4.2 there is an unstable $\mathscr{L}$-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$. By Ramsey and compactness (see e.g. [TZ12, Lemma 7.1.1]) we may assume that $(a_i)_{i\in\mathbb{Z}}$, $(b_i)_{i\in\mathbb{Z}}$ are two indiscernible sequences in $\mathbb{M}$ that witness the unstability of $\phi(x, y)$, i.e., $\phi(a_i, b_j)$ if and only if $i < j$. Assume towards a contradiction that $\phi(x, b_0)$ is definable by an $\mathscr{L}^-$-formula $\psi(x, c_0)$ with parameters c_0 in $\mathbb{M}$. For each $k \in \mathbb{Z}\setminus \{0\}$, as $tp(b_k/\emptyset) = tp(b_0/\emptyset)$ there is an automorphism σ_k such that $\sigma_k(b_0) = b_k$. Let $c_k = \sigma_k(c_0)$. Then $\phi(x, b_k)$ is equivalent to $\psi(x, c_k)$, and hence $\psi(a_i, c_j)$ if and only if $i < j$, a contradiction to the stability of $T\restriction_{\mathscr{L}^-}$. □

There is a "geometric" characterization of stable theories, which appears first in [HH84]. We give a modern presentation, see [Cas11, Theorem 12.22].

Fact 1.4.4 (Harnik-Harrington, characterisation of forking and stable theories)**.** *Let T be a complete theory, and $\mathbb{M}$ a monster model. The theory T is stable if and only if there is a ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ defined over small subsets which satisfies:* Invariance, Finite Character, Symmetry, Closure, Monotonicity, Base Monotonicity, Transitivity, Extension, Local Character *and* Stationnarity *over models. If such a relation $\mathop{\smile\!\!\!\!\!\!|}$ exists, $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^f = \mathop{\smile\!\!\!\!\!\!|}^d$.*

Proof. One only needs to check that our set of axioms is equivalent to the set of axioms in [Cas11, Theorem 12.22]. An *independence relation* in the sense of [Cas11] satisfies SYMMETRY ([Cas11, Corollary 12.6]). We need to check first that our set of axioms implies the property *Normality*: $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ implies $AC \mathop{\smile\!\!\!\!\!\!\mid}_C B$, which follows from INVARIANCE, EXTENSION, MONOTONICITY and SYMMETRY (using Lemma 1.2.5). We also need to check that the set of axioms in [Cas11] implies our, the only property one needs to check is that CLOSURE follows from the set of axioms in [Cas11], which is Lemma 1.2.5. □

In a stable theory, $\mathop{\smile\!\!\!\!\!\!\mid}^f$ coincides with the coheir independence $\mathop{\smile\!\!\!\!\!\!\mid}^u$ over models (see e.g. [TZ12]).

Fact 1.4.5. *Let T be a stable theory, $\mathbb{M}$ a monster model for T, $\mathscr{M} \prec \mathbb{M}$ and a, b tuples from $\mathbb{M}$. Then $a \mathop{\smile\!\!\!\!\!\!\mid}^f_{\mathscr{M}} b$ if and only if $a \mathop{\smile\!\!\!\!\!\!\mid}^u_{\mathscr{M}} b$.*

There is also a classical "geometric" characterization of simple theories ([KP97]).

Fact 1.4.6 (Kim-Pillay, characterization of forking and simple theories)**.** *Let T be a complete theory, and $\mathbb{M}$ a monster model. The theory T is simple if and only if there is a ternary relation $\mathop{\smile\!\!\!\!\!\!\mid}$ defined over small subsets which satisfies:* INVARIANCE, FINITE CHARACTER, SYMMETRY, CLOSURE, MONOTONICITY, BASE MONOTONICITY, TRANSITIVITY, EXTENSION, LOCAL CHARACTER *and* $\mathop{\smile\!\!\!\!\!\!\mid}$-AMALGAMATION *over models. If such a relation $\mathop{\smile\!\!\!\!\!\!\mid}$ exists, $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^f = \mathop{\smile\!\!\!\!\!\!\mid}^d$.*

Proof. This follows from [Cas11, Theorem 12.21]. Indeed, the first nine axioms in our statement are equivalent to the ones of an *independence relation* in the sense of [Cas11], as we saw in the proof of Fact 1.4.4. $\mathop{\smile\!\!\!\!\!\!\mid}$-AMALGAMATION over models is equivalent to the Independence Theorem over models, for any relation $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfying SYMMETRY, BASE MONOTONICITY, TRANSITIVITY and which is stronger than $\mathop{\smile\!\!\!\!\!\!\mid}^a$, which is the case for $\mathop{\smile\!\!\!\!\!\!\mid}^f$ is a simple theory by Fact 1.2.3. □

1.4.2 NSOP$_1$ theories and Kim-independence

Definition 1.4.7. Let T be a theory, $\mathbb{M}$ a monster for T and $\phi(x, y)$ a formula in the language of T. We say that $\phi(x, y)$ has the 1-*strong order property* (SOP$_1$) if there exists a tree of tuple $(b_\eta)_{\eta \in 2^{<\omega}}$ such that

- for all $\eta \in 2^\omega$ $\{\phi(x, b_{\eta \restriction \alpha} \mid \alpha < \omega\}$ is consistent
- for all $\eta \in 2^{<\omega}$ if $\eta^\frown 0 \lhd \nu$ then $\{\phi(x, b_\nu), \phi(x, b_{\eta^\frown 1}\}$ is inconsistent.

If in any monster model $\mathbb{M}$ of T, no formula has SOP$_1$, then the theory is called NSOP$_1$.

Recall the definitions of Kim-dividing and Kim-forking for types.

Definition 1.4.8. (1) *Kim-dividing independence.* $a \mathop{\smile\!\!\!\!\!\!\mid}^{Kd}_C b$ if and only if $tp(a/Cb)$ does not Kim-divides over C if and only if for all global C-invariant extension p of $tp(b/C)$ and sequences $(b_i)_{i<\omega}$ such that $b_i \models p \restriction Cb_{<i}$ for all $i < \omega$, there exists a' such that $a'b_i \equiv_C ab$ for all $i < \omega$;

(2) *Kim-(forking)independence.* $a \mathop{\smile\!\!\!\!\!\!\mid}^K_C b$ if and only if $tp(a/Cb)$ does not Kim-fork over C if and only if for any $b' \supseteq b$ there exists $a' \equiv_{Cb} a$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}^{Kd}_C b'$;

(3) A formula $\phi(x,b)$ *Kim-divides* over C if there is a global C-invariant extension p of $tp(b/C)$ and a sequence $(b_i)_{i<\omega}$ such that $b_i \models p \upharpoonright Cb_{<i}$ for all $i < \omega$, with $\{\phi(x,b_i) \mid i < \omega\}$ inconsistent;

(4) A formula $\phi(x,b)$ *Kim-forks* over C if it implies a finite disjunction of Kim-dividing formulae.

Note that $a \mathop{\smile\!\!\!\!\!\!\mid}^K_C b$ if and only if for all finite $a' \subseteq a$, no formula in $tp(a'/Cb)$ Kim-forks over C.

Remark 1.4.9. Given any b and C, a global C-invariant extension of $tp(b/C)$ need not exists. When considering Kim-independence, we will in general assume that the base set is a model, so that $tp(b/\mathscr{M})$ has a global extension finitely satisfiable in $\mathscr{M}$ hence $\mathscr{M}$-invariant. If $tp(b/C)$ has no global C invariant extension, then $a \mathop{\smile\!\!\!\!\!\!\mid}^K_C b$ for all a.

Fact 1.4.10 (Kim's Lemma for Kim-dividing [KR17, Theorem 3.16])**.** *Let T be an* NSOP$_1$ *theory. Then for all formula $\phi(x,b)$, with b in a monster $\mathbb{M}$ of T and $C \subseteq \mathbb{M}$, $\phi(x,b)$ Kim-divides over C if and only if for all global C-invariant extension p of $tp(b/C)$ and a sequences $(b_i)_{i<\omega}$ such that $b_i \models p \upharpoonright Cb_{<i}$ for all $i < \omega$, the set $\{\phi(x,b_i) \mid i < \omega\}$ inconsistent. This is actually equivalent to T being* NSOP$_1$.

There is also a recent "geometric" characterisation of NSOP$_1$ by Kim-independence (Definition 1.2.1), see [CR16], [KR17], [KR18].

Fact 1.4.11 (Chernikov-Kaplan-Ramsey, characterisation of Kim-independence and NSOP$_1$ theories)**.** *Let T be a complete theory, and $\mathbb{M}$ a monster model. The theory T is* NSOP$_1$ *if and only if there is a ternary relation $\mathop{\smile\!\!\!\!\!\!\mid}$ which is defined when the base set is a model, which satisfies:* Invariance*,* Symmetry*,* Monotonicity*,* Existence*,* Strong Finite Character *over models,* $\mathop{\smile\!\!\!\!\!\!\mid}$-amalgamation *over models and* Witnessing*. If such a relation $\mathop{\smile\!\!\!\!\!\!\mid}$ exists, $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^K = \mathop{\smile\!\!\!\!\!\!\mid}^{Kd}$.*

Proof. Only Witnessing and $\mathop{\smile\!\!\!\!\!\!\mid}$-amalgamation differs from the properties in the statement of [KR17, Theorem 9.1]. It is clear that our system of axioms is stronger than the one in [KR17, Theorem 9.1], we need to show that they are equivalent. If T is NSOP$_1$, by [KR18, Theorem 2.21] $\mathop{\smile\!\!\!\!\!\!\mid}^K$ satisfies the *Algebraically reasonable independence theorem*, which is exactly $\mathop{\smile\!\!\!\!\!\!\mid}$-amalgamation over models. Also, $\mathop{\smile\!\!\!\!\!\!\mid}^K = \mathop{\smile\!\!\!\!\!\!\mid}^{Kd}$. Assume that $a \not\mathop{\smile\!\!\!\!\!\!\mid}^{Kd}_{\mathscr{M}} b$, then there is a formula $\Lambda(x,b) \in tp(a/\mathscr{M}b)$ and a sequence $(b_i)_{i<\omega}$ in a global $\mathscr{M}$-invariant extension of $tp(b/\mathscr{M})$ such that $\{\Lambda(x,b_i) \mid i < \omega\}$ is inconsistent. By Kim's Lemma (Fact 1.4.10, this actually holds for all global $\mathscr{M}$-invariant extension of $tp(b/\mathscr{M})$ hence in particular for any global coheir of $tp(b/\mathscr{M})$ (which exists), hence our version of Witnessing holds. $\square$

Finally, NSOP$_1$ and simple theories are linked by the following.

Fact 1.4.12 ([KR17, Propositions 8.4 and 8.8])**.** *Let T be an* NSOP_1 *theory and* $\mathop{\smile\!\!\!\!\!\!|}^K$ *the Kim-independence. Then T is simple if and only if* $\mathop{\smile\!\!\!\!\!\!|}^K = \mathop{\smile\!\!\!\!\!\!|}^f$ *over models, if and only if* $\mathop{\smile\!\!\!\!\!\!|}^K$ *satisfies* BASE MONOTONICITY.

1.4.3 dp-rank, dp-minimality

We first review two equivalent definitions of dp-rank. More details about dp-rank can be found, e.g. in [Sim15]. Let $\mathbb{M}$ be a monster model of some complete $\mathscr{L}$-theory T.

Definition 1.4.13. A familly of sequences $(I_i)_{i\in S}$ is called *mutually indiscernible over* B if for all $i \in S$, the sequence I_i is indiscernible over $B(I_j)_{j\neq i}$.

Definition 1.4.14. Let $\phi(x, b)$ be an $\mathscr{L}$-formula, with parameters b from $\mathbb{M}$, and let λ be a (finite or infinite) cardinal. We say dp-rank$(\phi(x, b)) < \lambda$ if for every family $(I_i : i < \lambda)$ of mutually indiscernible sequences over b and $a \models \phi(x, b)$, there is $i < \lambda$ such that I_i is indiscernible over ab. We say that dp-rank$(\phi(x, b)) = \lambda$ if dp-rank$(\phi(x, b)) < \lambda^+$ but not dp-rank$(\phi(x, b)) < \lambda$. We say that dp-rank$(\phi(x, b)) \leq \lambda$ if dp-rank$(\phi(x, b)) < \lambda$ or dp-rank$(\phi(x, b)) = \lambda$. For a theory T we denote by dp-rank(T) the dp-rank of $(x = x)$ where $|x| = 1$. If dp-rank$(T) = 1$ we say that T is *dp-minimal*.

Note that if λ is a limit cardinal, it may happen that dp-rank$(\phi(x, b)) < \lambda$ but dp-rank$(\phi(x, b)) \geq \mu$ for all $\mu < \lambda$ (see [Sim15, Section 4.2]).

Definition 1.4.15. Let κ be some cardinal. An *ict-pattern of length* κ consists of:

- a collection of formulas $(\phi_\alpha(x; y_\alpha) : \alpha < \kappa)$, with $|x| = 1$;
- an array $(b_i^\alpha : i < \omega, \alpha < \kappa)$ of tuples; with $|b_i^\alpha| = |y_\alpha|$

such that for every $\eta : \kappa \to \omega$ there exists an element $a_\eta \in \mathbb{M}$ such that

$$\models \phi_\alpha(a_\eta; b_i^\alpha) \iff \eta(\alpha) = i.$$

We define κ_{ict} as the minimal κ such that there does not exist an ict-pattern of length κ.

Fact 1.4.16 ([Sim15, Proposition 4.22])**.** *For any cardinal κ, we have dp-rank$(T) < \kappa$ if and only if $\kappa_{ict} \leq \kappa$.*

Lemma 1.4.17. *Let $\mathscr{L} = \bigcup_{\alpha<\kappa} \mathscr{L}_\alpha$ be a language such that every atomic formula in $\mathscr{L}$ is in $\mathscr{L}_\alpha$ for some α. Let T be an $\mathscr{L}$-theory that eliminates quantifiers, and for $\alpha < \kappa$ let T_α be its reduction to $\mathscr{L}_\alpha$. Let μ_α be cardinals such that dp-rank$(T_\alpha) \leq \mu_\alpha$. Then dp-rank$(T) \leq \sum_{\alpha<\kappa} \mu_\alpha$, where $\sum$ is the cardinal sum.*

Proof. Suppose not. Let $\lambda := \sum_{\alpha<k} \mu_\alpha$. Then there is a family $(\mathcal{I}_t : t < \lambda^+)$ of mutually indiscernible sequences over $\emptyset$, $\mathcal{I}_t = (a_{t,i} : i \in I_t)$, and a singleton b, such that for all t, $\mathcal{I}_t$ is not indiscernible over b. For every $t < \lambda^+$, let $\phi_t(\bar{x}) = \phi_t(\bar{x}, b)$ be a formula over b and let $\bar{c}_{t,1}$ and $\bar{c}_{t,2}$ be two finite tuples of elements of $\mathcal{I}_t$ of length $|\bar{x}|$ such that $\phi_t(\bar{c}_{t,1})$ and $\neg\phi_t(\bar{c}_{t,2})$, i.e. witnessing the non-indiscernibility of $\mathcal{I}_t$ over b. By quantifier elimination in T, we may assume that ϕ_t is quantifier-free. Hence there must be an atomic formula $\psi_t(\bar{x}) = \psi_t(\bar{x}, b)$ such that $\psi_t(\bar{c}_{t,1})$ and $\neg\psi_t(\bar{c}_{t,2})$. By the assumption on $\mathscr{L}$, there is an $\alpha_t < \kappa$ such that $\psi_t(\bar{x}, y)$ is in $\mathscr{L}_{\alpha_t}$. Therefore, there must be an $\alpha < \kappa$ such that $|\{t < \lambda^+ : \alpha_t = \alpha\}| > \mu_\alpha$, as otherwise we get

$$\lambda^+ = \left|\bigcup_{\alpha<\kappa} \{t < \lambda^+ : \alpha_t = \alpha\}\right| \leq \sum_{\alpha<\kappa} \left|\{t < \lambda^+ : \alpha_t = \alpha\}\right| \leq \sum_{\alpha<\kappa} \mu_\alpha = \lambda,$$

a contradiction. But then $(\mathcal{I}_t : t < \lambda^+, \alpha_t = \alpha)$ is a family of more than μ_α mutually indiscernible sequences over $\emptyset$ with respect to $\mathscr{L}_\alpha$, and for all t such that $\alpha_t = \alpha$, $\mathcal{I}_t$ is not indiscernible over b with respect to $\mathscr{L}_\alpha$, a contradiction to dp-rank$(T_\alpha) \leq \mu_\alpha$. $\square$

1.5 Preliminaries on fields

1.5.1 Generalities

We recall some definitions from classical field theory, as can be found e.g. in [FJ05, Chapter 2]. We assume that all fields considered are subfields of a big algebraically closed field, and we denote by $\mathbb{F}$ the prime field. For a field K we will denote by K^{alg} or $\overline{K}$ and K^s respectively the algebraic closure and the separable closure of K, i.e. the field consisting of all elements algebraic (respectively separably algebraic) over K. We denote by K^{ins} the maximal purely inseparable extension of K, if $char(K) = 0$ then $K = K^{ins}$, if $char(K) = p > 0$, K^{ins} is the field generated by $\left\{\alpha \mid \alpha^{p^{-n}} \in K,\ n \in \mathbb{N}\right\}$. We denote by L/K the fact that L is an extension of the field K. Given two fields L and K, we denote by LK the compositum of L and K. For a tuple a, $K(a)$ is the field generated by K and a. Given a prime number p and $n \in \mathbb{N}$, the field of cardinality p^n will be denoted by $\mathbb{F}_{p^n}$.

Definition 1.5.1. Let K, L be two field extensions of a field E.

(1) We say that K is *linearly disjoint* from L over E (denoted by $K \mathop{\smile\!\!\!\!\!\!|}^{ld}_E L$) if every finite tuple from K which is linearly independent over E is also linearly independent over L in the compositum KL.

(2) We say that K is *algebraically independent* from L over E (denoted by $K \mathop{\smile\!\!\!\!\!\!|}^{alg}_E L$) if every finite tuple from K which is algebraically independent over E is also algebraically independent over L in the compositum KL.

(3) An extension L/K is called *separable* if $L \mathop{\smile\!\!\!\!\!\!|}^{ld}_K K^{ins}$. It is called *regular* if $L \mathop{\smile\!\!\!\!\!\!|}^{ld}_K K^{alg}$.

The definitions of $\mathop{\smile\!\!\!\!\!\!|}^{ld}$ and $\mathop{\smile\!\!\!\!\!\!|}^{alg}$ turn out to be symmetric, and we will sometimes say that K and L are linearly disjoint (or algebraically independent) over E. These are notions of independence only defined over fields. An easy way of extending their definition is by setting for every A, B, E subsets of some big field, $A \mathop{\smile\!\!\!\!\!\!|}^{ld}_E B$ if and only if $\mathbb{F}(AE) \mathop{\smile\!\!\!\!\!\!|}^{ld}_{\mathbb{F}(E)} \mathbb{F}(BE)$, and similarly with $\mathop{\smile\!\!\!\!\!\!|}^{acl}$. With this extended definition, in any field

F with prime field $\mathbb{F}$, the ternary relations $\mathop{\smile\hskip-0.9em ^| \ }^{ld}$ and $\mathop{\smile\hskip-0.9em ^| \ }^{alg}$ are defined over every subsets of F. Note that if K is an algebraically closed field, $\mathop{\smile\hskip-0.9em ^| \ }^{alg}$ defined over subsets of K is the independence relation associated with the pregeometry described in Example 1.3.6.

Fact 1.5.2. *$\mathop{\smile\hskip-0.9em ^| \ }^{ld}$ and $\mathop{\smile\hskip-0.9em ^| \ }^{alg}$ satisfy* Symmetry, Finite Character, Monotonicity, Transitivity *and* Base Monotonicity*. Furthermore $\mathop{\smile\hskip-0.9em ^| \ }^{alg}$ satisfies* Closure*: if $K \mathop{\smile\hskip-0.9em ^| \ }^{alg}_{E} L$ then $K \mathop{\smile\hskip-0.9em ^| \ }^{alg}_{\overline{E}} \overline{L}$. We have $\mathop{\smile\hskip-0.9em ^| \ }^{ld} \to \mathop{\smile\hskip-0.9em ^| \ }^{alg}$.*

Proof. For $\mathop{\smile\hskip-0.9em ^| \ }^{ld}$, Symmetry is [FJ05, Lemma 2.5.1], Monotonicity, Base Monotonicity and Transitivity follow from [FJ05, Lemma 2.5.3]. Finite Character is by definition. For $\mathop{\smile\hskip-0.9em ^| \ }^{alg}$, it is Fact 1.3.3. The last assertion follows from the simple fact that a tuple is algebraically dependent over some field if and only if the familly of monomials of this family is linearly independent over this field [FJ05, p. 41]. $\square$

Remark 1.5.3. Note that $A \mathop{\smile\hskip-0.9em ^| \ }^{ld}_{C} B$ implies $\mathbb{F}(AC) \cap \mathbb{F}(BC) = \mathbb{F}(C)$ whereas $A \mathop{\smile\hskip-0.9em ^| \ }^{alg}_{C} B$ implies $\overline{\mathbb{F}(AC)} \cap \overline{\mathbb{F}(BC)} = \overline{\mathbb{F}(C)}$.

We deduce the following classical fact:

Fact 1.5.4. *Let $E \subset K \subset L$ be three fields. Assume that L/K is separable (respectively regular). Then L/E is separable (resp. regular) if and only if K/E is separable (resp. regular).*

The relations $\mathop{\smile\hskip-0.9em ^| \ }^{ld}$ and $\mathop{\smile\hskip-0.9em ^| \ }^{alg}$ coincide when one of the extension is regular.

Fact 1.5.5 ([FJ05, Lemma 2.6.7])**.** *Let $E \subset K \cap L$ be three fields. If K/E is regular, then $K \mathop{\smile\hskip-0.9em ^| \ }^{ld}_{E} L$ if and only if $K \mathop{\smile\hskip-0.9em ^| \ }^{alg}_{E} L$.*

Fact 1.5.6 ([Cha99, Lemma 3.1 (1)])**.** *Let $E \subset K \cap L$ be three fields. If K/E, L/E are regular and $K \mathop{\smile\hskip-0.9em ^| \ }^{ld}_{E} L$ then $K^s \mathop{\smile\hskip-0.9em ^| \ }^{ld}_{E^s} L^s$.*

Lemma 1.5.7. *Let A, B be two extensions of some field E, such that AB/E is regular and $A \mathop{\smile\hskip-0.9em ^| \ }^{ld}_{E} B$. Then $(A^s + B^s) \cap AB = A + B$.*

Proof. First, observe that $A^sB \cap B^s = E^sB$. Indeed A/E and B/E are regular so by Fact 1.5.6, we have that $A^s \mathop{\smile\hskip-0.9em ^| \ }^{ld}_{E^s} B^s$ hence $A^sB \mathop{\smile\hskip-0.9em ^| \ }^{ld}_{E^sB} B^s$ and so $A^sB \cap B^s = E^sB$. Symmetrically, we have $AB^s \cap A^s = E^sA$. If $v \in AB$ is such that $v = \alpha + \beta$ for $\alpha \in A^s$ and $\beta \in B^s$, then $\alpha = v - \beta \in AB^s \cap A^s = E^sA$. Similarly $\beta \in E^sB$. Let L be a finite extension of E inside E^s such that $\alpha \in AL$ and $\beta \in BL$. We can complete $\{1\}$ to a basis $\{1, u_2, \ldots, u_n\}$ of the E-vector space L. As $AB \mathop{\smile\hskip-0.9em ^| \ }^{ld}_{E} L$, it is also a basis of the AB-vector space LAB. As $AB \mathop{\smile\hskip-0.9em ^| \ }^{ld}_{A} LA$ and $AB \mathop{\smile\hskip-0.9em ^| \ }^{ld}_{B} LB$, it is also a basis of the A-vector space LA and of the B-vector space LB. Now the coordinates of $v \in AB$ in the AB-vector space LAB are $(v, 0, \ldots, 0)$ as $v = v + 0u_2 + \cdots + 0u_n$. Let $(a_1, \ldots, a_n)$ (respectively $(b_1, \ldots, b_n)$) be the coordinates of α with respect to the basis $(1, u_2, \ldots, u_n)$ of the A-vector space LA (respectively of β in this basis of the B-vector space LB). As $v = \alpha + \beta$, we have, looking at the first coordinate that $v = a_1 + b_1$, so $v \in A + B$. $\square$

Lemma 1.5.8. *Let K be a field and $K(X, Y)$ be a rational function field in two variables (in other words $X \mathop{\smile\!\!\!\!\!\!\!\!|}^{alg}_{K} Y$ and $X, Y \notin \overline{K}$). Then*

$$XY \notin K(X) + K(Y);$$

$$X + Y \notin K(X) \cdot K(Y);$$

where $K(X) \cdot K(Y) = \{uv \mid u \in K(X), v \in K(Y)\}$.

Proof. There exists a derivative $D : K(X, Y) \to K(X, Y)$ such that $D(K(Y)) = \{0\}$ and D extends the canonical derivation on $K(X)$ (namely the *partial* derivative with respect to X, see [Mor96, Proposition 23.11]). Let $u \in K(X)$ and $v \in K(Y)$. If $XY = u+v$ then applying D we get $Y = Du \in K(X)$ a contradiction. If $X+Y = uv$ then applying D we get $1 = vDu$ hence, as $Du \in K(X)$, $v \in K(X) \cap K(Y) = K$. Now $Y = uv - X \in K(X)$ a contradiction. □

1.5.2 Fields and model theory

We denote by $\mathscr{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$ the language of rings. The following is [Cha99, (1.17)].

Fact 1.5.9. *Let T be any theory of fields in any language $\mathscr{L} \supseteq \mathscr{L}_{\text{ring}}$. Let $F \models T$ and $A \subseteq F$. Then $F/acl_T(A)$ is a regular extension.*

The following gives a behaviour of the Kim-independence in *any* theory of fields.

Fact 1.5.10 ([KR17, Proposition 9.28], [Cha99, Theorem 3.5])**.** *Let T be an arbitrary theory of fields, and $E \prec F \models T$. Let A, B be acl_T-closed subsets of F containing E, such that $A \mathop{\smile\!\!\!\!\!\!\!\!|}^{K}_{E} B$. Then*

(1) $A \mathop{\smile\!\!\!\!\!\!\!\!|}^{ld}_{E} B$;

(2) F/AB is a separable extension;

(3) $\mathrm{acl}_T(AB) \cap A^s B^s = AB$.

Lemma 1.5.11. *Let T be an arbitrary theory of fields, and $F \models T$. Let A, B, C, D be subsets of F, containing some set $k \subseteq F$, and such that $A, B \subseteq D$. Assume that A and B are acl_T-closed and $D \mathop{\smile\!\!\!\!\!\!\!\!|}^{u}_{k} C$, (i.e. $tp^T(D/C)$ is finitely satisfiable in k). Then we have the following results.*

(1) $(F \cap (AC)^s + F \cap (BC)^s) \cap D = A + B$;

(2) $[(F \cap \overline{AC}) \cdot (F \cap \overline{BC}))] \cap D = A \cdot B$.

Assume now that $A, B \mathop{\smile\!\!\!\!\!\!\!\!|}^{u}_{A \cap B} F \cap \overline{AC} \cap \overline{BC}$. Then $F \cap \overline{AC} \cap \overline{BC} = F \cap \overline{(A \cap B)C}$.

Proof. We give the idea for *(1)*, the others are proved by a similar argument. Let $v_1 \in F \cap (AC)^s$, $v_2 \in F \cap (BC)^s$ and $u \in D$ be such that $u = v_1 + v_2$. There exist nontrivial separable polynomials $P(X, a, c)$ and $Q(X, b, c')$ with leading coefficients 1 such that v_1 is a root of $P(X, a, c)$ and v_2 is a root of $Q(X, b, c')$, a a tuple in A, b a tuple in B. The formula $\phi(z_1, z_2, z_3, c, c')$

$$\exists x \exists y \; x + y = z_1 \wedge P(x, z_2, c) = 0 \wedge Q(y, z_3, c') = 0$$

is in $tp^T(u, a, b/C)$, which is finitely satisfiable in k. Hence, there exists $d, d' \in k$ such that $\phi(z_1, z_2, z_3, d, d') \in tp^T(u, a, b/k)$ and so $u \in A + B$ as A and B are acl_T-closed. □

The theory ACF. Let ACF be the theory of algebraically closed fields in $\mathscr{L}_{\mathrm{ring}}$. We recall here some basic facts about this well-known theory, as can be found e.g. in [Bou+98]. ACF is model-complete, it is the model-companion of the theory of fields in $\mathscr{L}_{\mathrm{ring}}$. It is not complete but its completions are given by specifying the characteristic p of the field, we denote the completion obtained by ACF_p. ACF_p is *strongly minimal*, so in particular it is stable. Let $K \models \mathrm{ACF}_p$. A *Zariski-closed* subset of K^n is the set of solutions of a finite number of polynomial equations in $(X_1, \cdots, X_n)$. Those are closed subsets of a topology on all cartesian powers of K called the Zariski topology. An *affine (irreducible) variety* is a Zariski-closed set that cannot be written as the union of two proper Zariski-closed sets. Every Zariski-closed set can be decomposed into the union of finitely many affine varieties (the topology is *Noetherian*). A *quasi-affine variety* is an open subset of an affine variety, hence a set of solutions of some polynomial equations and some polynomial inequations. The theory ACF_p has quantifier eliminations in the language of rings, this means that every definable set in an algebraically closed field K is a finite union of quasi-affine varieties. A *generic* $\mathbf{x}$ of some quasi-affine variety V is a tuple in an elementary extension of K such that if $P(\mathbf{x}) = 0$ for some polynomial P with coefficients in K, then V is included in the Zariski-closed set defined by P. Informally, $\mathbf{x}$ satisfies no other equations than the one defining V. Generic points of a variety $V \subset K^n$ always exists in elementary extensions of K. We will not need much those notions except in Section 3.3. For a field K of characteristic p, the *Frobenius endomorphism* is the field endomorphism of K defined by $\mathrm{Frob} : x \mapsto x^p$.

Fact 1.5.12. *Let $K \models \mathrm{ACF}_p$. If $\xi : K \to K$ is an additive definable endomorphism, then ξ is of the form $\xi(x) = a_1 \mathrm{Frob}^{n_1}(x) + \cdots + a_k \mathrm{Frob}^{n_k}(x)$, with $n_1, \cdots, n_k \in \mathbb{Z}$.*

Proof. By [Bou+98, Chapter 4, Corollary 1.5], a definable map is given by a composition of powers of the Frobenius and rational maps, on a definable partition of K, and by [Hum98, Lemma A, VII, 20.3], additive polynomials are p-polynomials. It is easy to see that the fact follows. □

The theory SCF. If K is of characteristic $p > 0$, we denote by K^p the image of K by the Frobenius endomorphism. If K is separably closed and *perfect* (i.e. if K is of characteristic 0 or K is of characteristic p and $K^p = K$), K is an algebraically closed field. We assume that the characteristic of K is $p > 0$. Let $A \subseteq K$, the *p-closure of A* is the field $K^p(A)$. This defines a pregeometry on K (see for instance [Bou06a, Chapitre 5, §13]), a basis for this pregeometry is called a p-basis, and an independent set is called a p-independent set. A set A is p-independent if and only if for all finite tuple $a_1, \cdots, a_n$ from A, the set of monomials $a_1^{e_1} \cdot \cdots \cdot a_n^{e_n}$ are K^p-linearly independent, where $0 \leq n_k < p$. If K/K^p is a finite extension, it has degree p^e for some integer e, which we call the *Ershov* invariant of K (or imperfection degree). If K/K^p is infinite, we write $e = \infty$. Let $\mathscr{L}$ be the language of rings extended by n-ary relations Q_n. Let $\mathrm{SCF}_{p,e}$ be the theory of separably closed field of characteristic p and Ershov invariant e in the language $\mathscr{L}$ in which the relations Q_n represent p-independence.

Fact 1.5.13. *For all $e \leq \infty$, the theory $\mathrm{SCF}_{p,e}$ is complete, model-complete and stable. Furthermore, $\mathrm{SCF}_{p,e}$ eliminates $\exists^\infty$.*

Proof. The first part is Theorems 1 and 3 of [Woo79]. In [Del88, Proposition 61.] is proved that $\mathrm{SCF}_{p,e}$ has the NFCP, which implies elimination of $\exists^\infty$. ☐

Note that any model of $\mathrm{SCF}_{p,e}$ is existentially closed in every separable extension ([Bou+98, Chapter 9, Claim 2.2]). We have the following description of nonforking in the sense of $\mathrm{SCF}_{p,e}$ (see the remark after [Cha02, (1.2)]).

Fact 1.5.14. *Assume that A, B, C are separably closed subfields of a separably closed field F such that $C \subseteq A \cap B$. If $A \mathop{\smile\!\!\!\!\!\!\,|}^{ld}_C B$ and F/AB is separable, then $tp^{SCF_{p,e}}(A/B)$ does not fork over C.*

The theories ACFA_p, DCF_p. The theory ACFA_p is the model-companion of the theory of difference fields (i.e. fields with a distinguished endomorphism) of characteristic p, it was proved to be model-complete in [Mac97] and unstable but supersimple in [CH99] for any p prime or zero. DCF_p is the model-companion of differential fields of characteristic p (for $p = 0$, see [MMP96] and for $p > 0$, see [Woo73]) and is proved to be stable in [Woo76]. The theory ACFA_p eliminates $\exists^\infty$ in all characteristic, this follows easily from the definability of the σ-degree (see [CH99, Section 7]). For all p prime or 0, the theory DCF_p eliminates the quantifier $\exists^\infty$, this follows from the proof of this result in [MMP96, Theorem 2.13, p51], although it was proved in the characteristic 0 case, the proof works in all characteristics.

The theory PAC. A *pseudo algebraically closed field* is a field K which is existentially closed in every regular extension[2]. The property for a field to be pseudo algebraically closed is first order (see [FJ05]), we denote by PAC the corresponding theory. It is an incomplete theory, even when specifying the characteristic of the field (we denote

[2]The classical definition of a pseudo algebraically bounded field is the following: K is pseudo algebraically closed if every absolutely irreducible variety defined over K has a K-rational point (see [FJ05] or [TZ12]). We do not use this definition here and prefer the equivalent in term of regular extension since it is the main property that we will use about these fields. Note that these fields were also called *regularly closed*, which would be a better name for our purpose.

the corresponding theory by PAC_p). The theory of a PAC field K is described by the isomorphism type of the field $\mathrm{acl}(\emptyset)$, the imperfection degree of K and the "first-order theory of the absolute Galois group" (in a suitable ω-sorted language, for more details, see [CDM81]). A PAC field K is *bounded* if it has finitely many algebraic extensions of degree n, for all n. It is known that a PAC field has a simple theory if and only if it is bounded (see [CP98] for the "if" and [Cha99] for the "only if"). An *ω-free* PAC field is a PAC field K which has an elementary substructure K_0 whose absolute Galois group is isomorphic to the free profinite group with countably many generators. In [FJ05, Chapter 27] is presented a language and a theory of fields for which ω-free PAC_p fields of imperfection degree 1 (if $p > 0$) are the existentially closed models: expand $\mathscr{L}_{\mathrm{ring}}$ by n-ary predicates $R_n(x_1, \cdots, x_n)$ expressing that $\exists z\ z^n + x_1 z^{n-1} + \cdots + x_n = 0$. In this expanded laguage, K is a substructure of L if and only if K is algebraically closed in L. Then the theory of ω-free PAC fields of imperfection degree 1 (if $p > 0$) is the model-companion of the theory of fields in this expanded language.

Fact 1.5.15 ([Cha02], [CR16]). *Every ω-free* PAC *field has an* NSOP_1 *theory.*

A recent result from Nick Ramsey states that a PAC field is NSOP_1 provided its Galois group has an NSOP_1 theory.

A theory of fields T in an expansion of the language of rings is *algebraically bounded* if for all formula $\phi(x, y)$ with $|x| = 1$ there are polynomials $P_1(X, Y), \cdots, P_n(X, Y)$ in $\mathbb{Z}[X, Y]$ with $|X| = 1$ and $|Y| = |y|$ such that for all $K \models T$, and b a $|y|$-tuple from K, if $\phi(K, b)$ is finite then there exists i such that $P_i(X, b)$ is finite and $\phi(K, b)$ is contained in the set of roots of $P_i(X, b)$. In particular, an algebraically bounded field eliminates the quantifier $\exists^\infty$. This notion was introduced in [Dri89], it leads to the existence of a well-behaved notion of dimension on the definable sets, in particular, any algebraically bounded field must be perfect.

Fact 1.5.16 ([CH04]). *Every perfect* PAC *field is algebraically bounded.*

The theory Psf. It is the theory of *pseudo-finite fields* (see [Ax68] or [TZ12]), fields which are PAC, perfect and 1-free (i.e. has only one extension of degree n for all n). In particular, from Fact 1.5.16 it eliminates the quantifier $\exists^\infty$. From [TZ12, Proposition B.4.13], an extension L of a pseudo-finite field K is regular if and only if K is relatively algebraically closed in L (i.e. $L \cap \overline{K} = K$), hence a Psf field K is existentially closed in every extension L in which it is relatively algebraically closed. Any non-principal ultraproduct of finite fields is a pseudo-finite field. Let $\mathscr{L}$ be the language of rings expanded by constants symbols $(c_{i,j})_{i<\omega, j<i}$, and let Psf_c be the expansion of Psf expressing that the polynomial $X^n + c_{n,n-1}X + \cdots + c_{n,0}$ is irreducible. The theory Psf_c is model-complete, see [Cha97, Section 3].

1.6 The Chabauty topology on $\mathrm{Sg}(\overline{\mathbb{F}_p})$

Recall some standard facts about the topology on the Cantor space 2^ω, which can be found for instance in [Mos09] or [Kec95]. The Cantor space 2^ω is endowed with the product topology coming from the discrete topology on 2. This topology turns 2^ω into a *Polish* space; i.e. the topology admits a countable basis (*second-countable*) and admits a complete metric (*complete metrizable*). By Tychonoff's theorem, it is also *compact*, and it is *perfect*, i.e. without isolated points. It is also *totally disconnected* i.e. has no nontrivial connected subsets. Finally, a theorem of Brouwer states that 2^ω is the unique (up to homeomorphism) non-empty, totally disconnected, perfect, compact, metrizable set. Such a set is henceforth referred to as *a Cantor space*.

We fix a bijection : $e : \omega \to \overline{\mathbb{F}_p}$ and an enumeration $E = \{e_i := e(i) \mid i < \omega\}$. This enumeration induces a homeomorphism between the Cantor space 2^ω and $2^{\overline{\mathbb{F}_p}}$, hence turns the powerset $\mathscr{P}(\overline{\mathbb{F}_p})$ into a Cantor space. We give a description of the topology obtained on $\mathscr{P}(\overline{\mathbb{F}_p})$ based one the notion of *Cantor scheme*, i.e. a topology on the branchs of an infinite binary tree, as can be seen in [Kec95, Subsection I.6.A].

For each $A \subseteq \overline{\mathbb{F}_p}$, let 1_A be the function $1_A : E \to 2 := \{0, 1\}$ defined by

$$1_A(b) = 1 \iff b \in A.$$

For $k < \omega$ we adopt the following notation:

$$A \restriction k := (1_A(e_0), \cdots, 1_A(e_{k-1})) \in 2^k.$$

Let $k < \omega$ and $s \in 2^k$, we define $\mathcal{B}_s = \left\{A \subseteq \overline{\mathbb{F}_p} \mid A \restriction k = s\right\}$ and for $A \subseteq \overline{\mathbb{F}_p}$, let $\mathcal{B}(A, k)$ be $\left\{B \subseteq \overline{\mathbb{F}_p} \mid B \restriction k = A \restriction k\right\} = \mathcal{B}_{A \restriction k}$. The family $(\mathcal{B}_s)_{s \in 2^k, k < \omega}$ forms a basis of the topology on $\mathscr{P}(\overline{\mathbb{F}_p})$. A convenient way of getting a picture of this topology is by representing the subsets of $\mathscr{P}(\overline{\mathbb{F}_p})$ by branchs of a binary tree in which each level of nodes represent an element of the enumeration E of $\overline{\mathbb{F}_p}$. A branch representing $A \in \mathscr{P}(\overline{\mathbb{F}_p})$ takes the value 1 at the node of level e_i if and only if $e_i \in A$. Hence, in Figure 1.1, the set A contains e_0, e_1, e_2 but not e_3. For $s \in 2^k$, the ball $\mathcal{B}_s$ contains all the sets that are represented by branchs that start with the sequence s. It is easy to see that each ball is clopen (closed and open).

The set of all subgroups of some countable group can be endowed with a topology that is compact, it is called the *Chabauty* topology. In the case of the group $(\overline{\mathbb{F}_p}, +)$, this topology has a very explicit description, in particular, it is the topology of a Cantor set. More generally the Chabauty topology of any countable group is the one of a Cantor space provided that the group is not *minimax*, see [CGP10, Proposition A].

Lemma 1.6.1. *Let* $\mathrm{Sg}(\overline{\mathbb{F}_p}) \subseteq \mathscr{P}(\overline{\mathbb{F}_p})$ *be the set of all subgroups of* $(\overline{\mathbb{F}_p}, +)$*. Then* $\mathrm{Sg}(\overline{\mathbb{F}_p})$ *is a compact subset of* $\mathscr{P}(\overline{\mathbb{F}_p})$*. Furthermore, it is a Cantor space, the topology is generated by clopens sets of the form* $\mathcal{B}(H_0, \mathbb{F}_{p^n}) = \left\{H \in \mathrm{Sg}(\overline{\mathbb{F}_p}) \mid H \cap \mathbb{F}_{p^n} = H_0\right\}$*, for some finite group* $H_0 \in \mathrm{Sg}(\overline{\mathbb{F}_p})$*.*

Proof. First, we show that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is compact. As $\mathscr{P}(\overline{\mathbb{F}_p})$ is compact, it is enough to show that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is closed. We show that its complement is open. A set $A \in \mathscr{P}(\overline{\mathbb{F}_p})$ is a not a group if and only if at least one of the following three conditions is satisfied:

- $0 \notin A$;

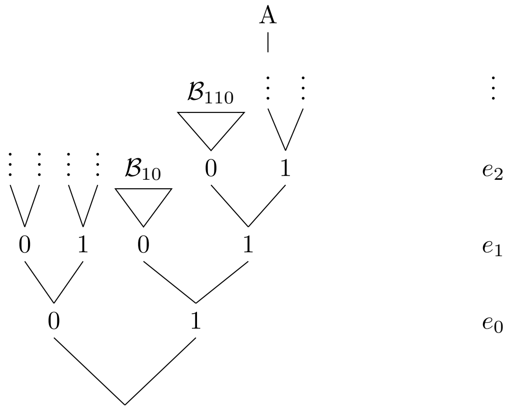

Figure 1.1: Topology on $2^{\overline{\mathbb{F}_p}}$.

- $a \in A$ and $-a \notin A$;
- $a, b \in A$ and $a + b \notin A$.

The first condition is clearly open since in a metric space every singleton is closed, let $\mathscr{O}_0$ be $\mathscr{P}(\overline{\mathbb{F}_p}) \setminus \{0\}$. Let $a, b \in \overline{\mathbb{F}_p}$, let $i, j, k < \omega$ be such that $e_i = a, e_j = b$ and $e_k = a + b$. Let $S(a, b)$ be the set of all finite sequence $s \in 2^{\max(i,j,k)}$ such that $s_i = s_j = 1$ and $s_k = 0$ (see Figure 1.2). Then $\mathscr{O}_1 = \bigcup_{a,b \in \overline{\mathbb{F}_p}} \bigcup_{s \in S(a,b)} \mathcal{B}_s$ is the set of all subsets A of $\overline{\mathbb{F}_p}$ such that for some $a, b \in \overline{\mathbb{F}_p}$ we have $a, b \in A$ and $a + b \notin A$. This is clearly an open set. Similarly there is an open set $\mathscr{O}_2$ which is the set of all $A \in \mathscr{P}(\overline{\mathbb{F}_p})$ such that there exists $a \in \overline{\mathbb{F}_p}$ with $a \in A$ and $-a \notin A$. Then $\mathscr{P}(\overline{\mathbb{F}_p}) \setminus \mathrm{Sg}(\overline{\mathbb{F}_p}) = \mathscr{O}_0 \cup \mathscr{O}_1 \cup \mathscr{O}_2$ is open.

It is clear that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is again metrizable and totally disconnected. Assume that it is not perfect, and let H be an isolated point in $\mathrm{Sg}(\overline{\mathbb{F}_p})$, and $\mathcal{B}(H, k)$ a clopen containing H, for some k. Then consider the finite subgroup H_0 generated by $\{e_i \in H \mid 0 \leq i \leq k\}$. It is clear that $H_0 \in \mathcal{B}(H, k)$ since $H_0 \restriction k = H \restriction k$. As H_0 is finite, there exists $n \geq k$ such that for all $m \geq n$ we have $1_{H_0}(m) = 0$. If $1_H(e_n) = 0$, then $e_n \notin H$ and consider G the group generated by H_0 and e_n. If $1_H(e_n) = 1$ consider $G = H_0$. In any case we have $G \neq H$ and $G, H \in \mathcal{B}(H, k)$ hence H is not isolated. It follows that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is perfect. As it is clearly nonempty it follows that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is a Cantor space. The topology on $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is generated by $\mathcal{B}(H, k)$, as for $\mathscr{P}(\overline{\mathbb{F}_p})$. By the same agument as above, if H_0 is the subgroup generated by $\{e_i \in H \mid 0 \leq i \leq k\}$, then $H_0 \in \mathcal{B}(H, k)$ hence for some $k' \geq k$, we have $\mathcal{B}(H_0, k') \subseteq \mathcal{B}(H, k)$. Similarly, there is some $n \in \mathbb{N}$ such that $\mathcal{B}(H_0, \mathbb{F}_{p^n}) \subseteq \mathcal{B}(H_0, k')$, hence the topology is spanned by balls of the form $\mathcal{B}(H_0, \mathbb{F}_{p^n})$. $\square$

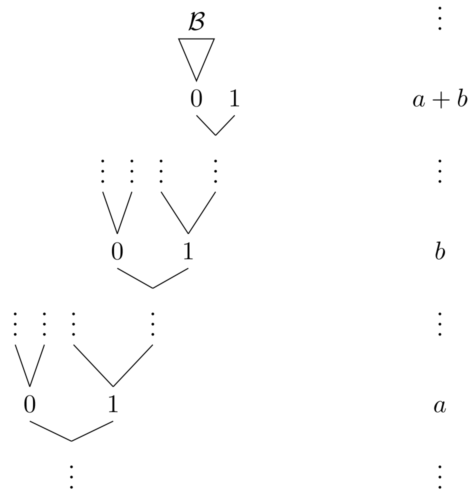

Figure 1.2: A ball that does not contain any groups.

Part A

Generic expansions

CHAPTER 2

Generic expansions by a reduct

Let T be an $\mathscr{L}$-theory. Let $\mathscr{L}_0 \subseteq \mathscr{L}$ and let T_0 be a reduct of T to the language $\mathscr{L}_0$. Let S be a new unary predicate symbol and set $\mathscr{L}_S = \mathscr{L} \cup \{S\}$. We denote by T_S the $\mathscr{L}_S$-theory of $\mathscr{L}_S$-structures $(\mathscr{M}, \mathscr{M}_0)$ where $\mathscr{M} \models T$ and $S(\mathscr{M}) = \mathscr{M}_0 \models T_0$ is a substructure of $\mathscr{M} \upharpoonright \mathscr{L}_0$. The main result of this chapter is an answer to the following question:

Under which conditions on T and T_0 does the model-companion of the theory T_S exist?

Contents

2.1 The main result

We denote by acl_0 the algebraic closure in the sense of T_0. Assume that T_0 is pregeometric. By Section 1.3, there is an associated independence relation $\mathop{\smile\!\!\!\!\!\!|}^0$. It is defined over every subset of any model of T_0 and satisfies the properties Finite Character, Symmetry, Closure, Monotonicity, Base Monotonicity, Transitivity. In particular, $\mathop{\smile\!\!\!\!\!\!|}^0$ is defined over every subset of any model of T, and we will only use it over small subsets of a monster model $\mathbb{M}$ of T. The property Symmetry of $\mathop{\smile\!\!\!\!\!\!|}^0$ will be tacitly used throughout this chapter.

Definition 2.1.1. Let t be a single variable and x, y two tuples of variables. We say that a formula $\psi(t, y)$ is *n-algebraic in t* (or just *algebraic in t*) if for all tuple b the number of realisations of $\psi(t, b)$ is at most n. In that context we say that a formula $\psi(t, x, y)$ is *strict in y* if whenever b is an $\mathop{\smile\!\!\!\!\!\!|}^0$-independent tuple over a, the set of realisations of $\psi(t, a, b)$ is in $\mathrm{acl}_0(a, b) \setminus \mathrm{acl}_0(a)$.

If $\psi(t, b)$ is an $\mathscr{L}_0$-algebraic formula, there exists an $\mathscr{L}_0$-formula $\tilde{\psi}(t, x)$ algebraic in t such that $\psi(\mathscr{M}, b) \subseteq \tilde{\psi}(\mathbb{M}, b)$.

Example 2.1.2. In the language of vector spaces, the formula $t = \lambda x + \mu y$ is strict in y if and only if $\mu \neq 0$.

Lemma 2.1.3. *Assume that T_0 is pregeometric. Then for u a singleton and tuples a and b, if $u \in \mathrm{acl}_0(a, b) \setminus \mathrm{acl}_0(a)$, there exists an $\mathscr{L}_0$-formula $\tau(t, x, y)$ algebraic in t and strict in y such that $u \models \tau(t, a, b)$.*

Proof. Assume that $b = b_1, \dots, b_n$. By hypothesis and using Exchange, we may assume that $b_1 \in \mathrm{acl}_0(u, a, b_2, \dots, b_n)$. Let $\tau_1(t, a, b)$ be an $\mathscr{L}_0$-formula algebraic in t isolating the type $tp^{T_0}(u/ab)$ and $\tau_2(y_1, u, a, b_2, \dots, b_n)$ algebraic in y_1 isolating $tp^{T_0}(b_1/u, a, b_2, \dots, b_n)$. Then $\tau(t, x, y) = \tau_1(t, x, y) \wedge \tau_2(y_1, t, x, y_2, \dots, y_n)$ is strict in y. Indeed assume that for some independent tuple b' over a', and singleton u' we have $\models \tau(u', a', b')$. It follows that $u' \in \mathrm{acl}_0(a'b')$ and $b'_1 \in \mathrm{acl}_0(u', a', b'_2 \dots, b'_n)$. If $u' \in \mathrm{acl}_0(a')$ then $b'_1 \in \mathrm{acl}_0(a', b'_2, \dots, b'_n)$ contradicting that b' is $\mathop{\smile\!\!\!\!\!\!|}^0$-independent over a', so $u' \notin \mathrm{acl}_0(a')$. □

Definition 2.1.4. An expansion $(\mathscr{M}, \mathscr{M}_0) \subseteq (\mathscr{N}, \mathscr{N}_0)$ is *strong* if $\mathscr{N}_0 \mathop{\smile\!\!\!\!\!\!|}^0_{\mathscr{M}_0} \mathscr{M}$.

Theorem 2.1.5. *Assume that the following holds:*

(H_1) *T is model complete;*

(H_2) *T_0 is model complete and for all infinite A, $\mathrm{acl}_0(A) \models T_0$;*

(H_3) *T_0 is pregeometric;*

(H_4) *for all $\mathscr{L}$-formula $\phi(x, y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for $b \in \mathscr{M} \models T$,*

$$\mathscr{M} \models \theta_\phi(b) \iff \begin{array}{l}\textit{there exists } \mathscr{N} \succ \mathscr{M} \textit{ and } a \in \mathscr{N} \textit{ such that}\\ \phi(a, b) \textit{ and } a \textit{ is an } \mathop{\smile\!\!\!\!\!\!|}^0\textit{-independent tuple over } \mathscr{M}.\end{array}$$

Then there exists a theory TS containing T_S such that

- *every model of T_S has a strong extension which is a model of TS;*
- *if $(\mathscr{M}, \mathscr{M}_0) \models TS$ and $(\mathscr{N}, \mathscr{N}_0) \models T_S$ is a strong extension of $(\mathscr{M}, \mathscr{M}_0)$ then $(\mathscr{M}, \mathscr{M}_0)$ is existentially closed in $(\mathscr{N}, \mathscr{N}_0)$.*

An axiomatization of TS is given by adding to T_S the following axiom scheme: for each tuple of variables $x = x^0x^1$, for $\mathscr{L}$-formula $\phi(x, y)$, and $\mathscr{L}_0$-formulae $(\tau_i(t, x, y))_{i<k}$ which are algebraic in t and strict in x^1,

$$\forall y(\theta_\phi(y) \to (\exists x \phi(x, y) \wedge x^0 \subseteq S \wedge \bigwedge_{i<k} \forall t\ (\tau_i(t, x, y) \to t \notin S))).$$

Proof. We prove the first assertion. Let $(\mathscr{M}, \mathscr{M}_0)$ be a model of T_S, $\phi(x, y)$ an $\mathscr{L}$-formula and a partition $x = x^0x^1$. Assume that for some tuple b from $\mathscr{M}$ we have $\theta_\phi(b)$. We show that the conclusion of the axiom can be satisfied in a strong extension $(\mathscr{N}, \mathscr{N}_0)$ with $\mathscr{N} \succ \mathscr{M}$. Then the result will follow by taking the union of a chain of models of T_S, which is again a model of T_S because it is an elementary chain of models of T with a predicate for models of T_0 which is inductive, by model-completeness. The fact that the union of a chain of strong extensions is again strong follows from Finite Character and Transitivity of $\mathop{\smile\!\!\!\!\!\mid}^0$, and the model-completeness of T_0.

By (H_4) there exists an extension $\mathscr{N} \succ \mathscr{M}$, and a tuple $a \in \mathscr{N}$ satisfying $\phi(x, b)$ and such that a is $\mathop{\smile\!\!\!\!\!\mid}^0$-independent over $\mathscr{M}$. Set $\mathscr{N}_0 = \mathrm{acl}_0(\mathscr{M}_0 a^0)$. Then using Monotonicity, Base Monotonicity and Closure of $\mathop{\smile\!\!\!\!\!\mid}^0$, $a^0\mathscr{M}_0 \mathop{\smile\!\!\!\!\!\mid}^0_{\mathscr{M}_0} \mathscr{M}$. This means that the extension $(\mathscr{M}, \mathscr{M}_0) \subseteq (\mathscr{N}, \mathscr{N}_0)$ is strong. Now clearly $a^0 \subseteq S$. Using Base Monotonicity and Closure, it follows that $ab \mathop{\smile\!\!\!\!\!\mid}^0_{a^0b} \mathscr{M}_0 a^0$. Take any $\mathscr{L}_0$-formula $\tau(t, x, y)$ algebraic in t and strict in x^1, and assume that $u \in \mathscr{N}$ satisfies $\tau(t, a, b)$. As τ is strict in x^1 and a^1 is $\mathop{\smile\!\!\!\!\!\mid}^0$ -independent over ba^0, we have $u \in \mathrm{acl}_0(ab) \setminus \mathrm{acl}_0(a^0b)$. If $u \in \mathscr{N}_0$ then it belongs to $\mathrm{acl}_0(ab) \cap \mathrm{acl}_0(\mathscr{M}_0 a^0) \subseteq \mathrm{acl}_0(a^0 b)$, a contradiction, hence $u \notin S$. It follows that $(\mathscr{N}, \mathscr{N}_0) \models \phi(a, b) \wedge a^0 \subseteq S \wedge \bigwedge_{i<k} \forall t\ (\tau_i(t, a, b) \to t \notin S)))$.

We now prove the second assertion.

Let $(\mathscr{M}, \mathscr{M}_0) \models TS$ and $(\mathscr{N}, \mathscr{N}_0) \models T_S$, a strong extension of $(\mathscr{M}, \mathscr{M}_0)$. Take finite tuples $a \in \mathscr{N}$ and $b \in \mathscr{M}$. To understand the quantifier-free $\mathscr{L}_S$-type of a over b, it is sufficient to deal with formulae of the form

$$\psi(x, b) \wedge \bigwedge_{i \in I} x_i \in S \wedge \bigwedge_{j \in J} x_j \notin S$$

with $\psi(x, y)$ an $\mathscr{L}$-formula. The reduction to formulae of this form is done by increasing the length of x (replacing $\mathscr{L}$-terms by variables), which may be greater than $|a|$. We assume that a satisfies the formula above.

Claim. There exists an $\mathop{\smile\!\!\!\!\!\mid}^0$-independent tuple $a' = a^{0\prime}a^{1\prime}$ such that $\mathrm{acl}_0(\mathscr{M} a) = \mathrm{acl}_0(\mathscr{M} a')$ with

(1) $a' \mathop{\smile\!\!\!\!\!\mid}^0 \mathscr{M}$;

(2) $\mathrm{acl}_0(a') \cap \mathscr{N}_0 = \mathrm{acl}_0(a^{0\prime})$;

(3) $\mathscr{N}_0 \cap \mathrm{acl}_0(\mathscr{M}, a') = \mathrm{acl}_0(\mathscr{M}_0, a^{0\prime})$;

Proof of the claim. Take a tuple $a^{0\prime}$ in $\mathscr{N}_0 \cap \mathrm{acl}_0(\mathscr{M}, a)$ maximal $\mathop{\smile\!\!\!\!\!\!|}^0$-independent over $\mathscr{M}_0$. We have $a^{0\prime} \mathop{\smile\!\!\!\!\!\!|}^0 \mathscr{M}_0$, and as the extension is strong we also have $a^{0\prime} \mathop{\smile\!\!\!\!\!\!|}^0 \mathscr{M}$ by TRANSITIVITY. Now take a tuple $a^{1\prime}$ in $\mathrm{acl}_0(\mathscr{M}a)$ maximal $\mathop{\smile\!\!\!\!\!\!|}^0$-independent over $\mathrm{acl}_0(\mathscr{M}a^{0\prime})$. We have $a^{1\prime} \mathop{\smile\!\!\!\!\!\!|}^0 \mathscr{M}a^{0\prime}$ and so $a^{0\prime}a^{1\prime} \mathop{\smile\!\!\!\!\!\!|}^0 \mathscr{M}$. Set $a' = a^{0\prime}a^{1\prime}$ and the claim holds. □

Now as $a \subseteq \mathrm{acl}_0(\mathscr{M}, a')$ there exists a finite tuple m^1 from $\mathscr{M}$ $\mathop{\smile\!\!\!\!\!\!|}^0$-independent over $\mathscr{M}_0 a'$ such that $a \subseteq \mathrm{acl}_0(\mathscr{M}_0 m^1 a')$. Similarly there exists a finite tuple m^0 from $\mathscr{M}_0$ with $m^0 \mathop{\smile\!\!\!\!\!\!|}^0 m^1 a'$ such that $a \subseteq \mathrm{acl}_0(m^0 m^1 a')$.

If $i \in I$, using *(3)*, we have $a_i \in \mathrm{acl}_0(\mathscr{M}_0 a^{0\prime}) \cap \mathrm{acl}_0(m^0 m^1 a') = \mathrm{acl}_0(m^0 a^{0\prime})$. Hence there is an $\mathscr{L}_0$-formula $\tau_i(t, a^{0\prime}, m^0)$ algebraic in t such that $a_i \models \tau_i(t, a^{0\prime}, m^0)$.

Let J_1 be the set of indices $j \in J$ such that $a_j \in \mathrm{acl}_0(a^{0\prime}, m^0, m^1)$. As $a_j \notin S$, by Lemma 2.1.3 there is an $\mathscr{L}_0$-formula $\tau_j(t, x^0, y, z)$ algebraic in t and strict in z such that $a_j \models \tau_j(t, a^{0\prime}, m^0, m^1)$.

Let $J_2 = J \setminus J_1$. Then for $j \in J_2$, we have $a_j \notin \mathrm{acl}_0(a^{0\prime}, m^0, m^1)$ so there is an $\mathscr{L}_0$-formula $\tau_j(t, x^0, x^1, y, z)$ algebraic in t and strict in x^1 such that $a_j \models \tau_j(t, a^{0\prime}, a^{1\prime}, m^0, m^1)$.

We now set $b' = bm^0m^1$ and set $\phi(a', b')$ to be the following formula

$$\begin{aligned} \exists v \psi(v, b) \quad &\wedge \bigwedge_{i \in I} \tau_i(v_i, a^{0\prime}, m^0) \\ &\wedge \bigwedge_{j \in J_1} \tau_j(v_j, a^{0\prime}, m^0, m^1) \\ &\wedge \bigwedge_{j \in J_2} \tau_j(v_j, a^{0\prime}, a^{1\prime}, m^0, m^1) \end{aligned}$$

Step $(\star)$. By model-completeness we have that $\mathscr{N} \succ \mathscr{M}$. As a' is $\mathop{\smile\!\!\!\!\!\!|}^0$ independent over $\mathscr{M}$ it follows that $\mathscr{M} \models \theta_\phi(b')$. Using one instance of the axiom scheme, there exists $d' \in \mathscr{M}$ such that $d' \models \phi(x, b')$ with $d^{0\prime} \subseteq \mathscr{M}_0$ and for all $j \in J_2$, all the realizations of $\tau_j(t, d', m)$ are not in $\mathscr{M}_0$. Let d be the tuple whose existence is stated in $\phi(d', b')$, in particular $\mathscr{M} \models \psi(d, b)$. For $i \in I$, we have $d_i \in \mathrm{acl}_0(d^{0\prime}m^0) \subseteq \mathscr{M}_0$. For $j \in J_2$ we already saw that $d_j \notin \mathscr{M}_0$. For $j \in J_1$, as $\tau_j(t, d^{0\prime}, m^0, m^1)$ is strict in the variable of m^1 and m^1 is $\mathop{\smile\!\!\!\!\!\!|}^0$-independent over $\mathscr{M}_0$, we have that $d_j \notin \mathrm{acl}_0(d^{0\prime}, m^0)$. Recall that $m^1 \mathop{\smile\!\!\!\!\!\!|}^0 \mathscr{M}_0$, so $m^1 \mathop{\smile\!\!\!\!\!\!|}^0_{d^{0\prime}, m^0} \mathscr{M}_0$ hence $\mathrm{acl}_0(d^{0\prime}, m^0, m^1) \cap \mathscr{M}_0 = \mathrm{acl}_0(d^{0\prime}, m^0)$, so d_j cannot belong to $\mathscr{M}_0$. We conclude that

$$(\mathscr{M}, \mathscr{M}_0) \models \psi(d, b) \wedge \bigwedge_{i \in I} d_i \in S \wedge \bigwedge_{j \in J} d_j \notin S$$

which proves that $(\mathscr{M}, \mathscr{M}_0)$ is existentially closed in $(\mathscr{N}, \mathscr{N}_0)$. □

Remark 2.1.6. Notice that if we consider $\mathscr{L}_0 = \{=\}$, the previous Theorem gives nothing more than the generic predicate (see[CP98]). The hypothesis (H_4) becomes equivalent to elimination of $\exists^\infty$ in that case. Note also that if T_0 is strongly minimal and has quantifier elimination in $\mathscr{L}_0$, the conditions (H_2) and (H_3) are satisfied.

We can forget hypothesis (H_1) to get this adapted version of Theorem 2.1.5.

Proposition 2.1.7. *Assume that the following holds.*

(H_2) *T_0 is model complete and for all A infinite, $\mathrm{acl}_0(A) \models T_0$;*

(H_3) *T_0 is pregeometric;*

(H_4) *for all $\mathscr{L}$-formula $\phi(x, y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for $b \in \mathscr{M} \models T$*

$$\mathscr{M} \models \theta_\phi(b) \iff \begin{array}{l}\text{there exists } \mathscr{N} \succ \mathscr{M} \text{ and } a \in \mathscr{N} \text{ such that}\\ \phi(a, b) \text{ and } a \text{ is an } \mathop{\downarrow}\limits^{0}\text{-independent tuple over } \mathscr{M}\end{array}$$

Then there exists a theory TS containing T_S such that

- *every model $(\mathscr{M}, \mathscr{M}_0)$ of T_S has a strong extension $(\mathscr{M}', \mathscr{M}_0')$ which is a model of TS, such that $\mathscr{M} \prec \mathscr{M}'$;*
- *assume that $(\mathscr{M}, \mathscr{M}_0) \models TS$ and $(\mathscr{N}, \mathscr{N}_0)$ is a model of T_S which is a strong extension of $(\mathscr{M}, \mathscr{M}_0)$. If $\mathscr{M}$ is existentially closed in $\mathscr{N}$ then $(\mathscr{M}, \mathscr{M}_0)$ is existentially closed in $(\mathscr{N}, \mathscr{N}_0)$.*

An axiomatization of TS is given by adding to T_S the following axioms, for each tuple of variables $x = x^0x^1$, for $\mathscr{L}$-formula $\phi(x, y)$, and $\mathscr{L}_0$-formulae $(\tau_i(t, x, y))_{i<k}$ which are algebraic in t and strict in x^1,

$$\forall y(\theta_\phi(y) \to (\exists x\phi(x, y) \wedge x^0 \subseteq S \wedge \bigwedge_{i<k} \forall t\ (\tau_i(t, x, y) \to t \notin S))).$$

Proof. The same proof as for Theorem 2.1.5 works. In the proof of Theorem 2.1.5, the model-completeness of T was used to ensure that given any model $\mathscr{N}$ of T extending $\mathscr{M}$, then $\mathscr{M}$ is existentially closed in $\mathscr{N}$, which is now part of the second bullet. In the first bullet, the model $\mathscr{M}'$ of T extending $\mathscr{M}$ is the union of an elementary chain of extensions hence is an elementary extension of $\mathscr{M}$, this condition does not use the model-completeness of T. □

Remark 2.1.8. Assume that T, T_0 satisfies (H_1), (H_2) and (H_3). Assume that there is a class $\mathscr{C}$ of $\mathscr{L}$-formula such that for all $\mathscr{M} \models T$, for all $\mathscr{L}$-formula $\phi(x, b)$ with parameters in $\mathscr{M}$, there exists a tuple c from $\mathscr{M}$ and formulae $\vartheta_1(x, z), \cdots, \vartheta_n(x, z) \in \mathscr{C}$ such that

$$\phi(\mathscr{M}, b) = \vartheta_1(\mathscr{M}, c) \cup \cdots \cup \vartheta_n(\mathscr{M}, c).$$

Assume that condition (H_4) holds only for formulae $\vartheta(x, z) \in \mathscr{C}$. Then the conclusion of Theorem 2.1.5 applies, with the axiom-scheme restricted to formulae in $\mathscr{C}$. It is clear that the proof of the first assertion works similarly, considering only formulae in $\mathscr{C}$. For the second assertion, the proof changes at Step $(\star)$, we need to show that there exists a realisation of $\phi(x,b')$ that satisfies the right properties using the axioms. By assumption $\phi(\mathscr{M},b') = \vartheta_1(\mathscr{M},c)\cup\cdots\cup\vartheta_n(\mathscr{M},c)$ for some $\vartheta_1(x,z),\cdots,\vartheta_n(x,z)\in\mathscr{C}$ and tuple c from $\mathscr{M}$. This decomposition holds also in $\mathscr{N}$ by model-completeness of T. Now as $a'\models\phi(x,b')$, there is some $i\leq n$ such that $a'\models\vartheta_i(x,c)$ hence $\mathscr{M}\models\theta_{\vartheta_i}(c)$. Using one instance of the axiom, there exists d' in $\mathscr{M}$ satisfying $\vartheta_i(x,c)$, hence also $\phi(x,b')$, and that satisfies the right properties, and the end of the proof is similar. The main example for the class $\mathscr{C}$ is the class of quasi-affine varieties in the theory ACF, see Theorem 3.3.5.

2.2 A weak converse

In this subsection, Lemmas 2.2.2 and 2.2.3 give some insight on the condition (H_4), and Proposition 2.2.4 gives a weak converse statement for the existence of TS.

In this section, we assume that T and T_0 satisfies the following conditions:

(H_1) T is model-complete;

(H_2^-) T_0 is model-complete;

(H_3) T_0 is pregeometric;

Given two tuples of variables x and y, the condition "x is $\mathop{\smile\!\!\!\!\!\!|}^0$-independent over $\mathrm{acl}_T(y)$" is type-definable, it is given by the set of formulae of the form

$$\forall t_1,\cdots,t_n\left(\bigwedge_{i=1}^{n}\psi_i(t_i,y)\to\bigwedge_{k=1}^{|x|}\neg\tau_k(x_k,t_1,\cdots,t_n,x_1,\cdots,x_{k-1},x_{k+1},\cdots,x_{|x|})\right)$$

for all $n\in\mathbb{N}$, $\psi_i(t,y)$ $\mathscr{L}$-formula algebraic in t and $\tau_j(t,t_1,\cdots,t_n,z)$ $\mathscr{L}_0$-formula algebraic in t with $|z|=|x|-1$. As algebraic formulae are closed under finite disjunction and conjunction, it is clear that the previous type is equivalent to the set of all formulae of the form

$$\forall t_1,\cdots,t_n\left(\bigwedge_{i=1}^{n}\psi(t_i,y)\to\bigwedge_{k=1}^{|x|}\neg\tau(x_k,t_1,\cdots,t_n,x_1,\cdots,x_{k-1},x_{k+1},\cdots,x_{|x|})\right)$$

for all $n\in\mathbb{N}$, $\psi(t,y)$ $\mathscr{L}$-formula algebraic in t and $\tau(t,t_1,\cdots,t_n,z)$ $\mathscr{L}_0$-formula algebraic in t with $|z|=|x|-1$. We call this type $\Sigma(x,y)$.

We work in a monster model $\mathbb{M}$ of T.

Lemma 2.2.1. *For all A,B,C acl_T-closed small sets in a monster model, then there exists $A'\equiv_C^T A$ such that $A'\mathop{\smile\!\!\!\!\!\!|}^0_C B$.*

Proof. The lemma follows from Fact 1.3.9, take q to be the type of an $\mathop{\smile\!\!\!\!\!\!|}^0$ basis of A over C. Note that we only use hypothesis (H_3) here. $\square$

Lemma 2.2.2. *Let $\phi(x, y)$ be an $\mathscr{L}$-formula, $\mathscr{M}$ an $\aleph_0$-saturated small model of T and b a $|y|$-tuple from $\mathscr{M}$. The following are equivalent:*

(1) there exists $\mathscr{N} \succ \mathscr{M}$ and some realisation a of $\phi(x, b)$ in $\mathscr{N}$ such that a is an $\mathop{\smile\!\!\!\!\!\!|}^0$-independent tuple over $\mathscr{M}$;

(2) there exists some realisation a of $\phi(x, b)$ in $\mathscr{M}$ such that a is $\mathop{\smile\!\!\!\!\!\!|}^0$-independent over $\mathrm{acl}_T(b)$.

Proof. *(1)* implies *(2)*. Let $\Sigma(x, b)$ be the partial type over b expressing that "x is an $\mathop{\smile\!\!\!\!\!\!|}^0$-independent tuple over $\mathrm{acl}_T(b)$". By *(1)*, $\Sigma(x, b)$ is finitely satisfiable in $\mathscr{M}$ hence by saturation it is realised in $\mathscr{M}$.
(2) implies *(1)*. Using Lemma 2.2.1, there exists $a' \equiv_b^T a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^0_{\mathrm{acl}_T(b)} \mathscr{M}$. Using TRANSITIVITY a' is $\mathop{\smile\!\!\!\!\!\!|}^0$-independent over $\mathscr{M}$. For any $\mathscr{N}$ containing a', the condition *(2)* holds. □

Lemma 2.2.3. *Let $\phi(x, y)$ be some $\mathscr{L}$-formula. The following are equivalent.*

(1) There exists a formula $\theta_\phi(y)$ such that $\theta_\phi(b)$ holds if and only if there exists some realisation a of $\phi(x, b)$ such that a is $\mathop{\smile\!\!\!\!\!\!|}^0$-independent over $\mathrm{acl}_T(b)$.

(2) There exists $n \in \mathbb{N}$, an $\mathscr{L}$-formula $\psi(t, y)$ algebraic in t and an $\mathscr{L}_0$-formula $\tau(t, t_1, \ldots, t_n, z)$ algebraic in t with $|z| = |x| - 1$ such that for all b, if some realisation a of $\phi(x, b)$ is not an $\mathop{\smile\!\!\!\!\!\!|}^0$-independent tuple over $\mathrm{acl}_T(b)$ then there exist n realizations $c_1, \ldots, c_n$ of $\psi(t, b)$ such that for some $1 \leq k \leq |x|$, we have that a_k satisfies $\tau(t, c_1, \cdots, c_n, a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_{|x|})$.

Proof. Recall that $\Sigma(x, y)$ is the set of all formula of the form

$$\phi(x, y) \wedge \forall t_1, \cdots, t_n \left(\bigwedge_{i=1}^{n} \psi(t_i, y) \to \bigwedge_{k=1}^{|x|} \neg\tau(x_k, t_1, \cdots, t_n, x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{|x|}) \right)$$

for all $n \in \mathbb{N}$, $\psi(t, y)$ $\mathscr{L}$-formula algebraic in t and $\tau(t, t_1, \cdots, t_n, z)$ $\mathscr{L}_0$-formula algebraic in t with $|z| = |x| - 1$. Let $\Sigma(y)$ be the (consistent) partial type $\{\exists x \Gamma(x, y) \mid \Gamma(x, y) \in \Sigma\}$. By compactness, if $\theta_\phi(y)$ exists, it is equivalent to a finite fragment of $\Sigma(y)$, hence to a single formula in $\Sigma(y)$. The existence of $\theta_\phi(y)$ is equivalent to the existence of a bound $n \in \mathbb{N}$, an $\mathscr{L}$-formula $\psi(t, y)$ algebraic in t and an $\mathscr{L}_0$-formula $\tau(t, t_1, \cdots, t_n, z)$ for $|z| = |x| - 1$ such that for all b if a realizes $\phi(x, b)$, a is not $\mathop{\smile\!\!\!\!\!\!|}^0$-independent over $\mathrm{acl}_T(b)$ (if and) only if there are n realisations $c_1, \ldots, c_n \in \mathrm{acl}_T(b)$ of $\psi(t, b)$ such that for some $1 \leq k \leq |x|$, a_k is in $\mathrm{acl}_0(c_1, \cdots, c_n, a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_{|x|})$, witnessed by τ. □

Proposition 2.2.4. *Assume that there exists a theory TS such that*

- *every model of T_S has a strong extension which is a model of TS;*
- *if $(\mathscr{M}, \mathscr{M}_0) \models TS$ and $(\mathscr{N}, \mathscr{N}_0) \models T_S$ is a strong extension of $(\mathscr{M}, \mathscr{M}_0)$ then $(\mathscr{M}, \mathscr{M}_0)$ is existentially closed in $(\mathscr{N}, \mathscr{N}_0)$.*

Then the following holds:

for all $\mathscr{L}$-formula $\phi(x,y)$ and all $1 \leq k \leq |x|$, there exists an $\mathscr{L}$-formula $\theta_\phi^k(y)$ such that for all tuple b in an $\aleph_0$-saturated model $\mathscr{M}$ of T,

$$\mathscr{M} \models \theta_\phi^k(b) \iff \text{there exists some realisation } a \text{ of } \phi(x,b) \text{ in } \mathscr{M} \text{ such that } a_k \notin \mathrm{acl}_0(\mathrm{acl}_T(b), a_1, \ldots, a_{k-1}, a_{k+1}, \cdots, a_{|x|}).$$

Proof. Given a single variable t and some tuple of variables y, we denote by $\mathscr{A}_\mathscr{L}(t,y)$ the set of all $\mathscr{L}$-formulae without parameters that are algebraic in t with free variables (other than t) in y. Assume that the conclusion doesn't hold. Similarly to Lemma 2.2.3 there is some formula $\phi(x,y)$, some $1 \leq k \leq |x|$ and an $\aleph_0$-saturated model $\mathscr{M}$ of T such that for all $n \in \mathbb{N}$, for all $\psi(t,y) \in \mathscr{A}_\mathscr{L}(t,y)$ and $\tau(t,t_1,\cdots,t_n,z) \in \mathscr{A}_{\mathscr{L}_0}(t,t_1,\cdots,t_n,z)$ (with $|z| = |x|-1$) there is some $b = b(n,\psi,\tau)$ and a realisation $a = a(n,\psi,\tau)$ of $\phi(x,b)$ in $\mathscr{M}$ such that $a_k \in \mathrm{acl}_0(\mathrm{acl}_T(b), a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{|x|})$ and for all realisations $c_1, \ldots, c_n$ of $\psi(t,b)$ and all k

$$\mathscr{M} \models \neg\tau(a_k, c_1, \cdots, c_n, a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_{|x|}).$$

For convenience, we assume that $k = 1$. We may assume that for all n, ψ and τ, all realisations of $\phi(x_1, a_{>1}, b)$ in $\mathscr{M}$ are in $\mathrm{acl}_0(\mathrm{acl}_T(b), a_{>1})$. Otherwise, for some n, ψ, τ as above, the formula

$$\exists x \phi(x,y) \wedge \forall t_1, \ldots, t_n \left(\bigwedge_{i=1}^n \psi(t_i, y) \to \neg\tau(x_1, t_1, \cdots, t_n, x_{>1}) \right)$$

would isolate the type $\exists x \phi(x,y) \wedge$ "$x_1 \notin \mathrm{acl}_0(\mathrm{acl}_T(y), x_{>1})$" which contradicts the hypotheses. By $\aleph_0$-saturation, as $\phi(\mathscr{M}, a_{>1}, b) \subseteq \mathrm{acl}_0(\mathrm{acl}_T(b), a_{>1})$, we have that $\phi(\mathscr{M}, a_{>1}, b)$ is finite, for all (n,ψ,τ).

We define the following subset of $\mathbb{N} \times \mathscr{A}_\mathscr{L}(t,y) \times \left(\bigcup_{n\in\mathbb{N}} \mathscr{A}_{\mathscr{L}_0}(t,t_1,\ldots,t_n,z)\right)$

$$I = \{(n,\psi,\tau) \mid n \in \mathbb{N}, \psi \in \mathscr{A}_\mathscr{L}(t,y), \tau \in \mathscr{A}_{\mathscr{L}_0}(t,t_1,\cdots,t_n,z)\}$$

By assumptions, $\mathscr{M}$ contains $\{a_{>1}b \mid (n,\psi,\tau) \in I\}$. We expand $\mathscr{M}$ to a model of T_S by setting $S(\mathscr{M}) = \mathscr{M}$. By hypothesis, there exists a model $(\mathscr{N}, \mathscr{N}_0)$ of TS which is a strong extension of $(\mathscr{M}, \mathscr{M})$. As T is model-complete, each realisation a_1' in $\mathscr{N}$ of $\phi(x_1, a_{>1}b)$ is still in $\mathrm{acl}_0(\mathrm{acl}_T(b), a_{>1})$, hence $\phi(\mathscr{N}, a_{>1}, b) \subset S$. Furthermore, for all $(n,\psi,\tau) \in I$, the following holds in $(\mathscr{N}, \mathscr{N}_0)$

$$\exists x_1 \phi(x_1, a_{>1}b) \wedge \forall t_1, \ldots, t_n \left(\bigwedge_{i=1}^n \psi(t_i, b) \to \neg\tau(x_1, t_1, \cdots, t_n, x_{>1}) \right).$$

Let $\mathscr{U}$ be a nonprincipal ultrafilter on I and consider the ultrapower $(\mathscr{N}, \mathscr{N}_0)^\mathscr{U}$ of $(\mathscr{N}, \mathscr{N}_0)$, which is also a model of TS. For $\overline{a_{>1}b}$ the class of $(a_{>1}b(n,\psi,\tau))_{(n,\psi,\tau)\in I}$ in

$(\mathcal{N},\mathcal{N}_0)^{\mathcal{U}}$, every realisation of $\phi(x_1,\overline{a_{>1}b})$ in $(\mathcal{N},\mathcal{N}_0)^{\mathcal{U}}$ is in S. On the other hand, the partial type consisting of all formulae of the form

$$\exists x_1\phi(x_1,\overline{a_{>1}b})\wedge\forall t_1,\dots,t_n\left(\bigwedge_{i=1}^{n}\psi(t_i,\bar b)\to\neg\tau(x_1,t_1,\cdots,t_n,\overline{a_{>1}})\right)$$

for $(n,\psi,\tau)\in I$, is consistent. Hence there exists a realisation $\tilde a_1$ of $\phi(x_1,\overline{a_{>1}b})$ in $\mathcal{N}^{\mathcal{U}}$ which is not in $\mathrm{acl}_0(\mathrm{acl}_T(\bar b)\overline{a_{>1}})$. By Lemma 2.2.1, there exists singleton $\tilde a_1'$ in some elementary extension $\mathcal{K}$ of $\mathcal{N}^{\mathcal{U}}$ such that $\tilde a_1'\equiv^T_{\mathrm{acl}_T(\bar b)\overline{a_{>1}}}\tilde a_1$ and $\tilde a_1'\mathop{\smile\!\!\!\!\!\mid}^0_{\mathrm{acl}_T(\bar b)\overline{a_{>1}}}\mathcal{N}^{\mathcal{U}}$. Now $\tilde a_1'\notin\mathrm{acl}_0(\mathrm{acl}_T(\bar b)\overline{a_{>1}})$ implies that $\tilde a_1'\mathop{\smile\!\!\!\!\!\mid}^0\mathrm{acl}_T(\bar b)\overline{a_{>1}}$, so by TRANSITIVITY $\tilde a_1'\notin\mathcal{N}^{\mathcal{U}}$. Finally observe that $(\mathcal{K},S(\mathcal{N}^{\mathcal{U}}))$ is a strong extension of $(\mathcal{N}^{\mathcal{U}},S(\mathcal{N}^{\mathcal{U}}))$, hence $(\mathcal{N}^{\mathcal{U}},S(\mathcal{N}^{\mathcal{U}}))$ is existentially closed in $(\mathcal{K},S(\mathcal{N}^{\mathcal{U}}))$, but

$$(\mathcal{K},S(\mathcal{N}^{\mathcal{U}}))\models\exists x_1\phi(x_1,\overline{a_{>1}b})\wedge x_1\notin S$$

a contradiction. □

Remark 2.2.5. A consequence of Proposition 2.2.4 is that if TS exists, then T eliminates $\exists^\infty$. A question one might ask is wether it is a sufficient condition for the existence of the theory TS. The answer is no, the theory ACF_0 eliminates $\exists^\infty$ but the model companion of the theory of algebraically closed fields of characteristic 0 with a predicate for an additive subgroup is not first order axiomatisable, see Proposition 3.2.7. On the other hand, the existence of TS under the reduction of the hypothesis (H_4) to formulae $\phi(x,y)$ with $|x|=2$ would be a good improvement, as it would be much easier to check.

2.3 Suitable triple

In Sections 2.1 and 2.2, we have listed minimal hypotheses in order to have (weakly) necessary and sufficient conditions for the existence of a generic theory TS. We now consider a stronger assumption on T_0 which encompass the conditions of Sections 2.1 and 2.2: the *modularity* of the pregeometry in T_0. This hypothesis make obsolete the notion of *strong* extension. As a consequence, the theory TS becomes the *model-companion* of the theory T_S.

Definition 2.3.1. We say that a triple $(T,T_0,\mathcal{L}_0)$ is *suitable* if it satisfies the following

(H_1) T is model complete;

(H_2) T_0 is model complete and for all infinite A, $\mathrm{acl}_0(A)\models T_0$;

(H_3^+) acl_0 defines a modular pregeometry;

(H_4) for all $\mathcal{L}$-formula $\phi(x,y)$ there exists an $\mathcal{L}$-formula $\theta_\phi(y)$ such that for $b\in\mathcal{M}\models T$

$$\mathcal{M}\models\theta_\phi(b)\iff\begin{array}{l}\text{there exists }\mathcal{N}\succ\mathcal{M}\text{ and }a\in\mathcal{N}\text{ such that}\\ \phi(a,b)\text{ and }a\text{ is }\mathop{\smile\!\!\!\!\!\mid}^0\text{-independent over }\mathcal{M}.\end{array}$$

Remark 2.3.2. Let $(T, T_0, \mathscr{L}_0)$ be a suitable triple. By Fact 1.2.3, in T, the relation $\mathop{\smile\!\!\!\!\!\!\mid}^a$ defined by $A \mathop{\smile\!\!\!\!\!\!\mid}^a_C B$ if and only if $\mathrm{acl}_T(AC) \cap \mathrm{acl}_T(BC) = \mathrm{acl}_T(C)$ satisfies FULL EXISTENCE, so for all A, B, C subsets of $\mathbb{M}$ there exists $A' \equiv^T_C A$ such that $\mathrm{acl}_T(A'C) \cap \mathrm{acl}_T(BC) = \mathrm{acl}_T(C)$. As acl_0 is modular, it follows that $\mathrm{acl}_T(A'C) \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathrm{acl}_T(C)} \mathrm{acl}_T(BC)$, this gives another proof of Lemma 2.2.1 in that context.

From Section 2.1, we immediately get the following.

Proposition 2.3.3. *Let $(T, T_0, \mathscr{L}_0)$ be a suitable triple. Then TS exists and is the model-companion of the theory T_S.*

Lemma 2.3.4. *Let $(\mathscr{M}, \mathscr{M}_0)$ and $(\mathscr{N}, \mathscr{N}_0)$ are two models of T_S, such that $\mathscr{M}_0 \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathscr{N}_0} \mathscr{N}$ and $\mathscr{N}_0 \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathscr{M}_0} \mathscr{M}$. Then, there exists a model $(\mathscr{K}, \mathscr{K}_0)$ of TS extending both $(\mathscr{M}, \mathscr{M}_0)$ and $(\mathscr{N}, \mathscr{N}_0)$. If furthermore $(\mathscr{M}, \mathscr{M}_0)$ and $(\mathscr{N}, \mathscr{N}_0)$ are models of TS, then $(\mathscr{K}, \mathscr{K}_0)$ is an elementary extension of both $(\mathscr{M}, \mathscr{M}_0)$ and $(\mathscr{N}, \mathscr{N}_0)$.*

Proof. Let $\mathscr{K}'$ be a model of T extending $\mathscr{M}$ and $\mathscr{N}$. Now set $\mathscr{K}_0' = \mathrm{acl}_0(\mathscr{M}_0, \mathscr{N}_0)$. Clearly $(\mathscr{K}', \mathscr{K}_0')$ is a model of T_S. By hypothesis we have $\mathscr{K}_0' \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathscr{M}_0} \mathscr{M}$ and $\mathscr{K}_0' \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathscr{N}_0} \mathscr{N}$. Using Theorem 2.1.5, there is a model $(\mathscr{K}, \mathscr{K}_0)$ of TS extending $(\mathscr{K}', \mathscr{K}_0')$, $(\mathscr{M}, \mathscr{M}_0)$ and $(\mathscr{N}, \mathscr{N}_0)$. We conclude by model-completeness. □

Proposition 2.3.5. *Let $(T, T_0, \mathscr{L}_0)$ be an adapted triple.*

(1) Let $(\mathscr{M}, \mathscr{M}_0)$ and $(\mathscr{N}, \mathscr{N}_0)$ be two models of TS and A be a common subset of $\mathscr{M}$ and $\mathscr{N}$. Then we have

$$(\mathscr{M}, \mathscr{M}_0) \equiv^{TS}_A (\mathscr{N}, \mathscr{N}_0) \iff \begin{array}{l} \text{there exists } f : \mathrm{acl}_T(A) \to \mathrm{acl}_T(A) \\ T\text{-elementary bijection over } A, \\ \text{such that } f(\mathscr{M}_0 \cap \mathrm{acl}_T(A)) = \mathscr{N}_0 \cap \mathrm{acl}_T(A). \end{array}$$

(2) For any a, b, A in a monster model of TS

$$a \equiv^{TS}_A b \iff \begin{array}{l} \text{there exists } f : \mathrm{acl}_T(Aa) \to \mathrm{acl}_T(Ab) \\ \text{a } T\text{-elementary bijection over } A \text{ with } f(a) = b, \\ \text{such that } f(S(\mathrm{acl}_T(Aa))) = S(\mathrm{acl}_T(Ab)).. \end{array}$$

We call such a function a T-elementary $\mathscr{L}_S$-isomorphism between $(\mathrm{acl}_T(Aa), S(\mathrm{acl}_T(Aa)))$ and $(\mathrm{acl}_T(Ab), S(\mathrm{acl}_T(Ab)))$.

(3) The completions of TS are given by the T-elementary $\mathscr{L}_S$-isomorphism types of

$$(\mathrm{acl}_T(\emptyset), S(\mathrm{acl}_T(\emptyset))).$$

(4) For all A, $\mathrm{acl}_{TS}(A) = \mathrm{acl}_T(A)$.

Proof. *(1)* The left to right implication is standard. From right to left. Note that, under hypotheses, we may assume that $A = \mathrm{acl}_T(A)$ is a subset of both $\mathscr{M}$ and $\mathscr{N}$ and that $\mathscr{M}_0 \cap A = \mathscr{N}_0 \cap A$. By Lemma 2.2.1, there exists $\mathscr{M}' \equiv^T_A \mathscr{M}$ such that $\mathscr{M}' \mathop{\smile\!\!\!\!\!\!\mid}^0_A \mathscr{N}$. There is an $\mathscr{L}$-isomorphism g between $\mathscr{M}'$ and $\mathscr{M}$ that fixes A, so we may define $\mathscr{M}_0' = g^{-1}(\mathscr{M}_0)$ and turn $(\mathscr{M}', \mathscr{M}_0')$ into a model of TS. By MONOTONICITY and BASE MONOTONICITY we have $\mathscr{M}_0' \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathscr{N}_0} \mathscr{N}$. Similarly we have $\mathscr{N}_0 \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathscr{M}_0'} \mathscr{M}'$ hence by Lemma 2.3.4 there exists a model $(\mathscr{K}, \mathscr{K}_0)$ of TS that is an elementary extension of both $(\mathscr{M}', \mathscr{M}_0')$ and $(\mathscr{N}, \mathscr{N}_0)$, hence $(\mathscr{M}', \mathscr{M}_0') \equiv^{TS}_A (\mathscr{K}, \mathscr{K}_0) \equiv^{TS}_A (\mathscr{N}, \mathscr{N}_0)$.

(2) This is similar to *(1)*.

(3) This is an obvious application of *(1)*.

(4) We only need to show that $\mathrm{acl}_{TS}(A) \subseteq \mathrm{acl}_T(A)$. Assume that $b \notin \mathrm{acl}_T(A)$. Let $(\mathscr{M}, \mathscr{M}_0)$ be a model of TS containing b. There exists a model $\mathscr{N}$ of T and a T-isomorphism $f : \mathscr{N} \to \mathscr{M}$ over A such that $\mathscr{N} \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathrm{acl}_T(A)} \mathscr{M}$. Consider $\mathscr{N}_0 = f^{-1}(\mathscr{M}_0)$, then $(\mathscr{N}, \mathscr{N}_0)$ and $(\mathscr{M}, \mathscr{M}_0)$ are $\mathscr{L}_S$-isomorphic. Now set $b' = f^{-1}(b)$, we have $b' \equiv^{TS}_A b$ and $b \neq b'$ because $b \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathrm{acl}_T(A)} b'$ and $b \notin \mathrm{acl}_T(A)$. Since $\mathscr{N} \mathop{\smile\!\!\!\!\!\!\mid}^0_{\mathrm{acl}_T(A)} \mathscr{M}$, we may do as in *(1)* and find a model of TS extending both $\mathscr{M}$ and $\mathscr{N}$ in which the condition *(3)* is satisfied. Similarly we can produce as many conjugates of b over A as we want inside some bigger model so $b \notin \mathrm{acl}_{TS}(A)$. □

Proposition 2.3.6. *Let $\mathbb{M}$ be a monster model of T. Let $\mathscr{M} \prec \mathbb{M}$ and $\mathscr{M}_0 \subseteq \mathscr{M}$ such that $(\mathscr{M}, \mathscr{M}_0)$ is a model of TS. Let $B \subset \mathscr{M}$, and X a small subset of $\mathbb{M}$. Let $S_{XB} \subseteq \mathrm{acl}_T(XB) \subset \mathbb{M}$ be some acl_0-closed set containing $S(\mathrm{acl}_T(B))$ and such that:*

(1) $S_{XB} \cap \mathscr{M} = S(\mathrm{acl}_T(B))$

(2) $\mathrm{acl}_T(XB) \cap \mathscr{M} = \mathrm{acl}_T(B)$.

Then the type (over B) associated to the T-elementary $\mathscr{L}_S$-isomorphism type of $(\mathrm{acl}_T(XB), S_{XB})$ *is consistent in $Th(\mathscr{M}, \mathscr{M}_0)$.*

Proof. Let $\mathbb{M}_0' = \mathrm{acl}_0(\mathscr{M}_0, S_{XB})$. We have that $(\mathbb{M}, \mathbb{M}_0')$ is a model of T_S and an extension of $(\mathscr{M}, \mathscr{M}_0)$. Indeed, $\mathbb{M}_0' \cap \mathscr{M} = \mathrm{acl}_0(\mathscr{M}_0, S_{XB}) \cap \mathscr{M} = \mathrm{acl}_0(\mathscr{M}_0, S_{XB} \cap \mathscr{M})$ by modularity. By hypothesis *(1)*, $S_{XB} \cap \mathscr{M} = S(\mathrm{acl}_T(B) \subseteq \mathscr{M}_0$ hence $\mathbb{M}_0' \cap \mathscr{M} = \mathscr{M}_0$. By Theorem 2.1.5 there exists a model $(\mathscr{N}, \mathscr{N}_0)$ of TS extending $(\mathbb{M}, \mathbb{M}_0')$ which is an elementary extension of $(\mathscr{M}, \mathscr{M}_0)$. Now

$$
\begin{aligned}
\mathrm{acl}_T(XB) \cap \mathscr{N}_0 &= \mathrm{acl}_T(XB) \cap \mathbb{M}_0 \\
&= \mathrm{acl}_T(XB) \cap \mathrm{acl}_0(\mathscr{M}_0, S_{XB}) \\
&= \mathrm{acl}_0(S_{XB}, \mathrm{acl}_T(XB) \cap \mathscr{M}_0) && \text{by modularity} \\
&= \mathrm{acl}_0(S_{XB}, \mathrm{acl}_T(B) \cap \mathscr{M}_0) && \text{by } (2) \\
&= \mathrm{acl}_0(S_{XB}, S(\mathrm{acl}_T(B))) \\
&= S_{XB}.
\end{aligned}
$$

It follows that in $(\mathscr{N}, \mathscr{N}_0)$, $tp^{TS}(X/B)$ is given by the T-elementary $\mathscr{L}_S$-isomorphism type of $(\mathrm{acl}_T(XB), S_{XB})$. □

2.4 Iterating the construction

Let T be an $\mathscr{L}$-theory, $\mathscr{L}_1, \cdots, \mathscr{L}_n$ be sublanguages of $\mathscr{L}$ and let $T_i = T \restriction \mathscr{L}_i$. Let $S_1, \cdots, S_n$ be new unary predicate and let $\mathscr{L}_{S_1 \dots S_n}$ be the language $\mathscr{L} \cup \{S_1, \cdots, S_n\}$. Let $T_{S_1 \dots S_n}$ be the $\mathscr{L}_{S_1 \dots S_n}$-theory which models are models $\mathscr{M}$ of T in which $\mathscr{M}_i := S_i(\mathscr{M})$ is an $\mathscr{L}_i$-substructure of $\mathscr{M}$ and a model of T_i. The following give a condition for the existence of a model companion for $T_{S_1 \dots S_n}$.

Proposition 2.4.1. *Assume inductively that $(TS_1 \dots S_i, T_{i+1}, \mathscr{L}_{i+1})$ is a suitable triple for $i = 0, \cdots, n-1$, and let $TS_1 \dots S_{i+1}$ be the model companion of the theory $TS_1, \dots, S_{i S_{i+1}}$ of models of $TS_1, \dots, S_i$ with a predicate S_{i+1} for an $\mathscr{L}_{i+1}$ submodel of T_{i+1}. Then $TS_1 \cdots S_n$ is the model-companion of the theory $T_{S_1 \dots S_n}$.*

Proof. We show the following:

(1) every model $(\mathscr{M}, \mathscr{M}_1, \dots \mathscr{M}_n)$ of $T_{S_1 \dots S_n}$ can be extended to a model $(\mathscr{N}, \mathscr{N}_1, \cdots, \mathscr{N}_n)$ of $TS_1 \dots S_n$;

(2) every model $(\mathscr{N}, \mathscr{N}_1, \dots, \mathscr{N}_n)$ of $TS_1 \dots S_n$ is existentially closed in an extension $(\mathscr{M}, \mathscr{M}_1, \dots, \mathscr{M}_n)$ model of $T_{S_1, \dots S_n}$.

(1) Start by extending $(\mathscr{M}, \mathscr{M}_1)$ to a model $(\mathscr{N}^1, \mathscr{N}_1^1)$ of TS_1. Then $(\mathscr{N}^1, \mathscr{N}_1^1, \mathscr{M}_2)$ is a model of $TS_{1 S_2}$ so can be extended to a model $(\mathscr{N}^2, \mathscr{N}_1^2, \mathscr{N}_2^2)$ of TS_1S_2. The structure $(\mathscr{N}^2, \mathscr{N}_1^2, \mathscr{N}_2^2)$ is also an extension of $(\mathscr{M}, \mathscr{M}_1, \mathscr{M}_2)$. We iterate this process to end with a model $(\mathscr{N}^n, \mathscr{N}_1^n, \cdots, \mathscr{N}_n^n)$ of $TS_1 \cdots S_n$ extending $(\mathscr{M}, \mathscr{M}_1, \cdots, \mathscr{M}_n)$.
(2) Let $(\mathscr{N}, \mathscr{N}_1, \cdots, \mathscr{N}_n)$ be a model of $TS_1 \cdots S_n$ and $(\mathscr{M}, \mathscr{M}_1, \cdots, \mathscr{M}_n)$ be a model of $T_{S_1 \dots S_n}$ extending it. By (1) there exists a model $(\mathscr{M}', \mathscr{M}_1', \cdots, \mathscr{M}_n')$ of $TS_1 \cdots S_n$ extending $(\mathscr{M}, \mathscr{M}_1, \cdots, \mathscr{M}_n)$. As $(\mathscr{N}, \mathscr{N}_1, \cdots, \mathscr{N}_n)$ is a model of $TS_1 \cdots S_n$ it is existentially closed in any model of $TS_1 \cdots S_{n-1 S_n}$ extending it, in particular, it is existentially closed in $(\mathscr{M}', \mathscr{M}_1', \cdots, \mathscr{M}_n')$ and hence also in $(\mathscr{M}, \mathscr{M}_1, \dots, \mathscr{M}_n)$. $\square$

In a model of $TS_1 \cdots S_n$, the relations between the S_i are very generic. For example, it is not possible that $S_i \subseteq S_j$ for some i, j, since one can always extend the predicate S_i by a new element which is not in S_j. In a sense, those generic predicates are invisible from one another. A way to impose relations between the S_i, is by considering, for instance, a slightly stronger version of the generic expansion by a reduct –analogously to the generic predicate in [CP98]. Consider a suitable triple $(T, T_0, \mathscr{L}_0)$ and P a 0-definable predicate in T such that in any model $\mathscr{M}$ of T, P is a model of T_0 which is a substructure of $\mathscr{M}$. One may do the construction of the generic expansion by a substructure S inside P. In that case, assume that $T_i = T_j$ for all $i, j \leq n$. One may construct TS_1 then add a generic substructure S_2 inside S_1 and iterate. This would be the model companion of the theory $T_{S_1 \dots S_n} \cup \{S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n\}$. One may also consider the case in which T_i is not the theory of a substructure but of a structure 0-definable in T.

CHAPTER 3

Examples of generic expansion by a reduct

In this chapter, we apply the results of Chapter 2 to construct new examples of generic expansions.

Contents

3.1 Generic vector subspaces over a finite field

Let $\mathbb{F}_q$ be a finite field. In this section, we let $\mathscr{L}_0 = \{(\lambda_\alpha)_{\alpha\in\mathbb{F}_q}, +, 0\}$, and $\mathscr{L}$ a language containing $\mathscr{L}_0$. We let T be a complete $\mathscr{L}$-theory which contains the $\mathscr{L}_0$-theory T_0 of infinite-dimensional $\mathbb{F}_q$-vector spaces. For A a subset of a model of T, the set $\mathrm{acl}_0(A)$ is the vector space spanned by A, and we denote it by $\langle A\rangle$. Let $\mathscr{L}_V = \mathscr{L}\cup\{V\}$, with V a unary predicate and T_V the $\mathscr{L}_V$-theory whose models are the models of T in which V is an infinite vector subspace.

Definability and notations. For $\alpha = \alpha_1, \dots, \alpha_n \in \mathbb{F}_q$ and any n-tuple x of variables let $\lambda_\alpha(x)$ be the term

$$\lambda_{\alpha_1}(x_1) + \cdots + \lambda_{\alpha_n}(x_n).$$

Let z be a tuple of variables of length $s = q^n - 1$ and $z' = z_0 z$ a tuple of length $s+1 = q^n$. Let $\psi(t)$ be any $\mathscr{L}_V$-formula, t a single variable. We fix an enumeration $\alpha^1, \dots, \alpha^s$ of $(\mathbb{F}_q)^n \setminus (0, \dots, 0)$. We denote by

$z = \langle x\rangle_0$	the formula	$\bigwedge_{i=1,\dots,s} z_i = \lambda_{\alpha^i}(x)$
$z' = \langle x\rangle$	the formula	$z_0 = 0 \wedge z_1, \cdots, z_s = \langle x\rangle_0$
$t \in \langle x\rangle$	the formula	$\forall z'\left(z' = \langle x\rangle \to \bigvee_{i=0}^{s} t = z_i\right)$
$t \in \langle xy\rangle \setminus \langle y\rangle$	the formula	$t \in \langle xy\rangle \wedge \neg t \in \langle y\rangle$
$\langle x\rangle \cap \psi = \langle y\rangle$	the formula	$\forall t\,(t\in\langle x\rangle \wedge \psi(t) \leftrightarrow t \in \langle y\rangle)$.

The formulae above have the obvious meaning, for instance, for any a, b in a model of T, if $\mathscr{M} \models b = \langle a\rangle_0$ then b is an enumeration of all non-trivial $\mathbb{F}_q$-linear combinations of a.

The following is [CP98, Lemma 2.3]:

Fact 3.1.1. *Assume that T is a theory that eliminates the quantifier $\exists^\infty$. Then for any formula $\phi(x, y)$ there is a formula $\theta_\phi(y)$ such that in any $\aleph_0$-saturated model $\mathscr{M}$ of T the set $\theta_\phi(\mathscr{M})$ consists of tuples b from $\mathscr{M}$ such that there exists a realisation a of $\phi(x, b)$ with $a_i \notin \mathrm{acl}_T(b)$ for all i.*

Theorem 3.1.2. *If T is model complete and eliminates the quantifier $\exists^\infty$, then $(T, T_0, \mathscr{L}_0)$ is a suitable triple. It follows that the theory T_V admits a model companion, which we denote by TV.*

Proof. We have to show that the triple $(T, T_0, \mathscr{L}_0)$ is suitable, the existence of the model-companion then follows from Proposition 2.3.3. We check the conditions of Definition 2.3.1:

(H_1) T is model complete;

(H_2) T_0 model complete and for all infinite A, $\langle A\rangle \models T_0$;

(H_3^+) $\langle\cdot\rangle$ defines a modular pregeometry;

(H_4) for all $\mathscr{L}$-formula $\phi(x,y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for $b \in \mathscr{M} \models T$

$$\mathscr{M} \models \theta_\phi(b) \iff \text{there exists a saturated } \mathscr{N} \succ \mathscr{M} \text{ and } a \in \mathscr{N} \text{ such that } \phi(a,b) \text{ and } a \text{ is } \mathop{\smile\!\!\!\!\!\!\mid}^0\text{-independent over } \mathscr{M}.$$

Condition (H_1) holds by hypothesis. Conditions (H_2) and (H_3^+) are also clear, these are basic properties of the theory of infinite dimensional vector spaces. As A is infinite, $\langle A\rangle$ is an infinite dimensional $\mathbb{F}_q$-vector space.

We prove condition (H_4). Let $\phi(x,y)$ be an $\mathscr{L}$-formula. For some tuple of variables z of suitable length, let $\tilde{\phi}(z,y)$ be the following formula

$$\exists x \; z = \langle x\rangle_0 \wedge \phi(x,y).$$

Now apply Fact 3.1.1 with $\tilde{\phi}(z,y)$. We get a formula $\theta_{\tilde{\phi}}(y)$ such that for any $\aleph_0$-saturated model $\mathscr{N}$ of T and $b \in \mathscr{N}$ we have that $\mathscr{N} \models \theta_{\tilde{\phi}}(b)$ if and only if there exist tuples a and c in $\mathscr{N}$ such that $\phi(a,b)$ holds, $c = \langle a\rangle_0$ and for all i, $c_i \notin \mathrm{acl}_T(b)$. Equivalently $\mathscr{N} \models \theta_{\tilde{\phi}}(b)$ if and only if there exists a tuple a from $\mathscr{N}$ such that a is $\mathbb{F}_q$-linearly independent over $\mathrm{acl}_T(b)$ and $\mathscr{N} \models \phi(a,b)$. By Lemma 2.2.2, this condition is equivalent to (H_4), hence the triple $(T, T_0, \mathscr{L}_0)$ is suitable. □

Lemma 3.1.3. *Let $\psi(x,y)$ be an $\mathscr{L}_V$-formula. Assume that in a saturated model $(\mathscr{M}, V)$ of T_V the following holds for some tuple b from $\mathscr{M}$, for all $\mathscr{L}$-formula $\phi(x,y)$:*

$$\theta_\phi(b) \to \exists x \phi(x,b) \wedge \psi(x,b).$$

Then for all $\phi(x,y)$, if $\mathscr{M} \models \theta_\phi(b)$ then there exists a realisation a of $\phi(x,b) \wedge \psi(x,b)$ such that a is linearly independent over $\mathrm{acl}_T(b)$.

Proof. Let $\Sigma(x,y)$ be the partial type expressing "x is linearly independent over $\mathrm{acl}_T(y)$" (see Section 2.2). We claim that $\{\phi(x,b) \wedge \psi(x,b)\} \cup \Sigma(x,b)$ is consistent. Indeed, let $\Lambda(x,b)$ be a finite conjunction of formulae in $\Sigma(x,b)$. As $\theta_\phi(b)$ holds, by Lemma 2.2.2 there exists a realisation a of $\phi(x,b)$ which is $\mathbb{F}_q$-linearly independent over $\mathrm{acl}_T(b)$, hence in particular a satisfies $\phi(x,b) \wedge \Lambda(x,b)$, hence $\mathscr{M} \models \theta_{\phi\wedge\Lambda}(b)$. By hypothesis, the formula $\phi(x,b) \wedge \Lambda(x,b) \wedge \psi(x,b)$ is consistent, hence we conclude by compactness. □

Proposition 3.1.4 (Axioms for TV)**.** *The theory TV is axiomatised by adding to T_V the following $\mathscr{L}_V$-sentences, for all tuples of variable $y_V \subset y$, $x_V \subset x$ and $\mathscr{L}$-formula $\phi(x,y)$*

$$\forall y(\langle y\rangle \cap V = \langle y_V\rangle \wedge \theta_\phi(y)) \to (\exists x \phi(x,y) \wedge \langle xy\rangle \cap V = \langle x_V y_V\rangle). \qquad (A_1)$$

Equivalently, the theory TV is axiomatised by adding to T_V the following $\mathscr{L}_V$-sentences, for all tuples of variable $y^1 \subseteq y$, $x_V \subset x$ and $\mathscr{L}$-formula $\phi(x,y)$

$$\forall y(\langle y^1\rangle \cap V = \{0\} \wedge \theta_\phi(y)) \to (\exists x \phi(x,y) \wedge \langle xy^1\rangle \cap V = \langle x_V\rangle). \qquad (A_2)$$

Proof. It is clear that the system of axioms (A_1) is equivalent to the one given in Theorem 2.1.5. It is also clear that the system of axioms (A_1) implies the system of axioms (A_2). We show that the two systems are equivalent. Assume that the system (A_2) is satisfied in an $\aleph_0$ saturated model $(\mathscr{M}, V)$ of T_V. Let $\phi(x, y)$ be given, and subtuples y_V of y and x_V of x. We show that $(\mathscr{M}, V)$ satisfies the axiom of the form (A_1) given by $y_V \subset y$, $x_V \subset x$ and $\phi(x, y)$. Assume that for some tuple b from $\mathscr{M}$, the formula $\langle b\rangle \cap V = \langle b_V\rangle \wedge \theta_\phi(b)$ holds. Let b^1 be a subtuple of b which is a basis of $\langle b\rangle$ over $\langle b_V\rangle$. We have $\langle b^1\rangle \cap V = \{0\}$ hence using an instance of an axiom (A_2), there exists a realisation a of $\phi(x, b)$ such that $\langle ab^1\rangle \cap V = \langle a_V\rangle$. Since $b_V \subseteq V$, it follows from Base Monotonicity that $\langle ab\rangle \cap V = \langle a_V b_V\rangle$. □

Lemma 3.1.5. *Assume that T is model complete and eliminates the quantifier $\exists^\infty$. Then TV eliminates the quantifier $\exists^\infty$, so $(TV, T_0, \mathscr{L}_0)$ is also a suitable triple.*

Proof. Assume that $|x| = 1$. From the description of types (see Proposition 2.3.5), types in TS are obtained by adding to the types in T the description of V on the algebraic closure. By compactness, every $\mathscr{L}_V$-formula $\phi(x, y)$ is equivalent to a disjunction of formulae of the form

$$\exists z \psi(x, z, y) \wedge \langle xz\rangle \cap V = \langle z_V\rangle$$

where $\psi(x, z, y)$ is an $\mathscr{L}$-formula (not necessarily quantifier-free) and z_V a subtuple of variables of z[1]. In order to prove elimination of $\exists^\infty$, by the pigeonhole principle , we may assume that $\phi(x, y)$ is equivalent to such a formula. Now let u, v be two tuples of variables such that $|u| + |v| \leq |z| + 1$, and let $u_V \subset u$, $v_V \subset v$ be two subtuples. Let $\Gamma^{uv}_{u_V v_V}(u, yv)$ be the following $\mathscr{L}$-formula

$$\exists xz \psi(x, z, y) \wedge \langle xz\rangle = \langle uv\rangle \wedge \langle z_v\rangle = \langle u_V v_V\rangle \wedge x \in \langle uv\rangle \setminus \langle v\rangle.$$

Let $\Lambda(y)$ be the formula

$$\bigvee_{|uv| \leq |z|+1, u_V \subseteq u, v_V \subseteq v, |u| \geq 1} \exists v(\langle v\rangle \cap V = \langle v_V\rangle \wedge \theta_{\Gamma^{uv}_{u_V v_V}}(yv)).$$

Claim: For all tuple b from a saturated model $(\mathscr{M}, V)$ of TV, $(\mathscr{M}, V) \models \Lambda(b)$ if and only if there exists $a \in \mathscr{M}$ such that $(\mathscr{M}, V) \models \phi(a, b)$ and $a \notin \mathrm{acl}_T(b)$.

From left to right. If $\Lambda(b)$ holds for some b, there exists a formula $\Gamma = \Gamma^{uv}_{u_V v_V}$ and some tuple e from $\mathscr{M}$ and a subtuple e_V of e such that $V \cap \langle e\rangle = \langle e_V\rangle$ and $\mathscr{M} \models \theta_\Gamma(be)$. Using one instance of the axioms (A_1) (Proposition 3.1.4) and Lemma 3.1.3, there exists a realisation d of $\Gamma(u, be)$ such that $\langle dbe\rangle \cap V = \langle d_V b_V e_V\rangle$, for d_V the subtuple associated to the variables u_V and such that d is linearly independent over $\mathrm{acl}_T(be)$. Using that d is linearly independent over $\langle de\rangle$, we obtain that $\langle de\rangle \cap V = \langle d_V e_V\rangle$. As $(\mathscr{M}, V) \models \Gamma(d, be)$, there exists a and a tuple c from $\mathscr{M}$ such that

[1] Actually we might assume that every realisation of z in ψ is algebraic over the realisations of x, y in ψ, but we don't need this fact here. Also, we may replace the condition $\langle xz\rangle \cap V = \langle z_V\rangle$ by $\langle z\rangle \cap V = \langle z_V\rangle$, but we assume that the formula gives a description of V on $\langle xz\rangle$ in order to simplify the proof.

- $\mathscr{M} \models \psi(a, c, b)$
- $\langle ac \rangle = \langle de \rangle$
- $\langle c_V \rangle = \langle d_V e_V \rangle$
- $a \in \langle de \rangle \setminus \langle e \rangle$.

Now as $\langle de \rangle \cap V = \langle d_V e_V \rangle$ we have $\langle ac \rangle \cap V = \langle c_V \rangle$ so $(\mathscr{M}, V) \models \phi(a, b)$. Now as d is linearly independent over $\mathrm{acl}_T(be)$ and $a \in \langle de \rangle \setminus \langle e \rangle$ we have $a \notin \mathrm{acl}_T(be)$ so $a \notin \mathrm{acl}_T(b)$.

From right to left. Assume that $(\mathscr{M}, V) \models \phi(a, b)$ and $a \notin \mathrm{acl}_T(b)$. Let c be such that $c \models \psi(a, z, b)$ and $\langle ac \rangle \cap V = \langle c_V \rangle$. Let e_V be a basis of $\mathrm{acl}_T(b) \cap V \cap \langle ac \rangle$, and complete it in a basis e of $\mathrm{acl}_T(b) \cap \langle ac \rangle$. Let d_V be a basis of a complement of $\langle e_V \rangle$ inside $\langle ac \rangle \cap V$ and complete it in a basis d of a complement of $\langle ed_V \rangle$ inside $\langle ac \rangle$. As $a \in \langle de \rangle \setminus \mathrm{acl}_T(b)$ we have $a \in \langle de \rangle \setminus \langle e \rangle$. It is clear that $(\mathscr{M}, V) \models \Gamma^{uv}_{u_V v_V}(d, be)$ for the appropriate choice of subtuple of variables $u_V \subseteq u$ and $v_V \subseteq v$. Furthermore, as d is linearly independent over $\mathrm{acl}_T(b) = \mathrm{acl}_T(be)$, we have $\theta_\Gamma(be)$, and so $\Lambda(b)$ holds. □

Corollary 3.1.6. *Assume that T is model-complete and eliminates $\exists^\infty$. Let $T_{V_1 \dots V_n}$ be the theory whose models are models of T in which V_i is a predicate for a vector subspace over $\mathbb{F}_q$. Then $T_{V_1 \dots V_n}$ admits a model companion $TV_1 \dots V_n$.*

Proof. This is an immediate consequence of Lemma 3.1.5 and Proposition 2.4.1. □

Example 3.1.7 (Generic vector subspace of a vector space)**.** Consider the theory T of infinite $\mathbb{F}_q$-vector spaces in the language $\mathscr{L} = \{(\lambda_\alpha)_{\alpha \in \mathbb{F}_q}, +, 0\}$. Applying Corollary 3.1.6 the theory $T_{V_1 \dots V_n}$ admits a model companion $TV_1 \dots V_n$. It is easy to check that TV_1 is the theory of belles paires (see [Poi83]) of the theory T, hence as T is NFCP, TV_1 is stable. One can easily show that TV_1 has U-rank 2, and one expects that $TV_1 \dots V_n$ has U-rank $n + 1$. This is a particular case of Proposition 3.4.1.

3.2 Fields with generic additive subgroups

Let $p > 0$ be a prime number. Let $\mathscr{L} = \{+, -, \cdot, 0, 1, \dots\}$ and T an $\mathscr{L}$-theory of an infinite field of characteristic p. Let $\mathbb{F}_{q_1}, \cdots, \mathbb{F}_{q_n}$ be finite subfields in any model of T. Consider the theory T' obtained by adding to the language a constant symbol for each element of $\mathbb{F}_{q_1} \cup \dots \cup \mathbb{F}_{q_n}$. Then T and T' have the same models. It follows that for each i we may consider that the theory of infinite $\mathbb{F}_{q_i}$-vector space in the language $\mathscr{L}_i = \left\{+, 0, (\lambda_\alpha)_{\alpha \in \mathbb{F}_{q_i}}\right\}$ is a reduct of T.

Proposition 3.2.1. *Let $\mathscr{L} \supseteq \mathscr{L}_{\mathrm{ring}}$ and T an $\mathscr{L}$-theory of an infinite field of characteristic p. Let $\mathbb{F}_{q_1}, \cdots, \mathbb{F}_{q_n}$ be finite subfields in any model of T. Assume that*

(1) T is model-complete;

(2) T eliminates $\exists^\infty$.

Let $T_{V_1 \ldots V_n}$ be the theory whose models are models of T in which each V_i is a predicate for an $\mathbb{F}_{q_i}$-vector subspace. By Corollary 3.1.6 the theory $T_{V_1 \ldots V_n}$ admits a model-companion.

An additive subgroup of a field of characteristic p is an $\mathbb{F}_p$-vector space, hence Proposition 3.2.1 translates as follows.

Proposition 3.2.2. *Let $\mathscr{L} \supseteq \mathscr{L}_{\mathrm{ring}}$ and T an $\mathscr{L}$-theory of an infinite field of characteristic p. Assume that*

(1) T is model-complete;

(2) T eliminates $\exists^\infty$.

Let $T_{G_1 \ldots G_n}$ be the theory whose models are models of T in which each G_i is a predicate for an additive subgroup. By Corollary 3.1.6 the theory $T_{G_1 \ldots G_n}$ admits a model-companion, which we denote by $TG_1 \ldots G_n$.

Example 3.2.3. The hypotheses of Propositions 3.2.1 and 3.2.2 are satisfied by the following theories by Subsection 1.5.2:

- ACF_p, $\mathrm{SCF}_{p,e}$ for e finite or infinite, Psf_c,
- ACFA_p, DCF_p.

Example 3.2.4 ($\mathrm{ACFV}_1 \cdots \mathrm{V}_n$ and ACFG)**.** Let $\mathbb{F}_{q_1}, \cdots, \mathbb{F}_{q_n}$ be any finite fields of characteristic p. We denote by $\mathrm{ACFV}_1 \cdots \mathrm{V}_n$ and ACFG respectively the theories $\mathrm{ACF}_p\mathrm{V}_1 \cdots \mathrm{V}_n$ and ACF_pG. Chapters 5, 6 and 7 are dedicated to a detailed study of the theory ACFG, which is NSOP_1 and not simple (see also Example 4.4.3).

Recall from Subsection 1.5.2 that a pseudo-algebraically closed field is a field K which is existentially closed in every regular extension. The theory PAC is incomplete but eliminates $\exists^\infty$ if the field is perfect (Fact 1.5.16).

Proposition 3.2.5. *Let PAC_{pG} be the theory whose models are perfect PAC_p-fields in $\mathscr{L}_{\mathrm{ring}}$ with a predicate G for an additive subgroup. Then there exists a theory PAC_pG such that*

(1) every model (F, G') of PAC_{pG} extends to a model (K, G) of PAC_pG such that K is a regular extension of F;

(2) every model (K, G) of PAC_pG is existentially closed in every extension (F, G') such that F is a regular extension of K.

Let T be a theory of perfect PAC_p-fields in a language containing $\mathscr{L}_{\mathrm{ring}}$ such that T is model-complete, and $T_{G_1 \cdots G_n}$ be the theory whose models are models of T with predicates G_i for additive subgroups. Then $T_{G_1 \cdots G_n}$ admits a model-companion, $TG_1 \cdots G_n$.

Proof. Perfect PAC_p-fields in $\mathscr{L}_{\mathrm{ring}}$ satisfies (H_4), the proof of this in Theorem 3.1.2 does not use the model-completeness of the theory T, so the first statement follows from Proposition 2.1.7. The second statement is Corollary 3.1.6. □

Remark 3.2.6. Note that the *perfect* assumption is only here to ensure that the fields eliminate the quantifier $\exists^\infty$. It should be true that all PAC fields eliminate the quantifier $\exists^\infty$ although we did not find any reference in the literature.

However, in the characteristic 0 case the model-companion does not exist.

Proposition 3.2.7. *Let T be the theory of a field of characteristic* 0 *in a language $\mathscr{L}$ containing $\mathscr{L}_{\text{ring}}$, such that T is inductive. Let $\mathscr{L}_G = \mathscr{L} \cup \{G\}$ and let T_G be the $\mathscr{L}_G$-theory of models of T in which G is a predicate for an additive subgroup of the field. Let (K, G) be an existentially closed model of T_G. Then*

$$S_K(G) := \{a \in K \mid aG \subseteq G\} = \mathbb{Z}.$$

In particular, the theory T_G does not admit a model-companion.

Proof. The right to left inclusion is trivial. Assume that $a \in K \setminus \mathbb{Z}$, let L be a proper elementary extension of K and $t \in L \setminus K$. Then $(L, G + \mathbb{Z}\frac{t}{a})$ is an $\mathscr{L}_G$-extension of (K, G). Furthermore, as $a \notin \mathbb{Z}$, we have $t \notin G + \mathbb{Z}\frac{t}{a}$. Then $\frac{t}{a} \in G + \mathbb{Z}\frac{t}{a}$ and $a\frac{t}{a} \notin G + \mathbb{Z}\frac{t}{a}$. As (K, G) is existentially closed in $(L, G + \mathbb{Z}\frac{t}{a})$, we have that

$$(K, G) \models \exists x (x \in G \wedge ax \notin G)$$

hence $a \notin S_K(G)$. The class of existentially closed models of T_G is not axiomatisable as the definable infinite set $S_L(G)$ is of fixed cardinality. As T_G is inductive, this is equivalent to saying that T_G does not admit a model-companion. $\square$

Remark 3.2.8. Let T be the theory of a field of characteristic 0 in a language $\mathscr{L}$ containing $\mathscr{L}_{\text{ring}}$, such that T is inductive. Let $\mathscr{L}_D = \mathscr{L} \cup \{D\}$ and let T_D be the $\mathscr{L}_D$-theory of models of T in which D is a predicate for a *divisible* additive subgroup of the field. Let (K, D) be an existentially closed model of T_D. A similar argument yields that $\{a \in K \mid aD = D\} = \mathbb{Q}$, so T_D does not admits a model-companion either.

Remark 3.2.9. Let $K = \mathbb{C}$ (or $\mathbb{R}$). Using Remark 3.2.8 and Lemmas 2.2.2 and 2.2.3, one deduces that there exist $k, l \in \mathbb{N}$ and a constructible set if $K = \mathbb{C}$ (or a semialgebraic set if $K = \mathbb{R}$) $V \subseteq K^k \times K^l$ such that for all polynomials $P(X, Y) \in K[X, Y]$ with $|X| = 1$, $|Y| = l$ and for all $n \in \mathbb{N}$ and all $q_1, \dots q_n, s_1, \dots, s_k \in \mathbb{Q}$ there exists $b \in K^l$ such that for all $a \in K^k$, if $(a, b) \in V$ then

(1) a is not $\mathbb{Q}$-linearly independent over $\overline{\mathbb{Q}(b)} \cap K$;

(2) $\sum_{i=1}^k s_i a_i \notin q_1 R + \cdots + q_n R$ for R the set of roots of $P(X, b)$ in K.

3.3 Algebraically closed fields with a generic multiplicative subgroup

We are now interested in using Theorem 2.1.5 to prove that the theory of algebraically closed fields of fixed arbitrary characteristic with a predicate for a multiplicative subgroup admits a model companion. Consider $\mathscr{L}_{\text{field}} = \{+, -, \cdot, ^{-1}, 0, 1\}$ and $\mathscr{L}_0 = \{\cdot, ^{-1}, 1\} \subseteq \mathscr{L}_{\text{field}}$.

The pure multiplicative group of any field is an $\aleph_1$-categorical abelian group, its model theory is described in [Mac71], see also [Che76, Chapter VI].

Fix p a prime or 0. Consider the theory ACF_p. The theory $\mathrm{ACF}_p \restriction \mathscr{L}_0$ is complete and we will identify it with the theory of the multiplicative group of an algebraically closed field of characteristic p, denoted by T_p. The theory T_p is axiomatised by adding to the theory of abelian groups the following sets of axiom:

- *If $p > 0$:* $\{\forall x\ \exists^{=n} y\ \ y^n = x \mid n \in \mathbb{N} \setminus p\mathbb{N}\} \cup \{\forall x \exists^{=1} y\ y^p = x\}$
- *If $p = 0$:* $\{\forall x\ \exists^{=n} y\ \ y^n = x \mid n \in \mathbb{N} \setminus \{0\}\}$.

Proposition 3.3.1. *The theory T_p has quantifier elimination in the language $\mathscr{L}_0$. It is strongly minimal hence $\aleph_1$-categorical. Furthermore for any subset A of a model M of T_p, the algebraic closure is given by*

$$\mathrm{acl}_p(A) := \{u \in M, u^n \in \langle A \rangle \text{ for some } n \in \mathbb{N} \setminus \{0\}\}$$

where $\langle A \rangle$ is the group spanned by A. Every algebraically closed set is a model of T_p. Furthermore acl_p *defines a pregeometry which is modular and the associated independence relation in T_p is given by*

$$A \mathop{\smile\!\!\!\!\!\!|}^{p}_{C} B :\iff \mathrm{acl}_p(AC) \cap \mathrm{acl}_p(BC) = \mathrm{acl}_p(C).$$

See Subsection 1.5.2 for basics about affine varieties and generics of a variety.

Lemma 3.3.2. *Let $K \models \mathrm{ACF}$, $V \subset K^n$ an affine (irreducible) variety, $\mathcal{O} \subset K^n$ a Zariski open set. The following are equivalent:*

(1) for all $k_1, \ldots, k_n \in \mathbb{N}$, $c \in K$ the quasi affine variety $V \cap \mathcal{O}$ is not included in the zero set of $x_1^{k_1} \cdot \cdots \cdot x_n^{k_n} = c$

(2) for all $k_1, \ldots, k_n \in \mathbb{N}$, $c \in K$ the variety V is not included in the zero set of $x_1^{k_1} \cdot \cdots \cdot x_n^{k_n} = c$

(3) there exist $L \succ K$ and a tuple a which is multiplicatively independent over K and with $a \in (V \cap \mathcal{O})(L)$

Proof. *(1)* implies *(2)* is trivial. We show that *(2)* implies *(3)*. Assume that *(3)* does not hold. Take a generic a over K of the variety V in some $L \succ K$. We have $a \in \mathcal{O}$. Then there exists $k_1, \ldots, k_n \in \mathbb{N}$ such that $a_1^{k_1} \cdot \cdots \cdot a_n^{k_n} = c$ for some $c \in K$. By genericity of a, it follows that V is included in the zero set of $x_1^{k_1} \cdot \cdots \cdot x_n^{k_n} = c$, hence *(2)* does not hold. *(3)* implies *(1)* follows easily from the fact that V and $\mathcal{O}$ are definable over K. $\square$

The following fact was first observed in the proof of Theorem 1.2 in [BGH13], it is also Corollary 3.12 in [Tra17].

Fact 3.3.3. *Let p be a prime number or 0. Let $\phi(x,y)$ an $\mathscr{L}_{\text{field}}$-formula such that for all tuple b in a model of ACF_p, $\phi(x,b)$ defines an affine variety. Then there exists an $\mathscr{L}_{\text{field}}$-formula $\theta_\phi(y)$ such that for any model K of ACF_p and tuple b from K, we have $K \models \theta_\phi(b)$ if and only if for all $k_1, \dots, k_n \in \mathbb{N}$, $c \in K$, the set $\phi(K,b)$ is not included in the zero set of $x_1^{k_1} \cdot \dots \cdot x_n^{k_n} = c$.*

By Subsection 1.5.2, every definable set in ACF_p can be written as a finite union of quasi-affine varieties. Furthermore, it is standard that given any $\mathscr{L}_{\text{ring}}$-formula $\vartheta(x,z)$, the set of c such that $\vartheta(x,c)$ is a quasi-affine variety is a definable set ([Tra17, Lemma 3.10]). Let $\mathscr{C}$ be the class of formulae $\vartheta(x,z)$ such that for all $K \models \mathrm{ACF}_p$ and c tuple from K, the set $\vartheta(K,c)$ is a quasi-affine variety.

Lemma 3.3.4. *Let p be a prime number or 0. For any $\vartheta(x,z) \in \mathscr{C}$ there exists an $\mathscr{L}_{\text{field}}$-formula $\theta_\vartheta(z)$ such that for any model K of ACF_p and tuple c from K, we have $K \models \theta_\vartheta(c)$ if and only if there exists a such that $\models \vartheta(a,c)$ and a is $\mathop{\smile\!\!\!\!\!\!\mid}^{p}$-independent over K.*

Proof. Let $K \models \mathrm{ACF}_p$ and $\vartheta(x,z) \in \mathscr{C}$. Using [Joh16, Theorem 10.2.1], there exists a formula $\tilde{\vartheta}(x,z)$ such that for all tuple c from K, the set $\tilde{\vartheta}(K,c)$ is the Zariski closure of $\vartheta(K,c)$. Now by Fact 3.3.3, there exists a formula $\theta(z)$ such that $K \models \theta(c)$ if and only if $\tilde{\vartheta}(K,c)$ is not included in the zero set of $x_1^{k_1} \cdot \dots \cdot x_n^{k_n} = d$, for all $d \in K$, $k_1, \cdots, k_n \in \mathbb{N}$. By Lemma 3.3.2, $K \models \theta(c)$ if and only if there exist $L \succ K$ and a tuple a which is multiplicatively independent over K and with $a \models \vartheta(x,c)$. □

If $G^\times$ is a symbol for a unary predicate, we denote by $\mathrm{ACF}_{G^\times}$ the theory in the language $\mathscr{L}_{\text{ring}} \cup \{G^\times\}$ whose models are algebraically closed fields of characteristic p in which the predicate $G^\times$ consists of a multiplicative subgroup.

Theorem 3.3.5. *The theory $\mathrm{ACF}_{G^\times}$ admits a model companion, which we denote by* $\mathrm{ACFG}^\times$.

Proof. We check the conditions of Definition 2.3.1

(H_1) ACF_p is model complete;

(H_2) T_p is model-complete and for all infinite A, $\mathrm{acl}_p(A) \models T_p$;

(H_3^+) acl_p defines a modular pregeometry;

(H_4) for all $\mathscr{L}_{\text{field}}$-formula $\phi(x,y)$ there exists an $\mathscr{L}_{\text{field}}$-formula $\theta_\phi(y)$ such that for $b \in K \models \mathrm{ACF}_p$

$$\mathscr{M} \models \theta_\phi(b) \iff \begin{array}{l} \text{there exists } L \succ K \text{ and } a \in L \text{ such that} \\ \phi(a,b) \text{ and } a \text{ is } \mathop{\smile\!\!\!\!\!\!\mid}^{p}\text{-independent over } K. \end{array}$$

ACF_p is model complete by quantifier elimination. Conditions (H_2) and (H_3) follow from Proposition 3.3.1. We don't have condition (H_4) for all formulae, but only for the formulae in $\mathscr{C}$ (Lemma 3.3.4), which is sufficient for the existence of the model-companion by Remark 2.1.8. □

3.4 Pairs of geometric structures

Let T be an $\mathscr{L}$-theory. Let $\mathscr{L}_S$ be the expansion of $\mathscr{L}$ by a unary predicate S. A *pair* of models of T is an $\mathscr{L}_S$-structure $(\mathscr{M}, \mathscr{M}_0)$, where $\mathscr{M} \models T$ and $S(\mathscr{M}) = \mathscr{M}_0$ is a substructure of $\mathscr{M}$ model of T. We call T_S the theory of the *pairs of models of* T. This is consistent with the notations in Chapter 2.

Proposition 3.4.1. *Let T be a model-complete geometric theory (see Section 1.3) in a language $\mathscr{L}$. Assume that every acl_T-closed set is a model of T. Then there exists an $\mathscr{L}_S$-theory TS containing T_S such that:*

(1) every model $(\mathscr{N}, \mathscr{N}_0)$ of T_S has a strong extension which is a model of TS;

(2) every model of TS is existentially closed in every strong extension model of T_S.

Furthermore, TS satisfies the conclusions of Proposition 2.3.5.

Proof. We check that $T, T_0, \mathscr{L}_0$ satisfies the hypotheses of Theorem 2.1.5. (H_1), (H_2) and (H_3) are clear, and (H_4) is Fact 1.3.10. □

We call this theory the *weak model companion of the pairs of models of* T. If the pregeometry is modular, it is the model-companion.

Example 3.4.2. The theory of pairs of any strongly minimal theory with quantifier elimination admits a weak model companion. For instance, the weak model companion of the theory of pairs of algebraically closed fields is the theory of proper pairs of algebraically closed fields and coincides with the theory of belle paires of algebraically closed fields (see [Del12], [Poi83]). The theory RCF also satisfies the hypotheses of Proposition 3.4.1, hence the theory of pairs of real closed fields admits a weak model-companion. Connections with lovely pairs of geometric structures ([BV10]) could be made, although we did not investigate.

CHAPTER 4

Preservation of NSOP_1

The aim of this chapter is to establish when the construction presented in Chapter 2 preserves NSOP_1. More precisely, given some suitable triple $(T, T_0, \mathscr{L}_0)$ such that T is NSOP_1, we establish a condition on the triple $(T, T_0, \mathscr{L}_0)$ so that TS is NSOP_1. This condition (see *(A)* in Theorem 4.2.1) expresses how the pregeometry given by acl_0 is controled by the Kim-independence in T, and how the latter interacts with $\mathop{\smile\!\!\!\!\!\mid}^0$.

Contents

4.1 Independence relations in T and TS

We set up the context for this section, Section 4.2 and Section 4.3. Let $(T, T_0, \mathscr{L}_0)$ be a suitable triple (see Definition 2.3.1 and Corollary 2.3.3). We work in a monster model $(\mathbb{M}, \mathbb{M}_0)$ of TS such that $\mathbb{M}$ is a monster model of T. In particular we fix some completion of TS. Also, $\mathbb{M}$ is a monster model for T (see Section 1.1). All small sets $A, B, C, \dots$ or models $\mathscr{M}, \mathscr{N}$ of T, or models $(\mathscr{M}, \mathscr{M}_0), (\mathscr{N}, \mathscr{N}_0)$ of TS are seen as subsets of $\mathbb{M}$, respectively elementary substructures of $\mathbb{M}$ or elementary substructures of $(\mathbb{M}, \mathbb{M}_0)$. For instance we have $S(\mathscr{M}) = \mathscr{M} \cap S(\mathbb{M}) = \mathscr{M} \cap \mathbb{M}_0 = \mathscr{M}_0$. We will start with a ternary relation $(\mathop{\perp\!\!\!\perp}^T)$ defined over subsets of $\mathbb{M}$ and construct from it a ternary relation $(\mathop{\perp\!\!\!\perp}^w)$ taking into account the predicate $S(\mathbb{M}) = \mathbb{M}_0$.

We denote by $\overline{A}$ the set $\mathrm{acl}_T(A)$ which, as we saw, equals $\mathrm{acl}_{TS}(A)$.

Assumption. There exists a ternary relation $\mathop{\perp\!\!\!\perp}^T$ defined over subsets of $\mathbb{M}$, such that $\mathop{\perp\!\!\!\perp}^T \to \mathop{\perp\!\!\!\perp}^a$, where $A \mathop{\perp\!\!\!\perp}^a_C B \iff \overline{AC} \cap \overline{BC} = \overline{C}$.

In particular, if $A \mathop{\perp\!\!\!\perp}^T_C B$ then $\mathrm{acl}_T(AC) \mathop{\perp\!\!\!\perp}^0_{\mathrm{acl}_T(C)} \mathrm{acl}_T(BC)$, by modularity.

Definition 4.1.1. We call *weak independence* the relation $\mathop{\perp\!\!\!\perp}^w$ defined by

$$A \mathop{\perp\!\!\!\perp}^w_C B \iff A \mathop{\perp\!\!\!\perp}^T_C B \text{ and } S(\mathrm{acl}_0(\overline{AC}, \overline{BC})) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC})).$$

We call *strong independence* the relation $\mathop{\perp\!\!\!\perp}^{st}$ defined by

$$A \mathop{\perp\!\!\!\perp}^{st}_C B \iff A \mathop{\perp\!\!\!\perp}^T_C B \text{ and } S(\overline{ABC}) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC})).$$

Obviously $\mathop{\perp\!\!\!\perp}^{st} \to \mathop{\perp\!\!\!\perp}^w$.

We will show that if $\mathop{\perp\!\!\!\perp}^T$ satisfies most of the properties listed in Section 1.2 relatively to the theory T, then so does $\mathop{\perp\!\!\!\perp}^w$ relatively to the theory TS. The property Symmetry of $\mathop{\perp\!\!\!\perp}^0$, $\mathop{\perp\!\!\!\perp}^T$ and $\mathop{\perp\!\!\!\perp}^w$ will be tacitly used throughout this chapter.

Lemma 4.1.2. *If $\mathop{\perp\!\!\!\perp}^T$ satisfies* Invariance*,* Closure*,* Symmetry*,* Existence *and* Monotonicity*, then so does $\mathop{\perp\!\!\!\perp}^w$.*

Proof. Invariance is clear because $S(\mathrm{acl}_0(\overline{AC}, \overline{BC})) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC}))$ is an $\mathscr{L}_S$-invariant condition. Closure, Symmetry and Existence are trivial.

For Monotonicity, let A, B, C, D such that $A \mathop{\perp\!\!\!\perp}^w_C BD$. By hypothesis, $A \mathop{\perp\!\!\!\perp}^T_C B$. Now

$$\begin{aligned} S(\mathrm{acl}_0(\overline{AC}, \overline{BC})) &= S(\mathrm{acl}_0(\overline{AC}, \overline{BCD})) \cap \mathrm{acl}_0(\overline{AC}, \overline{BC}) \\ &= \mathrm{acl}_0(S(\overline{AC}), S(\overline{BCD})) \cap \mathrm{acl}_0(\overline{AC}, \overline{BC}). \end{aligned}$$

Since $S(\overline{AC}) \subseteq \mathrm{acl}_0(\overline{AC}, \overline{BC})$, we have by modularity

$$\mathrm{acl}_0(S(\overline{AC}), S(\overline{BCD})) \cap \mathrm{acl}_0(\overline{AC}, \overline{BC}) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BCD}) \cap \mathrm{acl}_0(\overline{AC}, \overline{BC})).$$

Using that $\mathop{\smile\!\!\!\!\!\mid}^T \to \mathop{\smile\!\!\!\!\!\mid}^a$, it follows from the hypotheses that $\overline{AC} \mathop{\smile\!\!\!\!\!\mid}^0_{\overline{C}} \overline{BCD}$ hence by Base Monotonicity of $\mathop{\smile\!\!\!\!\!\mid}^0$ we have $\overline{BCD} \cap \mathrm{acl}_0(\overline{AB}, \overline{BC}) = \overline{BC}$ hence

$$S(\overline{BCD}) \cap \mathrm{acl}_0(\overline{AC}, \overline{BC}) = S(\overline{BC}).$$

It follows that $S(\mathrm{acl}_0(\overline{AC}, \overline{BCD})) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC}))$ and so $A \mathop{\smile\!\!\!\!\!\mid}^w_C B$. $\square$

Lemma 4.1.3. *If $\mathop{\smile\!\!\!\!\!\mid}^T$ satisfies* Full Existence, *then $\mathop{\smile\!\!\!\!\!\mid}^{st}$ and $\mathop{\smile\!\!\!\!\!\mid}^w$ satisfy* Full Existence.

Proof. We show that $\mathop{\smile\!\!\!\!\!\mid}^{st}$ satisfies Full Existence. Let A, B, C be contained in some model $(\mathscr{M}, \mathscr{M}_0)$ of TS. By Full Existence for $\mathop{\smile\!\!\!\!\!\mid}^T$, there exists $A' \equiv^T_C A$ with $A' \mathop{\smile\!\!\!\!\!\mid}^T_C \mathscr{M}$, in particular $\overline{A'C} \cap \overline{BC} = \overline{C}$. Using Full Existence of $\mathop{\smile\!\!\!\!\!\mid}^a$ we may assume that $\overline{A'BC} \cap \mathscr{M} = \overline{BC}$. Let $f : \overline{A'C} \to \overline{AC}$ be a T-elementary isomorphism over C and $S_{A'C} := f^{-1}(S(\overline{AC}))$. Let $S_{A'BC} = \mathrm{acl}_0(S_{A'C}, S(\overline{BC}))$. It is easy to see that

- $S_{A'BC} \cap \mathscr{M} = S_{A'BC} \cap \overline{BC} = S(\overline{BC})$
- $S_{A'BC} \cap \overline{A'C} = S_{A'C}$

Using $\overline{A'BC} \cap \mathscr{M} = \overline{BC}$ and the first item, the type over BC defined by the pair $(\overline{A'BC}, S_{A'BC})$ is consistent (see Proposition 2.3.6). We may assume that $A' \subseteq \mathbb{M}$ realizes this type. From the second item, we have that $A' \equiv^{TS}_C A$, and it is clear that $S(\overline{A'BC})$ is equal to $\mathrm{acl}_0(S(\overline{A'C}), S(\overline{BC}))$ so $A' \mathop{\smile\!\!\!\!\!\mid}^{st}_C B$. We conclude that Full Existence is satisfied by $\mathop{\smile\!\!\!\!\!\mid}^{st}$. As $\mathop{\smile\!\!\!\!\!\mid}^{st} \to \mathop{\smile\!\!\!\!\!\mid}^w$, Full Existence is also satisfied by $\mathop{\smile\!\!\!\!\!\mid}^w$. $\square$

Lemma 4.1.4. *If $\mathop{\smile\!\!\!\!\!\mid}^T$ satisfies* Strong Finite Character *over algebraically closed sets, then the relation $\mathop{\smile\!\!\!\!\!\mid}^w$ satisfies* Strong Finite Character *over algebraically closed sets.*

Proof. Assume that $a \not\mathop{\smile\!\!\!\!\!\mid}^w_C b$ and $C = \overline{C}$. If $a \not\mathop{\smile\!\!\!\!\!\mid}^T_C b$, we have a formula witnessing Strong Finite Character over C by hypothesis. Otherwise, assume that $a \mathop{\smile\!\!\!\!\!\mid}^T_C b$, set $A = \overline{Ca}$, $B = \overline{Cb}$ and assume that there exists $s \in S(\mathrm{acl}_0(A, B)) \setminus \mathrm{acl}_0(S(A), S(B))$. Let $u \in A \setminus S(A)$ and $v \in B \setminus S(B)$ be such that $s \in \mathrm{acl}_0(u, v)$. There exists $\mathscr{L}_S$-formulae $\psi_u(y, a, c)$ and $\psi_v(z, b, c)$ isolating respectively $tp^{TS}(u/Ca)$ and $tp^{TS}(v/Cb)$ for some tuple c in C. There is also an $\mathscr{L}_0$-formula $\phi(t, y, z)$ algebraic in t, strict in y and strict in z, such that $s \models \phi(t, u, v)$.

Claim. $v \notin \mathrm{acl}_0(S(B), C)$.

Proof of the claim. Assuming otherwise, by modularity there exists singletons $s_b \in S(B)$ and $c \in C$ such that $v \in \mathrm{acl}_0(s_b, c)$ and so $s \in \mathrm{acl}_0(s_b, c, u)$. As $cu \subseteq A$, by modularity there exists a singleton $u' \in A$ such that $s \in \mathrm{acl}_0(s_b, u')$ and by Exchange $u' \in \mathrm{acl}_0(s_b, s) \cap A \subseteq S(A)$, this contradicts the hypothesis on s. □

In particular for any other realisation v' of $\psi_v(z, b, c)$ we have $v' \notin \mathrm{acl}_0(S(B), C)$. Now let $\Lambda(x, b, c)$ be the following formula

$$\exists y \exists z \exists t \psi_u(y, x, c) \wedge \psi_v(z, b, c) \wedge \phi(t, y, z) \wedge t \in S.$$

We have that $\Lambda(x, b, c) \in tp^{TS}(a/bC)$. Assume that $a' \models \Lambda(x, b, c)$. If $a' \not\downarrow^T_C b$ then we are done, so we may assume that $a' \downarrow^T_C b$, in particular $\overline{Ca'} \cap B = C$ as C is algebraically closed. There exists $u' \in \overline{Ca'}$ and $v' \in B \setminus \mathrm{acl}_0(S(B), C)$ such that there is $s' \in \mathrm{acl}_0(u', v') \cap S$. In particular $v' \in \mathrm{acl}_0(s', u')$ as $\phi(t, y, z)$ is strict in z. Now assume that $s' \in \mathrm{acl}_0(S(B), S(\overline{Ca'}))$, then $v' \in \mathrm{acl}_0(\overline{Ca'}, S(B))$ and also $v' \in B$. By modularity,

$$\mathrm{acl}_0(S(B), \overline{Ca'}) \cap B = \mathrm{acl}_0(S(B), \overline{Ca'} \cap B) = \mathrm{acl}_0(S(B), C)$$

so $v' \in \mathrm{acl}_0(S(B), C)$, a contradiction. We conclude that

$$s' \in S(\mathrm{acl}_0(\overline{Ca'}, B)) \setminus \mathrm{acl}_0(S(\overline{Ca'}), S(B))$$

so $a' \not\downarrow^w_C B$. □

Theorem 4.1.5. *Assume that $\downarrow^T$ satisfies the hypotheses of Lemmas 4.1.2. Assume that for some subset E of $\mathbb{M}$, the following two properties hold:*

(A_1) *$\downarrow'$-*AMALGAMATION *over E for some $\downarrow' \to \downarrow^a$, $\downarrow'$ satisfying* MONOTONICITY*,* SYMMETRY *and* CLOSURE*;*

(A_2) *For all A, B, C algebraically closed containing E, if $C \downarrow^T_E A, B$ and $A \downarrow'_E B$ then*

$$(\overline{AC}, \overline{BC}) \underset{A,B}{\downarrow^0} \overline{AB}.$$

*Then $\downarrow^w$ satisfies $\downarrow'$-*AMALGAMATION *over E.*

Proof. Let c_1, c_2, A, B be in a $(\mathscr{M}, \mathscr{M}_0) \prec (\mathbb{M}, \mathbb{M}_0)$ such that

- $c_1 \equiv^{TS}_E c_2$
- $A \downarrow'_E B$
- $c_1 \downarrow^w_E A$ and $c_2 \downarrow^w_E B$

As $\mathop{\smile\!\!\!\!\!|}'$ satisfies Symmetry, Closure and Monotonicity, we have that $A \mathop{\smile\!\!\!\!\!|}'_E B \iff \overline{AE} \mathop{\smile\!\!\!\!\!|}'_E \overline{BE}$, hence we may assume that A, B are algebraically closed and contain E. By hypothesis there is a T-elementary $\mathscr{L}_S$-isomorphism $h : \overline{Ec_1} \to \overline{Ec_2}$ over E sending c_1 to c_2. Let C_1 be an enumeration of $\overline{Ec_1}$ and let C_2 be the enumeration $h(C_1)$. We have $C_1 \equiv^T_E C_2$.

We have $C_1 \mathop{\smile\!\!\!\!\!|}^T_E A$, $C_2 \mathop{\smile\!\!\!\!\!|}^T_E B$ and $C_1 \equiv^T_E C_2$. By (A_1), there exists C such that $C \equiv^T_A C_1$, $C \equiv^T_B C_2$ with $C \mathop{\smile\!\!\!\!\!|}^T_E AB$, $A \mathop{\smile\!\!\!\!\!|}^a_C B$, $C \mathop{\smile\!\!\!\!\!|}^a_B A$ and $C \mathop{\smile\!\!\!\!\!|}^a_A B$. We may assume that $\overline{ABC} \cap \mathscr{M} = \overline{AB}$ using Full Existence of $\mathop{\smile\!\!\!\!\!|}^a$. There exists two T-elementary bijections $f : \overline{AC} \to \overline{AC_1}$ over A and $g : \overline{BC} \to \overline{BC_2}$ over B such that $g \restriction C = h \circ (f \restriction C)$.

We define $S_{AC} = f^{-1}(S(\overline{AC_1})) \subseteq \overline{AC}$ and $S_{BC} = g^{-1}(S(\overline{BC_2})) \subseteq \overline{BC}$, and set $S_{ABC} = \mathrm{acl}_0(S_{AB}, S_{AC}, S_{BC})$, with $S_{AB} = S(\overline{AB})$. The following is easy to check, it uses that $A \mathop{\smile\!\!\!\!\!|}^a_C B$, $C \mathop{\smile\!\!\!\!\!|}^a_B A$ and $C \mathop{\smile\!\!\!\!\!|}^a_A B$:

- $S_{AB} \cap S_{AC} = S_{AB} \cap A = S_{AC} \cap A = S(A) =: S_A$;
- $S_{AB} \cap S_{BC} = S_{AB} \cap B = S_{BC} \cap B = S(B) =: S_B$;
- $S_{AC} \cap S_{BC} = S_{AC} \cap C = S_{BC} \cap C = f^{-1}(S(C_1)) = g^{-1}(S(C_2)) =: S_C$.

Furthermore, with $S^-_{AB} = S_{AB} \cap \mathrm{acl}_0(A, B)$, $S^-_{AC} = S_{AC} \cap \mathrm{acl}_0(A, C)$ and $S^-_{BC} = S_{BC} \cap \mathrm{acl}_0(B, C)$, it follows from $c_1 \mathop{\smile\!\!\!\!\!|}^w_E A$ and $c_2 \mathop{\smile\!\!\!\!\!|}^w_E B$ that

(1) $S^-_{AC} = \mathrm{acl}_0(S_A, S_C)$;

(2) $S^-_{BC} = \mathrm{acl}_0(S_B, S_C)$.

Claim. We have the following

- $S_{ABC} \cap \overline{AB} = S_{AB}$;
- $S_{ABC} \cap \overline{AC} = S_{AC}$;
- $S_{ABC} \cap \overline{BC} = S_{BC}$.

Proof of the claim. As $A \mathop{\smile\!\!\!\!\!|}^a_C B$, $C \mathop{\smile\!\!\!\!\!|}^a_B A$ and $C \mathop{\smile\!\!\!\!\!|}^a_A B$, we have that $\overline{AC} \mathop{\smile\!\!\!\!\!|}^0_C \overline{BC}$, $\overline{BC} \mathop{\smile\!\!\!\!\!|}^0_B \overline{AB}$ and $\overline{AC} \mathop{\smile\!\!\!\!\!|}^0_A \overline{AB}$. By hypothesis (A_2) and Transitivity of $\mathop{\smile\!\!\!\!\!|}^0$ we have the following:

- $(\overline{AC}, \overline{BC}) \mathop{\smile\!\!\!\!\!|}^0_{A,B} \overline{AB}$;
- $(\overline{AB}, \overline{BC}) \mathop{\smile\!\!\!\!\!|}^0_{A,C} \overline{AC}$;
- $(\overline{AC}, \overline{AB}) \mathop{\smile\!\!\!\!\!|}^0_{B,C} \overline{BC}$.

In order to prove the first item of the claim, by modularity, it suffices to show that $\mathrm{acl}_0(S_{AC}, S_{BC}) \cap \overline{AB} \subseteq S_{AB}$. We will in fact show that

$$\mathrm{acl}_0(S_{AC}, S_{BC}) \cap \overline{AB} = S^-_{AB}.$$

We have that $(\overline{AB}, \overline{BC}) \mathop{\smile\!\!\!\!\!\!|}^0_{A,C} \overline{AC}$. Since $S^-_{AC} = S_{AC} \cap \mathrm{acl}_0(A, C)$ and $S_{BC} \subseteq \overline{BC}$ we deduce $S_{AC} \mathop{\smile\!\!\!\!\!\!|}^0_{S^-_{AC}} \overline{AB}, S_{BC}$. Now since $S^-_{AC} = \mathrm{acl}_0(S_A, S_C)$ we can use Base Monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^0$ and the fact that $S_C \subseteq S_{BC}$ to get

$$S_{AC} \mathop{\smile\!\!\!\!\!\!|}^0_{S_A, S_B, S_{BC}} \overline{AB}.$$

On the other hand, $\overline{BC} \cap \overline{AB} = B$ so $S_{BC} \mathop{\smile\!\!\!\!\!\!|}^0_{S_B} \overline{AB}$. Using Base Monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^0$ we also have that $S_{BC} \mathop{\smile\!\!\!\!\!\!|}^0_{S_A, S_B} \overline{AB}$ so using Transitivity of $\mathop{\smile\!\!\!\!\!\!|}^0$ it follows that $(S_{AC}, S_{BC}) \mathop{\smile\!\!\!\!\!\!|}^0_{S_A, S_B} \overline{AB}$.

For the second item, it is sufficient to prove that $\mathrm{acl}_0(S_{AB}, S_{BC}) \cap \overline{AC} \subseteq S_{AC}$. We do similarly as before paying attention to the fact that S_{AB} and S_{AC} do not play a symmetric role. We get first that $S_{BC} \mathop{\smile\!\!\!\!\!\!|}^0_{S^-_{BC}} (\overline{AC}, S_{AB})$ using $(\overline{AC}, \overline{AB}) \mathop{\smile\!\!\!\!\!\!|}^0_{B,C} \overline{BC}$. Now $S^-_{BC} = \mathrm{acl}_0(S_B, S_C)$, so we deduce $S_{BC} \mathop{\smile\!\!\!\!\!\!|}^0_{S_C, S_B} (\overline{AC}, S_{AB})$ and by Base Monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^0$ and the fact that $S_B, S_A \subseteq S_{AB}$ we deduce

$$S_{BC} \mathop{\smile\!\!\!\!\!\!|}^0_{S_C, S_A, S_{AB}} \overline{AC}.$$

Now by Base Monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^0$, we have $S_{AB} \mathop{\smile\!\!\!\!\!\!|}^0_{S_A, S_C} \overline{AC}$. We conclude using Transitivity of $\mathop{\smile\!\!\!\!\!\!|}^0$ that $(S_{AB}, S_{BC}) \mathop{\smile\!\!\!\!\!\!|}^0_{S_A, S_C} \overline{AC}$. The proof of the last assertion is similar. □

We know that $\overline{ABC} \cap \mathscr{M} = \overline{AB}$. Moreover, it follows from the first point of the claim that $S_{ABC} \cap \mathscr{M} = S_{ABC} \cap \overline{AB} = S_{AB}$. Consequently, by Proposition 2.3.6, the type in the sense of the theory TS defined by the pair $(\overline{ABC}, S_{ABC})$ is consistent, so we may consider that it is realised in $(\mathbb{M}, \mathbb{M}_0)$, by say C. It follows that $C = \overline{Ec}$ with c such that $c \equiv^{TS}_A c_1$ and $c \equiv^{TS}_B c_2$. What remains to show is that $C \mathop{\smile\!\!\!\!\!\!|}^w_E A, B$. We already have that $C \mathop{\smile\!\!\!\!\!\!|}^T_E A, B$ so we will prove that

$$S(\mathrm{acl}_0(C, \overline{AB})) = \mathrm{acl}_0(S(C), S(\overline{AB})).$$

By modularity, it suffices to show that $\mathrm{acl}_0(S_{AC}, S_{BC}) \cap \mathrm{acl}_0(C, \overline{AB}) \subseteq \mathrm{acl}_0(S_C, S_{AB})$. We in fact prove that $(S_{AC}, S_{BC}) \mathop{\smile\!\!\!\!\!\!|}^0_{S_A, S_B, S_C} (\overline{AB}, C)$. As before, using $(\overline{AB}, \overline{BC}) \mathop{\smile\!\!\!\!\!\!|}^0_{A,C} \overline{AC}$ we have that $S_{AC} \mathop{\smile\!\!\!\!\!\!|}^0_{S^-_{AC}} (\overline{AB}, \overline{BC})$, so as $S^-_{AC} = \mathrm{acl}_0(S_A, S_C)$ we have

$$S_{AC} \mathop{\smile\!\!\!\!\!\!|}^0_{S_A, S_C} (\overline{AB}, S_{BC}, C).$$

Using Base Monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^0$, we have

$$S_{AC} \mathop{\smile\!\!\!\!\!\!|}^0_{S_A, S_B, S_C, S_{BC}} (\overline{AB}, C).$$

On the other hand, from $(\overline{AC},\overline{AB}) \mathop{\smile\!\!\!\!\!\!\,|}^0_{B,C} \overline{BC}$ and MONOTONICITY of $\mathop{\smile\!\!\!\!\!\!\,|}^0$, we have that $\overline{BC} \mathop{\smile\!\!\!\!\!\!\,|}^0_{B,C} (\overline{AB},C)$. It follows that $S_{BC}\cap \mathrm{acl}_0(\overline{AB},C)\subseteq S_{BC}^- = \mathrm{acl}_0(S_B,S_C)$ so $S_{BC} \mathop{\smile\!\!\!\!\!\!\,|}^0_{S_B,S_C} (\overline{AB},C)$. Using BASE MONOTONICITY of $\mathop{\smile\!\!\!\!\!\!\,|}^0$ we have

$$S_{BC} \mathop{\smile\!\!\!\!\!\!\,|}^0_{S_B,S_A,S_C} (\overline{AB},C).$$

Now using TRANSITIVITY of $\mathop{\smile\!\!\!\!\!\!\,|}^0$, we get $(S_{AC},S_{BC}) \mathop{\smile\!\!\!\!\!\!\,|}^0_{S_A,S_B,S_C} (\overline{AB},C)$. $\square$

Lemma 4.1.6. *Assume that $a \not\mathop{\smile\!\!\!\!\!\!\,|}^w_C b$ and $a \mathop{\smile\!\!\!\!\!\!\,|}^T_C b$ with $C=\overline{C}$. Then there is a formula $\Lambda(x,b,c)\in tp(a/Cb)$ such that for all sequence $(b_i)_{i<\omega}$ such that*

(1) $b_i \equiv^{TS}_C b$ for all $i<\omega$,

(2) $b_i \mathop{\smile\!\!\!\!\!\!\,|}^a_C b_j$ and $S(\mathrm{acl}_0(\overline{Cb_i},\overline{Cb_j})) = \mathrm{acl}_0(S(\overline{Cb_i}),S(\overline{Cb_j}))$ for all $i,j<\omega$,

the partial type $\{\Lambda(x,b_i,c)\mid i<\omega\}$ is inconsistent.

Proof. Let $A=\overline{Ca}$, $B=\overline{Cb}$. As $a \not\mathop{\smile\!\!\!\!\!\!\,|}^w_C b$ there exists $s\in S(\mathrm{acl}_0(A,B))\backslash \mathrm{acl}_0(S(A),S(B))$. As we saw in the proof of Lemma 4.1.4, there exist $u\in A\backslash S(A)$, $v\in B\backslash S(B)$ and $\mathscr{L}_S(C)$-formulae $\psi_u(y,a)$ algebraic in y and $\psi_v(z,b)$ algebraic in z, satisfied respectively by u and v. There is also an $\mathscr{L}_0$-formula $\phi(t,y,z)$ algebraic in t, strict in y and strict in z, such that $s\models \phi(t,u,v)$. Again, as $v\notin \mathrm{acl}_0(S(B),C)$ and $\psi_v(z,b)$ isolates the type $tp^{TS}(v/Cb)$, every v' satisfying $\psi_v(z,b)$ will satisfy $v'\notin \mathrm{acl}_0(S(B),C)$. Let $\Lambda(x,b,c)\in tp^{TS}(a/Cb)$ be the following formula, for a tuple c from C

$$\exists y\exists z\exists t \psi_u(y,x)\wedge\psi_v(z,b)\wedge\phi(t,y,z)\wedge t\in S.$$

As we saw in the proof of Lemma 4.1.4, it witnesses STRONG FINITE CHARACTER over C. Note that if $b'\equiv^{TS}_C b$, then no realization of $\psi_v(y,b')$ is in $\mathrm{acl}_0(S(\overline{Cb'}),C)$.

Now let $(b_i)_{i<\omega}$ be as in the hypothesis. By contradiction, assume that $\{\Lambda(x,b_i,c)\mid i<\omega\}$ is consistent, and realised by some a'. Assume that $\psi_u(t,a')$ does not have more than k distinct realisations. As

$$\bigwedge_{i<k+1}\Lambda(a',b_i,c)$$

is consistent, there is $u'\in\overline{Ca'}$ and $i<j<k+1$ such that v_i,v_j are two realisations of $\psi_v(z,b_i)$ and $\psi_v(z,b_j)$ respectively –we assume $i=1,j=2$ for convenience– and such that there exist $s_1\in \mathrm{acl}_0(u',v_1)\cap S$ and $s_2\in\mathrm{acl}_0(u',v_2)\cap S$. As $v_2\notin \mathrm{acl}_0(S(\overline{Cb_2}),C)$ it follows that $v_2\notin\mathrm{acl}_0(u')$, hence $u'\in\mathrm{acl}_0(s_2,v_2)$ so $s_1\in\mathrm{acl}_0(s_2,v_1,v_2)$. By modularity, it means that there is some $w\in\mathrm{acl}_0(v_1,v_2)$ such that $s_1\in\mathrm{acl}_0(s_2,w)$. We have that $w\in\mathrm{acl}_0(s_1,s_2)$, so $w\in\mathrm{acl}_0(v_1,v_2)\cap S$. As $S(\mathrm{acl}_0(\overline{Cb_1}),\mathrm{acl}_0(\overline{Cb_2})) = \mathrm{acl}_0(S(\mathrm{acl}_0(\overline{Cb_1}),S(\mathrm{acl}_0(\overline{Cb_2})))$ there is some $s_1^b\in S(\overline{Cb_1})$ and $s_2^b\in S(\overline{Cb_2})$ such that $w\in\mathrm{acl}_0(s_1^b,s_2^b)$. Now, as $v_1\notin C$, it follows that $v_1\notin\mathrm{acl}_0(v_2)$ hence $v_1\in\mathrm{acl}_0(w,v_2)$, and so $v_1\in\mathrm{acl}_0(s_1^b,s_2^b,v_2)$. So there is $v_2'\in\mathrm{acl}_0(s_2^b,v_2)\subseteq\overline{Cb_2}$ such that $v_1\in\mathrm{acl}_0(s_1^b,v_2')$. It follows that $v_2'\in\mathrm{acl}_0(s_1^b,v_1)$ so $v_2'\in\overline{Cb_1}\cap\overline{Cb_2}=C$, hence $v_2'\in C$. Now $v_1\in\mathrm{acl}_0(S(\overline{Cb_1}),C)$ and this is a contradiction. $\square$

Lemma 4.1.7. *Assume that $\mathop{\smile\hspace{-0.9em}\mid}^T$ satisfies the hypothesis of Lemma 4.1.2 and 4.1.4. If $\mathop{\smile\hspace{-0.9em}\mid}^T$ satisfies* Witnessing*, then so does $\mathop{\smile\hspace{-0.9em}\mid}^w$.*

Proof. Assume that $a \mathop{\not\smile\hspace{-0.9em}\mid}^w_{\mathscr{M}} b$, and let $\Lambda(x, b, m)$ be as in Lemma 4.1.6 and set $p(x) = tp^{TS}(a/\mathscr{M}b)$, $p_{\mathscr{L}} = p \restriction \mathscr{L} = tp^T(a/\mathscr{M}b)$. Let $q(x)$ be a global extension of $tp^{TS}(b/\mathscr{M})$ finitely satisfiable in $\mathscr{M}$, $q_{\mathscr{L}} = q \restriction \mathscr{L}$. It is clear that $q_{\mathscr{L}}$ is finitely satisfiable in $\mathscr{M}$. Let $(b_i)_{i<\omega}$ be a sequence in $\mathbb{M}$ such that $b_i \models q \restriction \mathscr{M}b_{<i}$ for all $i < \omega$. Observe that for $j < i$ we have $tp^{TS}(b_i/\mathscr{M}b_j)$ is finitely satisfiable in $\mathscr{M}$. By hypothesis, $\mathop{\smile\hspace{-0.9em}\mid}^w$ satisfies in particular Symmetry, Monotonicity, Existence, and Strong Finite Character over models, hence by Lemma 1.2.4, $b_i \mathop{\smile\hspace{-0.9em}\mid}^w_{\mathscr{M}} b_j$. In particular $b_i \mathop{\smile\hspace{-0.9em}\mid}^a_{\mathscr{M}} b_j$ and $S(\mathrm{acl}_0(\overline{\mathscr{M}b_i}, \overline{\mathscr{M}b_j})) = \mathrm{acl}_0(S(\overline{\mathscr{M}b_i}), S(\overline{\mathscr{M}b_j}))$ for all $i, j < \omega$. If $\{\Lambda(x, b_i, m) \mid i < \omega\}$ is inconsistent, we conclude. If $\{\Lambda(x, b_i, m) \mid i < \omega\}$ is consistent, by Lemma 4.1.6 we have $a \mathop{\not\smile\hspace{-0.9em}\mid}^T_{\mathscr{M}} b$. Now also $b_i \models q_{\mathscr{L}} \restriction \mathscr{M}b_{<i}$, hence as $\mathop{\smile\hspace{-0.9em}\mid}^T$ satisfies Witnessing, we conclude. □

Lemma 4.1.8. *Assume that $\mathop{\smile\hspace{-0.9em}\mid}^T$ satisfies* Base Monotonicity*. The following are equivalent.*

(1) $\mathop{\smile\hspace{-0.9em}\mid}^w$ satisfies Base Monotonicity*;*

(2) For all algebraically closed sets A, B, C, D such that A, B, D contain C and $A \mathop{\smile\hspace{-0.9em}\mid}^T_C BD$, the following holds

$$\mathrm{acl}_0(A, \overline{BD}) \cup \overline{AD} = \mathrm{acl}_0(\overline{AD}, \overline{BD}).$$

In particular if acl_0 *is trivial or if* $\mathrm{acl}_0 = \mathrm{acl}_T$ *then $\mathop{\smile\hspace{-0.9em}\mid}^w$ satisfies* Base Monotonicity.

Proof. Assume that there exist A, B, C, D that does not satisfy *(2)*. Let $w \in \mathrm{acl}_0(\overline{AD}, \overline{BD}) \setminus (\mathrm{acl}_0(A, \overline{BD}) \cup \overline{AD})$, and $S_0 := S(\mathrm{acl}_T(\emptyset))$. We define $S_{ABD} = \mathrm{acl}_0(S_0, w)$. The type (over $\emptyset$) defined by the pair $(\overline{ABD}, S_{ABD})$ is consistent. As $S_{ABD} \cap \mathrm{acl}_0(A, \overline{BD}) = S_{ABD} \cap A = S_{ABD} \cap \overline{BD} = S_0$ and $A \mathop{\smile\hspace{-0.9em}\mid}^T_C BD$ we have that $A \mathop{\smile\hspace{-0.9em}\mid}^w_C BD$. Now $w \in S_{ABD} \cap \mathrm{acl}_0(\overline{AD}, \overline{BD})$ whereas $S_{ABD} \cap \overline{AD} = S_{ABD} \cap \overline{BD} = S_0$, hence

$$S_0 = \mathrm{acl}_0(S_{ABD} \cap \overline{AD}, S_{ABD} \cap \overline{BD}) \subsetneq S_{ABD} \cap \mathrm{acl}_0(\overline{AD}, \overline{BD}).$$

It follows that $A \mathop{\not\smile\hspace{-0.9em}\mid}^w_D B$, so $\mathop{\smile\hspace{-0.9em}\mid}^w$ doesn't satisfies Base Monotonicity.

Conversely if $\mathop{\smile\hspace{-0.9em}\mid}^w$ doesn't satisfies Base Monotonicity, it means that there exist A, B, C, D such that $A \mathop{\smile\hspace{-0.9em}\mid}^w_C BD$ and $A \mathop{\not\smile\hspace{-0.9em}\mid}^w_{CD} B$. We may assume that A, B, D are algebraically closed and contains C. As $\mathop{\smile\hspace{-0.9em}\mid}^T$ satisfies Base Monotonicity we have that

$$S(\mathrm{acl}_0(\overline{AD}, \overline{BD})) \supsetneq \mathrm{acl}_0(S(\overline{AD}), S(\overline{BD})).$$

Let w be in $S(\mathrm{acl}_0(\overline{AD}, \overline{BD})) \setminus \mathrm{acl}_0(S(\overline{AD}), S(\overline{BD}))$. As $w \in S$ we have that $w \notin \overline{AD}$ and $w \notin \overline{BD}$. It remains to show that $w \notin \mathrm{acl}_0(A, \overline{BD})$. Assume that $w \in \mathrm{acl}_0(A, \overline{BD})$. As $w \in S$ we have that $w \in S(\mathrm{acl}_0(A, \overline{BD}))$. From $A \mathop{\smile\hspace{-0.9em}\mid}^w_C BD$ we have that $S(\mathrm{acl}_0(A, \overline{BD})) = \mathrm{acl}_0(S(A), S(\overline{BD}))$ so $w \in \mathrm{acl}_0(S(A), S(\overline{BD}))$ which contradicts that $w \notin \mathrm{acl}_0(S(\overline{AD}), S(\overline{BD}))$. So it follows that $w \in \mathrm{acl}_0(\overline{AD}, \overline{BD}) \setminus (\mathrm{acl}_0(A, \overline{BD}) \cup \overline{AD})$. □

4.2 Preservation of NSOP_1

In this section, we use the results of the previous section to prove that if T is NSOP_1 and T satisfies an additional hypothesis then TS is also NSOP_1. This additional hypothesis (namely (A) below) translates how $\mathop{\smile\!\!\!\!\!\mid}^0$ in the reduct T_0 is controlled by $\mathop{\smile\!\!\!\!\!\mid}^T$ in T. We work in the same context as the previous section, with small sets and small models in a monster model for TS, when $(T, \mathscr{L}_0, T_0)$ is a suitable triple.

Theorem 4.2.1. *Assume that $(T, \mathscr{L}_0, T_0)$ is a suitable triple. Assume that T is NSOP_1 and that $\mathop{\smile\!\!\!\!\!\mid}^T$ is the Kim-independence relation in T. If*

(A) *all $\mathscr{M} \models T$ and A, B, C algebraically closed containing $\mathscr{M}$, if $C \mathop{\smile\!\!\!\!\!\mid}^T_{\mathscr{M}} A, B$ and $A \mathop{\smile\!\!\!\!\!\mid}^T_{\mathscr{M}} B$ then*
$$(\overline{AC}, \overline{BC}) \mathop{\smile\!\!\!\!\!\mid}^0_{A,B} \overline{AB}.$$

Then TS is NSOP_1 and the Kim-independence relation in TS is given by $\mathop{\smile\!\!\!\!\!\mid}^w$, i.e. the relation
$$A \mathop{\smile\!\!\!\!\!\mid}^T_{\mathscr{M}} B \text{ and } S(\mathrm{acl}_0(\overline{A\mathscr{M}}, \overline{B\mathscr{M}})) = \mathrm{acl}_0(S(\overline{A\mathscr{M}}), S(\overline{B\mathscr{M}})).$$

Proof. From [KR17], if T is NSOP_1 the Kim-independence $\mathop{\smile\!\!\!\!\!\mid}^T$ satisfies Invariance, Symmetry, Monotonicity, Existence and Strong Finite Character all over models. Furthermore, by [KR18, Theorem 2.21], it also satisfies $\mathop{\smile\!\!\!\!\!\mid}^T$-amalgamation over models. By Lemmas 4.1.2, 4.1.4 and Theorem 4.1.5, all these properties are also satisfied over models by $\mathop{\smile\!\!\!\!\!\mid}^w$ (relatively to the theory TS). By Proposition 5.3 in [CR16], TS is NSOP_1. As $\mathop{\smile\!\!\!\!\!\mid}^T$ satisfies Witnessing, so does $\mathop{\smile\!\!\!\!\!\mid}^w$ by Lemma 4.1.7. Using [KR17, Theorem 9.1] (and [KR17, Remark 9.2]), it follows that $\mathop{\smile\!\!\!\!\!\mid}^w$ and Kim-independence in TS coincide over models. □

The results of the previous section give more than the previous Theorem. Indeed, most of the nice features that may happen in T for $\mathop{\smile\!\!\!\!\!\mid}^T$ are preserved when expanding T to TS. For instance, if $\mathop{\smile\!\!\!\!\!\mid}^T$ is defined over every small base set, so is $\mathop{\smile\!\!\!\!\!\mid}^w$. If the independence theorem in T is satisfied by $\mathop{\smile\!\!\!\!\!\mid}^T$ not only over models but over a wider class of small sets then the same holds in TS for $\mathop{\smile\!\!\!\!\!\mid}^w$. We summarize these features in the next result.

Theorem 4.2.2. *Assume that $(T, \mathscr{L}_0, T_0)$ is a suitable triple. Assume that there is a ternary relation $\mathop{\smile\!\!\!\!\!\mid}^T$ over small sets of a monster model of T that satisfies*

- Invariance;
- Symmetry;
- Closure;
- Monotonicity;

- *Existence;*
- *Full Existence;*
- *Strong Finite Character over E for $E = \overline{E}$;*
- *$\mathop{\smile\!\!\!\!\!\!|}'$-amalgamation over E for $E = \overline{E}$, where $\mathop{\smile\!\!\!\!\!\!|}'$ is such that $\mathop{\smile\!\!\!\!\!\!|}^T \to \mathop{\smile\!\!\!\!\!\!|}' \to \mathop{\smile\!\!\!\!\!\!|}^a$ and $\mathop{\smile\!\!\!\!\!\!|}'$ satisfies Monotonicity, Symmetry and Closure;*

(A) *For $E = \overline{E}$ and A, B, C algebraically closed containing E, if $C \mathop{\smile\!\!\!\!\!\!|}^T_E A, B$ and $A \mathop{\smile\!\!\!\!\!\!|}^T_E B$ then $\overline{AC} \mathop{\smile\!\!\!\!\!\!|}^0_C \overline{BC}$ and*

$$(\overline{AC}, \overline{BC}) \mathop{\smile\!\!\!\!\!\!|}^0_{A,B} \overline{AB};$$

- *Witnessing.*

(In particular T is NSOP_1, and $\mathop{\smile\!\!\!\!\!\!|}^T$ coincide with Kim-independence over models of T, by [CR16, Proposition 5.3] and [KR17, Theorem 9.1]).

Then any completion of TS is NSOP_1 and $\mathop{\smile\!\!\!\!\!\!|}^w$ and the Kim-forking independence relation in TS coincide over models. Furthermore $\mathop{\smile\!\!\!\!\!\!|}^w$ satisfies all these properties, relatively to the theory TS.

Finally, using [KR17, Proposition 8.8] we give a condition on $(T, T_0, \mathscr{L}_0)$ that characterizes the simplicity of TS, assuming that T satisfies the hypotheses of Theorem 4.2.2.

Corollary 4.2.3. *Let $(T, \mathscr{L}_0, T_0)$ be a suitable triple satisfying all the assumptions of Theorem 4.2.2. The following are equivalent.*

(1) Any completion of TS is not simple

(2) T is not simple or there exist algebraically closed sets A, B, C, D such that A, B, D contain C and $A \mathop{\smile\!\!\!\!\!\!|}^T_C BD$, and such that

$$\mathrm{acl}_0(A, \overline{BD}) \cup \overline{AD} \neq \mathrm{acl}_0(\overline{AD}, \overline{BD}).$$

In particular if acl_0 is trivial or if $\mathrm{acl}_0 = \mathrm{acl}_T$ the theory TS is simple if and only if T is simple. If TS is simple, $\mathop{\smile\!\!\!\!\!\!|}^w$ is forking independence over models.

Proof. From Theorem 4.2.2, we know that the relation $\mathop{\smile\!\!\!\!\!\!|}^w$ is Kim-independence over models. By [KR17, Proposition 8.8], TS is simple if and only if $\mathop{\smile\!\!\!\!\!\!|}^w$ satisfies Base Monotonicity. The equivalence follows from Lemma 4.1.8. The fact that Kim-independence and forking independence coincide is [KR17, Proposition 8.4]. $\square$

Corollary 4.2.4. *Assume that T is a complete $\mathscr{L}$-theory and $\mathscr{L}_1, \ldots, \mathscr{L}_n$ are sublanguages of $\mathscr{L}$. Let $T_1 = T \restriction \mathscr{L}_1, \ldots, T_n = T \restriction \mathscr{L}_n$ such that $(TS_1 \ldots S_i, T_{i+1}, \mathscr{L}_{i+1})$ is a suitable triple for each $i = 0, \cdots, n-1$. By Proposition 2.4.1, let $TS_1 \ldots S_n$ be the model companion of the theory of models of T with a predicate S_i for an $\mathscr{L}_i$ substructure.*

(1) *Assume that T is* NSOP_1*, with Kim-independence* $\mathop{\smile\!\!\!\!\!\!\mid}^T$ *in T and that for all i we have (for A, B, C algebraically closed containing $\mathscr{M} \models T$)*

$$\text{if } C \mathop{\smile\!\!\!\!\!\!\mid}^T_{\mathscr{M}} A, B \text{ and } A \mathop{\smile\!\!\!\!\!\!\mid}^T_{\mathscr{M}} B \text{ then } (\overline{AC}, \overline{BC}) \mathop{\smile\!\!\!\!\!\!\mid}^i_{A,B} \overline{AB}.$$

Then $TS_1 \dots S_n$ is NSOP_1 *and Kim-independence in TS is given by*

$$A \mathop{\smile\!\!\!\!\!\!\mid}^T_{\mathscr{M}} B \text{ and for all } i \leq n \ S_i(\mathrm{acl}_i(\overline{A\mathscr{M}}, \overline{B\mathscr{M}})) = \mathrm{acl}_i(S_i(\overline{A\mathscr{M}}), S_i(\overline{B\mathscr{M}}))$$

(for acl_i*,* $\mathop{\smile\!\!\!\!\!\!\mid}^i$ *the algebraic closure and independence in the sense of the pregeometric theory T_i).*

(2) *If there exists* $\mathop{\smile\!\!\!\!\!\!\mid}^T$ *that satisfies the hypotheses of Theorem 4.2.2 (relatively to each theory T_i), then $TS_1 \dots S_n$ is* NSOP_1 *and the relation*

$$A \mathop{\smile\!\!\!\!\!\!\mid}^T_{C} B \text{ and for all } i \leq n \ S_i(\mathrm{acl}_i(\overline{AC}, \overline{BC})) = \mathrm{acl}_i(S_i(\overline{AC}), S_i(\overline{BC}))$$

agrees with Kim-independence over models. Furthermore this relation satisfies all the properties listed in Theorem 4.2.2.

4.3 Mock stability and stability

We keep the same hypotheses on T and $\mathop{\smile\!\!\!\!\!\!\mid}^T$ as in the previous section. Mock stability is a notion introduced in [Adl08a] by Adler.

Definition 4.3.1. A theory T is *mock stable* if there is a relation satisfying Invariance, Finite Character, Closure, Symmetry, Monotonicity, Base Monotonicity, Transitivity, Full Existence, Stationnarity over models.

Remark 4.3.2. In the original definition of mock stability ([Adl08a]), Adler asks for slightly different properties but as in the proof of Fact 1.4.4, it is easy to see that our set of properties is equivalent to the one in [Adl08a].

Lemma 4.3.3. *Assume that* $\mathop{\smile\!\!\!\!\!\!\mid}^T$ *satisfies* Invariance*,* Finite Character*,* Symmetry*,* Closure*,* Monotonicity*,* Base Monotonicity*,*Transitivity*,* Full Existence *then so does* $\mathop{\smile\!\!\!\!\!\!\mid}^{st}$*. Furthermore, for any $E = \overline{E}$, if* $\mathop{\smile\!\!\!\!\!\!\mid}^T$ *satisfies* Stationnarity *over $E = \overline{E}$, so does* $\mathop{\smile\!\!\!\!\!\!\mid}^{st}$*. In particular if T is mock stable, so is TS.*

Proof. Invariance, Finite Character, Symmetry, Closure are trivial. Full Existence is Lemma 4.1.3. It remains to show Monotonicity, Base Monotonicity, Transitivity and Stationnarity over algebraically closed sets.

Monotonicity. Assume that $A \mathop{\smile\!\!\!\!\!\!|}^{st}_C BD$. We only need to check that $S(\overline{ABC}) = \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC})$. We have

$$\begin{aligned} S(\overline{ABC}) &= \mathrm{acl}_0(S(\overline{AC}), S(\overline{BCD})) \cap \overline{ABC} \\ &= \mathrm{acl}_0(S(\overline{AC}), S(\overline{BCD}) \cap \overline{ABC} && \text{by modularity)} \\ &= \mathrm{acl}_0(S(\overline{AC}), S(\overline{BC})) && \text{as } \overline{BCD} \cap \overline{ABC} = \overline{BC} \ (\mathop{\smile\!\!\!\!\!\!|}^{T} \to \mathop{\smile\!\!\!\!\!\!|}^{a}). \end{aligned}$$

Base Monotonicity. If $A \mathop{\smile\!\!\!\!\!\!|}^{st}_C BD$ then by Base Monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^T$ we have $A \mathop{\smile\!\!\!\!\!\!|}^T_{CD} B$. As $S(\overline{ABCD}) = \mathrm{acl}_0(S(\overline{CA}), S(\overline{CBD})$, in particular $S(\overline{ABCD}) \subseteq \mathrm{acl}_0(S(\overline{ACD}), S(\overline{BCD}) \subseteq S(\overline{ABCD})$, so $A \mathop{\smile\!\!\!\!\!\!|}^{st}_{CD} B$.

Transitivity. Assume that $A \mathop{\smile\!\!\!\!\!\!|}^{st}_{CB} D$ and $B \mathop{\smile\!\!\!\!\!\!|}^{st}_C D$. By Closure, we may assume that $A = \overline{ABC}, B = \overline{BC}, D = \overline{CD}$. By Monotonicity, it is sufficient to show that $A \mathop{\smile\!\!\!\!\!\!|}^{st}_C D$. We have $A \mathop{\smile\!\!\!\!\!\!|}^T_C D$ by Transitivity of $\mathop{\smile\!\!\!\!\!\!|}^T$. We show that $S(\overline{AD}) = \mathrm{acl}_0(S(A), S(D)$. By $A \mathop{\smile\!\!\!\!\!\!|}^{st}_B D$ we have $S(\overline{AD}) = \mathrm{acl}_0(S(A), S(\overline{BD}))$. By $B \mathop{\smile\!\!\!\!\!\!|}^{st}_C D$, $S(\overline{BD}) = \mathrm{acl}_0(S(B), S(D))$ hence $S(\overline{AD}) = \mathrm{acl}_0(S(A), S(B), S(D)) = \mathrm{acl}_0(S(A), S(D)$.

Stationnarity. Assume that $c_1 \mathop{\smile\!\!\!\!\!\!|}^{st}_E A$ and $c_2 \mathop{\smile\!\!\!\!\!\!|}^{st}_E A$ and $c_1 \equiv^{TS}_E c_2$. We may assume that A is algebraically closed and contains E. There is a T-elementary S-preserving map $f : \overline{Ec_1} \to \overline{Ec_2}$ over E. By Stationnarity over E, we can extend f to $\tilde{f} : \overline{Ac_1} \to \overline{Ac_2}$ T-elementary over A. But as $S(\overline{Ac_1})) = \mathrm{acl}_0(S(\overline{Ec_1}), S(A))$ and $S(\overline{Ac_2}) = \mathrm{acl}_0(S(\overline{Ec_2}), S(A))$, $\tilde{f}$ preserves S, so $c_1 \equiv^{TS}_B c_2$. □

Proposition 4.3.4. *If T is stable and $\mathrm{acl}_0 = \mathrm{acl}_T$, then the theory TS is stable.*

Proof. By Corollary 4.2.3, TS is simple and $\mathop{\smile\!\!\!\!\!\!|}^w$ is the forking independence, in particular it satisfies Local Character. As $\mathrm{acl}_T = \mathrm{acl}_0$ it follows that $\mathop{\smile\!\!\!\!\!\!|}^{st} = \mathop{\smile\!\!\!\!\!\!|}^w$, hence as $\mathop{\smile\!\!\!\!\!\!|}^T$ is stationnary over models, so is $\mathop{\smile\!\!\!\!\!\!|}^w$ by Lemma 4.3.3. Hence TS is stable by Fact 1.4.4. Note that the fact that forking independence is stationnary over models gives directly the stability. □

Remark 4.3.5. Assume that T is stable and that acl_0 is trivial, then TS is not necessary stable. From Corollary 4.2.3, TS is simple and $\mathop{\smile\!\!\!\!\!\!|}^w$ is forking independence. As acl_0 is trivial, we have $\mathop{\smile\!\!\!\!\!\!|}^w = \mathop{\smile\!\!\!\!\!\!|}^T$, (with $\mathop{\smile\!\!\!\!\!\!|}^T$ forking independence in T) which is not likely to be stationnary. The easiest example of a reduct T_0 for which acl_0 is trivial is the particular case of $\mathscr{L}_0 = \{=\}$. Then TS is the theory of the generic predicate on T (see Remark 2.1.6 and [CP98]), which does not preserve stability. Indeed [CP98, (2.10) Proposition, Errata] gives a sufficient condition on T so that TS have the independence property (hence is unstable): there exists a model $\mathscr{M}$ of T and two elements a and b such that $b \mathop{\smile\!\!\!\!\!\!|}^u_{\mathscr{M}} a$ and $\overline{\mathscr{M}ab} \neq \overline{\mathscr{M}a} \cup \overline{\mathscr{M}b}$. It follows that adding a generic predicate to an algebraically closed field result in a simple unstable theory (take a and b two generics independent over $\mathscr{M}$).

Example 4.3.6. We saw in Example 3.1.7 that the generic theory $TV_1 \cdots V_n$ of infinite $\mathbb{F}_q$-vector spaces with predicates for $\mathbb{F}_q$-vector subspaces $V_1, \cdots, V_n$ is stable for $n = 1$

as it is the theory of a belle paire of infinite $\mathbb{F}_q$-vector space. Proposition 4.3.4 gives us inductively that $TV_1 \cdots V_n$ is stable for all $n \in \mathbb{N}$.

Example 4.3.7. Assume that T is a model-complete geometric theory such that every acl_T-closed set is a model of T (Proposition 3.4.1). If T is stable, then the weak model-companion of the pairs of models of T is stable.

4.4 NSOP$_1$ expansions of fields

4.4.1 Fields with generic additive subgroups

In this section, we give some condition under which the theory obtained in Proposition 3.2.1 is NSOP$_1$. In this section, for A in some field, we denote by acl_T the model-theoretic algebraic closure, A^s the separable closure and $\overline{A}$ the field theoretic algebraic closure.

Theorem 4.4.1. *Let T be a model-complete theory of an* NSOP$_1$ *field that eliminates $\exists^\infty$ and let $\mathbb{F}_{q_1}, \cdots, \mathbb{F}_{q_n}$ be subfields. Assume that T satisfies the following assumption for all acl_T-closed A, B and $E \models T$ contained in A and B:*

$$\text{if } A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^T_E B \text{ then } \mathrm{acl}_T(AB) \subseteq \overline{AB}.$$

Then $TV_1 \dots V_n$ is NSOP$_1$ *and Kim-independence in $TV_1 \dots V_n$ is given by*

$$A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^w_E B \iff A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^T_E B \text{ and for all } i \leq n \ V_i(A+B) = V_i(A) + V_i(B)$$

(for A, B, C acl_T-closed, A, B containing E, $E \models T$).

Proof. We prove that $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^T$ satisfies the conditions of Corollary 4.2.4. Let $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^i$ the independence in the sense of $\mathbb{F}_{q_i}$-vector space, we want to show that for all $i = 1, \dots, n$,

(A) for all model E of T and A, B, C algebraically closed containing E, if $C \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^T_E A, B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^T_E B$ then

$$(\mathrm{acl}_T(AC), \mathrm{acl}_T(BC)) \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^i_{A,B} \mathrm{acl}_T(AB).$$

Let $F \models T$, let $E \prec F$ and A, B, C in F containing E, with $C \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^T_E A, B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^T_E B$. For all $i = 1, \cdots, n$, the condition $(\mathrm{acl}_T(AC), \mathrm{acl}_T(BC)) \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^i_{A,B} \mathrm{acl}_T(AB)$ is equivalent to

$$(\mathrm{acl}_T(AC) + \mathrm{acl}_T(BC)) \cap \mathrm{acl}_T(AB) = A + B.$$

From Fact 1.5.10 *(2)*, F/AB, F/BC and F/AC are separable extension. By our assumptions on T and A, B and C we have that $\mathrm{acl}_T(AB) \subseteq (AB)^s$, $\mathrm{acl}_T(AC) \subseteq (AC)^s$ and $\mathrm{acl}_T(BC) \subseteq (BC)^s$, so

$$(\mathrm{acl}_T(AC) + \mathrm{acl}_T(BC)) \cap \mathrm{acl}_T(AB) \subseteq ((AC)^s + (BC)^s) \cap (AB)^s.$$

Claim. $((AC)^s + (BC)^s) \cap (AB)^s = A^s + B^s$

Proof of the claim. First, observe that as fields, E^s is an elementary substructure of F^s. Indeed, by model completeness of $Th(E^s)$ (which is $\mathrm{SCF}_{p,e}$ for some $e \leq \infty$, see Subsection 1.5.2) we have to check that they have the same imperfection degree (which is clear as $F \succ E$) and that F^s/E^s is separable (the later follows from the fact that F/E is a regular extension). Now by Fact 1.5.10 *(1)* we have $C \mathop{\smile\!\!\!\!\!\!|}^{ld}_E AB$. As E is a model, C/E and AB/E are regular extensions[1], by Fact 1.5.6 we have that

$$C^s \mathop{\smile\!\!\!\!\!\!|}^{ld}_{E^s} (AB)^s. \quad (*)$$

Moreover F^s/ABC is separable, (as so are F^s/F and F/ABC, the latter using Fact 1.5.10 *(2)*) and so is $C^s(AB)^s/ABC$. It follows that the following extension is separable

$$F^s/C^s(AB)^s. \quad (**)$$

From $(*)$ and $(**)$, using Fact 1.5.14 we have that $tp_{\mathrm{SCF}}(C^s/(AB)^s)$ does not fork over E^s. By stability, as E^s is an elementary submodel of the ambiant model F^s of $\mathrm{SCF}_{p,e}$, $tp_{\mathrm{SCF}}(C^s/(AB)^s)$ is a coheir of $tp_{\mathrm{SCF}}(C^s/E^s)$ (Fact 1.4.5). From Lemma 1.5.11, it follows that $((AC)^s + (BC)^s) \cap (AB)^s = A^s + B^s$.

By the claim $(\mathrm{acl}_T(AC) + \mathrm{acl}_T(BC)) \cap \mathrm{acl}_T(AB) \subseteq (A^s + B^s) \cap \mathrm{acl}_T(AB)$. Now by Fact 1.5.10 *(3)*, we have that $A^sB^s \cap \mathrm{acl}_T(AB) = AB$ so $(A^s + B^s) \cap \mathrm{acl}_T(AB) \subseteq (A^s + B^s) \cap AB$. Finally, by Lemma 1.5.7, as AB/E is regular and $A \mathop{\smile\!\!\!\!\!\!|}^{ld}_E B$, we have $(A^s + B^s) \cap AB = A + B$. $\square$

Proposition 4.4.2. *Let T be a theory of fields satisfying the same hypotheses as Theorem 4.4.1. Then $TV_1 \cdots V_n$ is not simple.*

Proof. To prove that $TV_1 \cdots V_n$ is not simple, it is sufficient to prove that TV is not simple. Let $E \prec F$ be models of T and a, b, d elements of F be such that $a \mathop{\smile\!\!\!\!\!\!|}^{T}_E b, d$ and $b \mathop{\smile\!\!\!\!\!\!|}^{T}_E d$. We show that

$$ad + b \in [\mathrm{acl}_T(Ead) + \mathrm{acl}_T(Ebd)] \setminus [(\mathrm{acl}_T(Ea) + \mathrm{acl}_T(Ebd)) \cup \mathrm{acl}_T(Ead)],$$

then TV is not simple, byy Corollary 4.2.3. Since $b \notin \mathrm{acl}_T(Ead)$, it is clear that $ad + b \notin \mathrm{acl}_T(Ead)$. Assume that $ad+b \in \mathrm{acl}_T(Ea)+\mathrm{acl}_T(Ebd)$. Then $ad \in \mathrm{acl}_T(Ea)+\mathrm{acl}_T(Ebd)$, let $u \in \mathrm{acl}_T(Ea)$ and $v \in \mathrm{acl}_T(Ebd)$ be such that $ad = u + v$. From Fact 1.5.10, we have that $\mathrm{acl}_T(Ea) \mathop{\smile\!\!\!\!\!\!|}^{ld}_E \mathrm{acl}_T(Ebd)$, hence $\mathrm{acl}_T(Ea)(d) \mathop{\smile\!\!\!\!\!\!|}^{ld}_{E(d)} \mathrm{acl}_T(Ebd)$ so $\mathrm{acl}_T(Ea)(d) \cap \mathrm{acl}_T(Ebd) = E(d)$. Similarly, $\mathrm{acl}_T(Ebd)(a) \cap \mathrm{acl}_T(Ea) = E(a)$. It follows that

$$u = ad - v \in \mathrm{acl}_T(Ebd)(a) \cap \mathrm{acl}_T(Ea) = E(a)$$
$$v = ad - u \in \mathrm{acl}_T(Ea)(d) \cap \mathrm{acl}_T(Ebd) = E(d)$$

hence $ad \in E(a) + E(b)$, which contradicts Lemma 1.5.8. $\square$

[1]In fact here we only use that $E = \mathrm{acl}_T(E)$, and Fact 1.5.9.

Example 4.4.3 (The theories $\mathrm{ACFV}_1 \dots \mathrm{V}_n$ and ACFG)**.** Let $\mathrm{ACFV}_1 \dots \mathrm{V}_n$ and ACFG be the theories as in Example 3.2.4. By Theorem 4.4.1 and Proposition 4.4.2 those theories are NSOP_1 not simple. In $\mathrm{ACFV}_1 \dots \mathrm{V}_n$, Kim-independence agrees with the relation

$$A \mathop{\smile\!\!\!\!\!\!|}^{w}_{C} B \iff A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} B \text{ and for all } i \leq n,\ V_i(\overline{AC} + \overline{BC}) = V_i(\overline{AC}) + V_i(\overline{BC}).$$

Furthermore, $\mathop{\smile\!\!\!\!\!\!|}^{w}$ satisfies

- Strong Finite Character over algebraically closed sets. For algebraically closed E, if $a \not\mathop{\smile\!\!\!\!\!\!|}^{w}_{E} b$, then there is a formula $\phi(x, b, e) \in tp^{ACFV_1 \dots V_n}(a/bE)$ such that for all a', if $a' \models \phi(x, b, e)$ then $a' \not\mathop{\smile\!\!\!\!\!\!|}^{w}_{E} b$.
- $\mathop{\smile\!\!\!\!\!\!|}^{a}$-amalgamation over algebraically closed sets. For algebraically closed set E if there exists tuples c_1, c_2 and sets A, B such that
  - $c_1 \equiv_E^{ACFV_1 \dots V_n} c_2$
  - $\overline{AE} \cap \overline{BE} = E$
  - $c_1 \mathop{\smile\!\!\!\!\!\!|}^{w}_{E} A$ and $c_2 \mathop{\smile\!\!\!\!\!\!|}^{w}_{E} B$

  then there exists $c \mathop{\smile\!\!\!\!\!\!|}^{w}_{E} A, B$ such that $c \equiv_A^{ACFV_1 \dots V_n} c_1$, $c \equiv_B^{ACFV_1 \dots V_n} c_2$, $A \mathop{\smile\!\!\!\!\!\!|}^{a}_{Ec} B$, $c \mathop{\smile\!\!\!\!\!\!|}^{a}_{EA} B$ and $c \mathop{\smile\!\!\!\!\!\!|}^{a}_{EB} A$.

This is Theorem 4.2.2, knowing that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}$ is stationary over algebraically closed sets hence satisfies the independence theorem over algebraically closed sets without any assumption on the parameters.

Example 4.4.4. Perfect ω-free PAC_p fields are NSOP_1 (see Subsection 1.5.2), furthermore, as they are algebraically bounded, the condition on the algebraic closure in Theorem 4.4.1 is satisfied. If T is a theory of a perfect ω-free PAC_p-field in an expansion of the language $\mathscr{L}_{\mathrm{ring}}$ such that T is model-complete, then $TG_1 \cdots G_n$ (Proposition 3.2.5) is NSOP_1. This holds of course for any NSOP_1 perfect PAC_p field.

4.4.2 Algebraically closed fields with a generic multiplicative subgroup

Let $\mathrm{ACFG}^\times$ be the theory obtained in Theorem 3.3.5. We denote by $A \cdot B$ the product set $\{a \cdot b \mid a \in A,\ b \in B\}$.

Theorem 4.4.5. *Any completion of* $\mathrm{ACFG}^\times$ *is* NSOP_1 *and not simple. Furthermore, Kim-independence coincide over models with the relation*

$$A \mathop{\smile\!\!\!\!\!\!|}^{w}_{C} B \iff A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} B \text{ and } G^\times(\overline{AC} \cdot \overline{BC}) = G^\times(\overline{AC}) \cdot G^\times(\overline{BC}).$$

Furthermore, $\mathop{\smile\!\!\!\!\!\!|}^{w}$ *satisfies*

- Strong Finite Character over algebraically closed sets. *For algebraically closed E, if $a \not\mathop{\smile\!\!\!\!\!\!|}^w_E b$, then there is a formula $\phi(x,b,e) \in tp^{\mathrm{ACFG}^\times}(a/bE)$ such that for all a', if $a' \models \phi(x,b,e)$ then $a' \not\mathop{\smile\!\!\!\!\!\!|}^w_E b$.*
- $\mathop{\smile\!\!\!\!\!\!|}^a$-amalgamation over algebraically closed sets. *For algebraically closed set E if there exists tuples c_1, c_2 and sets A, B such that*
  - $c_1 \equiv^{\mathrm{ACFG}^\times}_E c_2$
  - $\overline{AE} \cap \overline{BE} = E$
  - $c_1 \mathop{\smile\!\!\!\!\!\!|}^w_E A$ *and* $c_2 \mathop{\smile\!\!\!\!\!\!|}^w_E B$

  then there exists $c \mathop{\smile\!\!\!\!\!\!|}^w_E A, B$ such that $c \equiv^{\mathrm{ACFG}^\times}_A c_1$, $c \equiv^{\mathrm{ACFG}^\times}_B c_2$, $A \mathop{\smile\!\!\!\!\!\!|}^a_{Ec} B$, $c \mathop{\smile\!\!\!\!\!\!|}^a_{EA} B$ and $c \mathop{\smile\!\!\!\!\!\!|}^a_{EB} A$.

Proof. Using Theorem 4.2.1, it is enough to show that for E algebraically closed and A, B, C algebraically closed containing E, if $C \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_E A, B$ and $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_E B$ then

$$\overline{AC} \cdot \overline{BC} \cap \overline{AB} = A \cdot B.$$

This easily follows from the fact that $tp^{\mathrm{ACF}}(C/AB))$ is finitely satisfiable in E, as in the proof of Theorem 4.4.1. The rest is Theorem 4.2.2, knowing that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}$ is stationnary over algebraically closed sets, similarly to Example 4.4.3. To prove that $\mathrm{ACFG}^\times$ is not simple, we use Corollary 4.2.3, as in the proof of Proposition 4.4.2. Let E be a model of ACF_p and a, b, d in an extension be such that $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_E b, d$ and $b \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_E d$. We claim that

$$(a+d)b \in \left[\overline{Ead} \cdot \overline{Ebd}\right] \setminus \left[(\overline{Ea} \cdot \overline{Ebd}) \cup \overline{Ead}\right].$$

Since $b \notin \overline{Ead}$, it is clear that $(a+d)b \notin \overline{Ead}$. Assume that $(a+d)b \in \overline{Ea} \cdot \overline{Ebd}$. Then $a+d \in \overline{Ea} \cdot \overline{Ebd}$, let $u \in \overline{Ea}$ and $v \in \overline{Ebd}$ be such that $a+d = uv$. We have that $\overline{Ea} \mathop{\smile\!\!\!\!\!\!|}^{ld}_E \overline{Ebd}$, hence $\overline{Ea}(d) \mathop{\smile\!\!\!\!\!\!|}^{ld}_{E(d)} \overline{Ebd}$ so $\overline{Ea}(d) \cap \overline{Ebd} = E(d)$. Similarly, $\overline{Ebd}(a) \cap \overline{Ea} = E(a)$. It follows that

$$\begin{aligned} u &= (a+d)v^{-1} \in \overline{Ebd}(a) \cap \overline{Ea} = E(a) \text{ and} \\ v &= (a+d)u^{-1} \in \overline{Ea}(d) \cap \overline{Ebd} = E(d) \end{aligned}$$

hence $a + d \in E(a) \cdot E(d)$, which contradicts Lemma 1.5.8. □

CHAPTER 5

A study of ACFG

Let $p > 0$ be a fixed prime number. Unless stated otherwise, every field we consider has characteristic p. Let $\mathscr{L}_{\mathrm{ring}}$ be the language of rings and $\mathscr{L}_G = \mathscr{L}_{\mathrm{ring}} \cup \{G\}$ for G a unary predicate. Let ACF_G be the $\mathscr{L}_G$-theory whose models are algebraically closed fields of characteristic p in which G is a predicate for an additive subgroup. Let ACFG be the model companion of ACF_G, see Examples 3.2.4 and 4.4.3. In this chapter, we give a basic study of the theory ACFG. First, we give a precise description of the Kim-independence. Then we investigate some algebraic properties of any models. Finally, we construct inductively a model inside $\overline{\mathbb{F}_p}$ and prove that such models are numerous in $\overline{\mathbb{F}_p}$.

Contents

5.1 Preliminaries, axioms and types

The following is Proposition 3.1.4.

Proposition 5.1.1 (Axiomatisation of ACFG)**.** *The theory* ACFG *is axiomatised by adding to* ACF_G *the following* $\mathscr{L}_G$*-sentences, for all tuples of variables* $x' \subset x$, $y' \subset y$ *and* $\mathscr{L}_{\mathrm{ring}}$*-formula* $\phi(x, y)$

$$\forall y(\langle y' \rangle \cap G = \{0\} \wedge \theta_\phi(y)) \to (\exists x \phi(x, y) \wedge \langle xy' \rangle \cap G = \langle x' \rangle),$$

where $\theta_\phi(y)$ *such that* $K \models \theta_\phi(b)$ *if and only if in an elementary extension of* K*, there exists a tuple of realisations of* $\phi(x, b)$ *which is* $\mathbb{F}_p$*-linearly independent over* K *(see Theorem 3.1.2).*

By Proposition 2.3.5 we have the following, for $(K, G) \models$ ACFG sufficiently saturated, and a, b, C in K

(1) $\mathrm{acl}_{\mathrm{ACFG}}(C) = \mathrm{acl}_{\mathrm{ACF}}(C) =: \overline{C}$;

(2) $a \equiv_C b$ if and only if there exists an $\mathscr{L}_G$-isomorphism $\sigma : \overline{Ca} \to \overline{Cb}$ over C such that $\sigma(a) = b$;

(3) the completions of ACFG are given by the $\mathscr{L}_G$-isomorphism type of $(\overline{\mathbb{F}_p}, G(\overline{\mathbb{F}_p}))$.

Let x be a tuple from a field extension of K and H be an additive subgroup of the field $\overline{Cx}$. If

$$\overline{Cx} \cap K = \overline{C} \text{ and } H \cap \overline{C} = G(\overline{C})$$

then, by Proposition 2.3.6, the type associated to the $\mathscr{L}_G$-isomorphism class of the pair $(\overline{Cx}, H)$ is consistent in (K, G), i.e. there exists a tuple a from K such that there is a $\mathscr{L}_G$-isomorphism over C

$$f : (\overline{Ca}, G(\overline{Ca})) \to (\overline{Cx}, H)$$

with $f(a) = x$.

Example 5.1.2 (Empty types)**.** Let (K, G) be a κ-saturated model of ACFG, $C \subseteq K$ such that $|C| < \kappa$ and x a finite tuple algebraically independent over K. By previously, the type associated to the pair $(\overline{Cx}, G(\overline{C}))$ is consistent. Hence there is some tuple a from K, algebraically independent over C such that $G(\overline{Ca}) = G(\overline{C})$. This type is unique if $G(\overline{C}) \subseteq C$: let a and a' realise this type, meaning that $G(\overline{Ca}) = G(\overline{Ca'}) = G(\overline{C})$. Then $a \equiv_C a'$. Indeed if σ is a field isomorphism over C between $\overline{Ca}$ and $\overline{Ca'}$, then it fixes $G(\overline{C})$ so it is an $\mathscr{L}_G$-isomorphism. The type is unique in particular if C is algebraically closed. This uniqueness is a special case of the stationarity of the strong independence (cf. Lemma 4.3.3).

5.2 Independence relations in (K, G)

We work in a monster model (K, G) of ACFG.

Definition 5.2.1 (Weak and strong independence)**.** Let A, B, C be subsets of K. Let $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}$ be the forking independence in the sense of ACF. Recall the *weak independence relation*:

$$A \mathop{\smile\!\!\!\!\!\!|}^{w}_{C} B \text{ if and only if } A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} B \text{ and } G(\overline{AC} + \overline{BC}) = G(\overline{AC}) + G(\overline{BC}),$$

and the *strong independence relation*:

$$A \mathop{\smile\!\!\!\!\!\!|}^{st}_{C} B \text{ if and only if } A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} B \text{ and } G(\overline{ABC}) = G(\overline{AC}) + G(\overline{BC}).$$

Theorem 5.2.2. *The relation* $\mathop{\smile\!\!\!\!\!\!|}^{w}$ *satisfies* INVARIANCE, CLOSURE, SYMMETRY, FULL EXISTENCE, MONOTONICITY, EXISTENCE, LOCAL CHARACTER, TRANSITIVITY, STRONG FINITE CHARACTER *over algebraically closed sets,* $\mathop{\smile\!\!\!\!\!\!|}^{a}$-AMALGAMATION *over algebraically closed sets.*

Proof. Apart from TRANSITIVITY and LOCAL CHARACTER, all properties has been proven in Theorem 4.2.2 and Example 4.4.3.

TRANSITIVITY. Assume that $A \mathop{\smile\!\!\!\!\!\!|}^{w}_{CB} D$ and $B \mathop{\smile\!\!\!\!\!\!|}^{w}_{C} D$. We may assume that $A = \overline{ABC}, B = \overline{CB}$ and $D = \overline{CD}$. By MONOTONICITY, it is sufficient to show that $A \mathop{\smile\!\!\!\!\!\!|}^{w}_{C} D$. We clearly have $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} D$ by TRANSITIVITY of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}$. We show that $G(A + D) = G(A) + G(D)$. By $A \mathop{\smile\!\!\!\!\!\!|}^{w}_{B} D$ we have $G(A + \overline{BD}) = G(A) + G(\overline{BD})$. It follows that $G(A + D)$ is included in $(A + D) \cap (G(A) + G(\overline{BD}))$, which, by modularity, is equal to

$$G(A) + (A + D) \cap G(\overline{BD}) = G(A) + G(A \cap \overline{BD} + D).$$

As $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{B} D$, $A \cap \overline{BD} = B$. By $B \mathop{\smile\!\!\!\!\!\!|}^{w}_{C} D$, $G(B + D) = G(B) + G(D)$ hence

$$G(A + D) = G(A) + G(B) + G(D) = G(A) + G(D).$$

LOCAL CHARACTER. We start with a claim.

Claim. Let A, B be subsets of (K, G) with B subgroup of $(K, +)$, then there exists $C \subseteq B$ with $|C| \leq |A|$ such that

$$G(A + B) = G(A + C) + G(B).$$

Proof of the claim. For each $a \in A$ define $C(a)$ to be the set of those $b \in B$ such that $a + b \in G$. Take $c(a) \in C(a)$ for each a such that $C(a)$ is nonempty, and set

$$C = \{c(a) \mid a \in A \text{ and } C(a) \neq \emptyset\}.$$

Now if $g \in G(A+B)$ then $g = a+b$ with $a \in A$, $b \in B$. We have $C(a)$ nonempty so we can write for $c = c(a)$

$$g = (a+c) + (b-c).$$

It follows that $b - c \in G(B)$ hence $g \in G(A+C) + G(B)$. The reverse inclusion is trivial. $\square$

Let a be a finite tuple and B an algebraically closed set. We construct two sequences $(A_i)_{i<\omega}$ and $(D_i)_{i<\omega}$ such that the following holds for all $n < \omega$:

(1) $A_n \subseteq A_{n+1} \subseteq \overline{Ba}$ and $D_n \subseteq A_n$

(2) $G(A_n + B) \subseteq G(A_{n+1}) + G(B)$

(3) $A_n \mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}_{D_n} B$

(4) $|A_n| \leq \aleph_0$

Using Local Character for $\mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}$ there exists a countable set $D_0 \subseteq B$ such that $a \mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}_{D_0} B$. We define $A_0 = \overline{aD_0}$. Assume that D_n and A_n has been constructed and that $|A_n| \leq \aleph_0$. By the claim there exists $C \subseteq B$ with $|C| \leq \aleph_0$ such that $G(A_n + B) = G(A_n + C) + G(B)$. Using Local Character[1] of $\mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}$ on the set A_nC there exists $D_{n+1} \subseteq B$ with $|D_{n+1}| \leq \aleph_0$ such that $A_nC \mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}_{D_{n+1}} B$. We set $A_{n+1} = \overline{A_nCD_{n+1}}$. Note that $A_n + C \subseteq A_{n+1}$ so $G(A_n + B) \subseteq G(A_{n+1}) + G(B)$.

Now set $A_\omega = \bigcup_{i<\omega} A_i$ and $D_\omega = \bigcup_{i<\omega} D_i$. We have $|A_\omega| \leq \aleph_0$ and $|D_\omega| \leq \aleph_0$. We claim that

$$A_\omega \mathop{\smile\!\!\!\!\mid}^{w}_{D_\omega} B.$$

If u is a finite tuple from A_ω, then $u \subseteq A_n$ for some n, so as $A_n \mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}_{D_n} B$ we have $u \mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}_{D_n} B$. Now as $D_\omega \subseteq B$, we use Base Monotonicity of $\mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}$ to conclude that $u \mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}_{D_\omega} B$. As this holds for every finite tuple u from A_ω, we conclude that

$$A_\omega \mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}_{D_\omega} B.$$

It remains to show that $G(A_\omega + B) = G(A_\omega) + G(B)$. If $g \in G(A_\omega + B)$ then there is some n such that $g \in A_n + B$ and so

$$g \in G(A_n + B) \subseteq G(A_{n+1}) + G(B) \subseteq G(A_\omega) + G(B).$$

The reverse inclusion being trivial, we conclude that $G(A_\omega + B) = G(A_\omega) + G(B)$, so $A_\omega \mathop{\smile\!\!\!\!\mid}^{w}_{D_\omega} B$. As $a \subseteq A_\omega$ we conclude by Monotonicity of $\mathop{\smile\!\!\!\!\mid}^{w}$. $\square$

[1] Here we use a stronger version of Local Character which holds in any simple (countable) theory (see [Cas11, Proposition 5.5]): for all countable set A and arbitrary set B there exists $B_0 \subseteq B$ with $|B_0| \leq \aleph_0$ with $A \mathop{\smile\!\!\!\!\mid}_{B_0} B$.

Proposition 5.2.3. *Assume that $C = \overline{C}$. If $a \mathop{\smile\!\!\!\!\!\!|}^w_C b$, then for all C-indiscernible sequence $(b_i)_{i<\omega}$ in $tp(b/C)$ such that $b_i \mathop{\smile\!\!\!\!\!\!|}^a_C (b_j)_{j<i}$ there exists a' such that $a'b_i \equiv_C ab$ for all $i < \omega$. In particular, the following are equivalent, for C algebraically closed and $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_C b$.*

(1) $a \mathop{\smile\!\!\!\!\!\!|}^w_C b$;

(2) for all C-indiscernible sequence $(b_i)_{i<\omega}$ in $tp(b/C)$ such that, $b_i \mathop{\smile\!\!\!\!\!\!|}^a_C (b_j)_{j<i}$ and $G(\overline{Cb_i} + \overline{Cb_k}) = G(\overline{Cb_i}) + G(\overline{Cb_k})$ there exists a' such that $a'b_i \equiv_C ab$ for all i;

(3) for some C-indiscernible sequence $(b_i)_{i<\omega}$ in $tp(b/C)$ such that, $b_i \mathop{\smile\!\!\!\!\!\!|}^a_C (b_j)_{j<i}$ and $G(\overline{Cb_i} + \overline{Cb_k}) = G(\overline{Cb_i}) + G(\overline{Cb_k})$ there exists a' such that $a'b_i \equiv_C ab$ for all i.

Proof. The first assertion holds because $\mathop{\smile\!\!\!\!\!\!|}^w$ satisfies $\mathop{\smile\!\!\!\!\!\!|}^a$-AMALGAMATION over algebraically closed sets (Theorem 5.2.2). The proof is a classical induction similar to the proof of Lemma 7.1.9 or [CK17, Proposition 4.11].

(1) implies *(2)* is a particular case of the first assertion. *(2)* implies *(3)* follows from the fact that such sequence exists, which follows from FULL EXISTENCE of $\mathop{\smile\!\!\!\!\!\!|}^w$. We show that *(3)* implies *(1)*. Assume that $a \not\mathop{\smile\!\!\!\!\!\!|}^w_C b$ and let $\Lambda(x, b, c)$ be as in Lemma 4.1.6. If *(3)* holds, then in particular $\{\Lambda(x, b_i, c) \mid i < \omega\}$ is consistent, for some $(b_i)_{i<\omega}$ such that $b_i \equiv_C b$ and $b_i \mathop{\smile\!\!\!\!\!\!|}^a_C b_j$. This contradicts Lemma 4.1.6. □

In particular, we have the following combinatorial characterization of $\mathop{\smile\!\!\!\!\!\!|}^w$ over algebraically closed sets.

Corollary 5.2.4. *The following are equivalent, for C algebraically closed*

(1) $a \mathop{\smile\!\!\!\!\!\!|}^w_C b$;

(2) for all C-indiscernible sequence $(b_i)_{i<\omega}$ in $tp(b/C)$ such that, $b_i \mathop{\smile\!\!\!\!\!\!|}^w_C (b_j)_{j<i}$ there exists a' such that $a'b_i \equiv_C ab$ for all i;

(3) for some C-indiscernible sequence $(b_i)_{i<\omega}$ in $tp(b/C)$ such that, $b_i \mathop{\smile\!\!\!\!\!\!|}^w_C (b_j)_{j<i}$ there exists a' such that $a'b_i \equiv_C ab$ for all i.

Proof. *(1)* implies *(2)* follows from Proposition 5.2.3, and *(2)* implies *(3)* holds since $\mathop{\smile\!\!\!\!\!\!|}^w$ satisfies FULL EXISTENCE. Assume that *(3)* holds for some a' and indiscernible sequence $(b_i)_{i<\omega}$ such that $b_i \mathop{\smile\!\!\!\!\!\!|}^w_C (b_j)_{j<i}$for all $i < \omega$. In particular, $(b_i)_{i<\omega}$ is a Morley sequence in the sense of ACF_p, and $a'b_i \equiv^{\mathrm{ACF}}_C ab$ for all $i < \omega$. As $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}$ is forking independence in the sense of ACF_p, we have $a \mathop{\smile\!\!\!\!\!\!|}^{ACF}_C b$. By Proposition 5.2.3 we have $a \mathop{\smile\!\!\!\!\!\!|}^w_C b$. □

The Kim-Pillay theorem (see Fact 1.4.6) states that if a relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfies INVARIANCE, SYMMETRY, MONOTONICITY, BASE MONOTONICITY, TRANSITIVITY, FULL EXISTENCE, LOCAL CHARACTER, $\mathop{\smile\!\!\!\!\!\!|}$-AMALGAMATION over models and FINITE CHARACTER[2], then the theory is simple and this relation is forking independence. From Theorem 5.2.2 and

[2]This property is trivial for $\mathop{\smile\!\!\!\!\!\!|}^w$ and $\mathop{\smile\!\!\!\!\!\!|}^{st}$.

Proposition 4.4.2, the weak independence $\mathop{\smile\!\!\!\!\!\!\vert}^w$ satisfies all the previous properties except Base Monotonicity. This is similar to the case of $K_{n,m}$-free bipartite graph [CK17, Remark 4.17].

The property $\mathop{\smile\!\!\!\!\!\!\vert}$-amalgamation over models is a special case of Stationnarity over algebraically closed sets, hence from Lemma 4.3.3, the strong independence $\mathop{\smile\!\!\!\!\!\!\vert}^{st}$ satisfies every property of the Kim-Pillay characterization except Local Character otherwise, ACFG would be simple. Example 7.1.4 shows directly that Local Character is not satisfied by $\mathop{\smile\!\!\!\!\!\!\vert}^{st}$, nor by any relation stronger than $\mathop{\smile\!\!\!\!\!\!\vert}^w$ which satisfies Base Monotonicity. As we saw in Lemma 4.3.3, ACFG is mock stable in the sense of Adler.

5.3 Some structural features of (K, G)

Let $P(X)$ be a polynomial in variables $X = X_1, \dots, X_n$ with coefficients in K. We say that P is *$\mathbb{F}_p$-flat over K* if whenever u is a zero of P in some field extension of K, there exists a non trivial $\mathbb{F}_p$-linear combination of u that falls in K.

Lemma 5.3.1. *Let (K, G) be an $\aleph_0$-saturated model of* ACFG*, and $P(X_1, \dots, X_n)$ a polynomial non-$\mathbb{F}_p$-flat over K. Then for every $I \subset \{1, \dots, n\}$ there exists a zero a of P in K such that $a_i \in G \iff i \in I$.*

Proof. Let $I \subset \{1, \dots, n\}$. As P is non-$\mathbb{F}_p$-flat, there exists a zero t of P in an extension of K such that no non nontrivial $\mathbb{F}_p$-combination of t falls in K. It follows that $(\overline{K(t)}, G + \langle t_i \mid i \in I\rangle)$ is an $\mathscr{L}_G$-extension of (K, G). Indeed $(G + \langle t_i \mid i \in I\rangle) \cap K = G$. Furthermore $t_j \in (G + \langle t_i \mid i \in I\rangle)$ if and only if $j \in I$. As (K, G) is existentially closed in $(\overline{K(t)}, G + \langle t_i \mid i \in I\rangle)$, we have that

$$(K, G) \models \exists x (P(x) = 0 \wedge \bigwedge_{i \in I} x_i \in G \wedge \bigwedge_{j \notin I} x_j \notin G).$$

□

Lemma 5.3.2. *A polynomial P in $K[X]$ is $\mathbb{F}_p$-flat over K if and only if all its irreducible factors in $K[X]$ are of the form $c(\lambda_1 X_1 + \dots + \lambda_n X_n - b)$ for some $\lambda_1, \dots, \lambda_n$ in $\mathbb{F}_p \setminus \{0\}$ and $b, c \in K$.*

Proof. Assume that P is $\mathbb{F}_p$-flat over K. If $|X| = 1$, then P satisfies the conclusion. Assume that $|X| > 1$. Let $t_2, \dots, t_n$ be algebraically independent over K, and consider $P(X_1, t_2, \dots, t_n)$. This polynomial has zeros in $\overline{K(t_2, \dots, t_n)}$ hence by $\mathbb{F}_p$-flatness each root u satisfies $\lambda_1 u + \lambda_2 t_2 + \dots + \lambda_n t_n = b$ for some non-zero tuple $\lambda_1, \dots, \lambda_n$ from $\mathbb{F}_p$ and $b \in K$. By hypothesis on $t_2, \dots, t_n$ we have that $\lambda_1 \neq 0$. It follows that $X_1 - \lambda_1^{-1}(\lambda_2 t_2 + \dots + \lambda_n t_n - b)$ divides $P(X_1, t_2, \dots, t_n)$ hence $\lambda_1 X_1 + \dots + \lambda_n X_n - b$ divides P, as $K[X_1, t_2, \dots, t_n] \cong K[X]$. If $\lambda_i = 0$ for some i, then the tuple $(0, \dots, t, \dots, 0)$ with t transcendental over K at the i-th coordinate, is a zero of P that contradicts the $\mathbb{F}_p$-flatness. It follows that P is of the desired form. The other direction is trivial. □

Example 5.3.3 ($\mathbb{F}_p$-flatness might depends on p)**.** Consider the polynomial $P = X^2+Y^2$, with $b \in K$. Then P is $\mathbb{F}_p$-flat over any algebraically closed field if and only if -1 is a square in $\mathbb{F}_p$. From [Fre01, Exercice 1.9.24], when $p > 2$ this is equivalent to $p \in 4\mathbb{Z}+1$. Using Lemmas 5.3.1 and 5.3.2 it follows that whenever $(K, G) \models \mathrm{ACFG}$, $p > 2$,

- if $p \notin 4\mathbb{Z} + 1$ there exists $g \in G$ and $u \in K \setminus G$ such that $g^2 + u^2 = 0$;
- if $p \in 4\mathbb{Z}+1$ such couple (u, g) does not exists in (K, G), as every couple of solution to $X^2 + Y^2 = 0$ are $\mathbb{F}_p$-linearly dependent.

Proposition 5.3.4. *Let (K, G) be a model of* ACFG*. The following holds:*

(1) $K = G \cdot G = G \cdot (K \setminus G) = (K \setminus G) \cdot (K \setminus G)$*;*

(2) *G is stably embedded in K;*

(3) *For $a \notin \mathbb{F}_p$ and $P \in K[X] \setminus (K + \mathbb{F}_p \cdot X)$, we have $K = G + aG = (K \setminus G) + aG = G + P(G)$.*

Proof. (1) For all $b \in K$ the polynomial $XY - b$ is not $\mathbb{F}_p$-flat by Lemma 5.3.2, so we conclude using Lemma 5.3.1.
(2) From *(1)*, every element in K is product of two elements in G, so any $\mathscr{L}_G$-formula $\phi(x, a_1, \dots, a_n)$ is equivalent to $\phi(x, g_1h_1, \dots, g_nh_n)$ with $g_i, h_i \in G$.
(3) For all $P \in K[X] \setminus (K + \mathbb{F}_p \cdot X)$, $b \in K$, the polynomial $Y + P(X) - b$ is not $\mathbb{F}_p$-flat, similarly to *(1)*. $\square$

Proposition 5.3.5. *Let $\zeta_1, \dots, \zeta_n$ be $\mathscr{L}_{\mathrm{ring}}$-definable endomorphisms of $(K, +)$, $\mathbb{F}_p$-linearly independent. Then*

$$K/(\zeta_1^{-1}(G) \cap \dots \cap \zeta_n^{-1}(G)) \cong K/\zeta_1^{-1}(G) \times \dots \times K/\zeta_n^{-1}(G).$$

Proof. Using the first isomorphism theorem, it is sufficient to prove that the function $\zeta : K \to K/\zeta_1^{-1}(G) \times \dots \times K/\zeta_n^{-1}(G)$ defined by $\zeta(u) = (u + \zeta_1^{-1}(G), \dots, u + \zeta_n^{-1}(G))$ is onto. Let $c_1, \dots, c_n \in K$, we want to show that there exists $c \in K$ such that for all i $\zeta_i(c-c_i) \in G$. Let t be a transcendental element over K, by model completeness of ACF_p, $\zeta_1, \dots, \zeta_n$ are $\mathbb{F}_p$-linearly independent definable endomorphisms of $(\overline{Kt}, +)$. Consider the $\mathscr{L}_G$-structure

$$(\overline{Kt}, G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle).$$

We have $(G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle) \cap K = G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle \cap K$. For $\lambda_1, \dots, \lambda_n \in \mathbb{F}_p$, if $\sum_i \lambda_i \zeta_i(t - c_i) \in K$ then $\sum_i \lambda_i \zeta_i(t) \in K$. By Fact 1.5.12, there is some k such that $t \mapsto (\sum_i \lambda_i \zeta_i(t))^{p^k}$ is polynomial, hence as t is transcendental over K, $(\sum_i \lambda_i \zeta_i)^{p^k} = 0$, so $\sum_i \lambda_i \zeta_i = 0$. As $\zeta_1, \dots, \zeta_n$ are $\mathbb{F}_p$-linearly independent, $\lambda_1 = \dots = \lambda_n = 0$. It follows that $(G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle) \cap K = G$, so $(\overline{Kt}, G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle)$ extends (K, G). As (K, G) is existentially closed in $(\overline{Kt}, G + \langle \zeta_i(t - c_i) \mid i \leq n \rangle)$ we have that $(K, G) \models \exists x \bigwedge_i \zeta_i(x - c_i) \in G$, hence ζ is onto. $\square$

If $\zeta_1, \ldots, \zeta_n$ are $\mathbb{F}_p$-linearly independent $\mathscr{L}_{\mathrm{ring}}$-definable isomorphisms of $(K, +)$, the previous result can be used to find canonical parameters for the quotient $K/(\zeta_1^{-1}(G) \cap \cdots \cap \zeta_n^{-1}(G))$ provided one have canonical parameters for the quotient K/G, see Example 6.0.1.

5.4 Models of ACFG in $\overline{\mathbb{F}_p}$

From Theorem 3.1.2, for any quantifier free $\mathscr{L}_{\mathrm{ring}}$-formula $\phi(x, y)$, there exists an $\mathscr{L}_{\mathrm{ring}}$-formula $\theta_\phi(y)$ such that for $K \models \mathrm{ACF}_p$ sufficiently saturated and b tuple in K such that $K \models \theta_\phi(b)$ if and only if there exists a realisation a of $\phi(x, b)$ which is $\mathbb{F}_p$-linearly independent over $\overline{\mathbb{F}_p(b)}$. By quantifier elimination in ACF_p, the formula θ_ϕ can be choosen quantifier-free.

Lemma 5.4.1. *If $\mathbb{F}_{p^n} \models \theta_\phi(b)$ then for all $m \geq n$ there exists $k > m$ such that*

$$\mathbb{F}_{p^k} \models \exists x \phi(x, b) \wedge x \textit{ is } \mathbb{F}_p\textit{-linearly independent over } \mathbb{F}_{p^m}.$$

Proof. Assume that $\mathbb{F}_{p^n} \models \theta_\phi(b)$. Then as θ_ϕ is quantifier free, $\overline{\mathbb{F}_p} \models \theta_\phi(b)$. It follows that for some elementary extension K of $\overline{\mathbb{F}_p}$, there is some realisation a of $\phi(x, b)$ which is $\mathbb{F}_p$-linearly independent over $\overline{\mathbb{F}_p}$. In particular for every non trivial polynomial $P(Z, Y) \in \mathbb{F}_p[Z, Y]$ (where Z is a single variable and Y a tuple of variables with $|Y| = |y|$), no nontrivial $\mathbb{F}_p$-linear combination of a is a root of $P(Z, b)$. As $\overline{\mathbb{F}_p} \equiv^{\mathrm{ACF}} K$, the following sentence holds in $\overline{\mathbb{F}_p}$:

$$\forall y (\theta_\phi(y) \to (\exists x \phi(x, y) \wedge \text{"no nontrivial } \mathbb{F}_p\text{-linear combination of } x \text{ is a root of } P(Z, y)\text{"})).$$

In particular, for the polynomial $X^{p^m} - X$ for some m we have

$$\overline{\mathbb{F}_p} \models \exists x \phi(x, b) \wedge \text{no non-trivial } \mathbb{F}_p\text{-linear combination of } x \text{ falls in } \mathbb{F}_{p^m}.$$

Hence for some $k > m, n$ there exists a tuple a from $\mathbb{F}_{p^k}$ such that

$$\overline{\mathbb{F}_p} \models \phi(a, b) \wedge a \text{ is } \mathbb{F}_p\text{-linearly independent over } \mathbb{F}_{p^m}.$$

As $\phi(x, y)$ is quantifier-free, we also have that

$$\mathbb{F}_{p^k} \models \phi(a, b) \wedge a \text{ is } \mathbb{F}_p\text{-linearly independent over } \mathbb{F}_{p^m}.$$

□

Proposition 5.4.2. *For any $n \in \mathbb{N}$ and any G_0 additive subgroup of $\mathbb{F}_{p^n}$ there exists a subgroup G of $\overline{\mathbb{F}_p}$ such that $G \cap \mathbb{F}_{p^n} = G_0$ and $(\overline{\mathbb{F}_p}, G) \models \mathrm{ACFG}$.*

Proof. Start with the following claim.

Claim. Let $n \in \mathbb{N}$, let $s \in \mathbb{N}$, let $k_1, \dots, k_s \in \mathbb{N}$ and let $\phi_1(x^1, y^1), \dots, \phi_s(x^s, y^s)$ be quantifier free formulae in $\mathscr{L}_{\text{ring}}$. For $i \leq s$, let $B_i = \left\{ b \in \mathbb{F}_{p^n}^{|y^i|} \mid b \models \theta_{\phi_i}(y) \right\}$. Then there exists $m > n$ such that for all $i \leq s$ and $b \in B_i$ there exists some $|x^i|$-tuples $a^{i,1}, \dots, a^{i,k_i}$ (depending on b) from $\mathbb{F}_{p^m}$ such that

(1) $(a_k^{i,j})_{i \leq s, j \leq k_i, k \leq |x^i|}$ is a $\mathbb{F}_p$-linearly independent tuple over $\mathbb{F}_{p^n}$

(2) $\mathbb{F}_{p^m} \models \phi_i(a^{i,1}, b), \dots, \mathbb{F}_{p^m} \models \phi_i(a^{i,k_i}, b)$.

Proof of the Claim. We do it step by step, as there are only a finite number of tuples to add. Start with $\phi_1(x^1, y^1)$. Take a first $b \in B_1$. As $\mathbb{F}_{p^n} \models \theta_{\phi_1}(b)$, we use Lemma 5.4.1 with $m = n$ to get a first $m_1 > n$ such that there exists $a^1 \in \mathbb{F}_{p^{m^1}}^{|x_1|}$ such that $\models \phi_1(a^1, b)$ and a^1 is $\mathbb{F}_p$-linearly independent over $\mathbb{F}_{p^n}$. Using again Lemma 5.4.1 with $m = m_1$ there exists $m_2 > m_1$ and a second $a^2 \in \mathbb{F}_{p^{m_2}}^{|x^1|}$ such that $\mathbb{F}_{p^{m_2}} \models \phi_1(a^2, b)$ and a^2 is $\mathbb{F}_p$-linearly independent over $\mathbb{F}_{p^{m_1}}$. In particular a^2 is $\mathbb{F}_p$-linearly independent from a^1 over $\mathbb{F}_{p^n}$. So we can construct as many (finitely) solution to $\phi_1(x^1, b)$ as we want which are $\mathbb{F}_p$-linearly independent over $\mathbb{F}_{p^n}$. Once we have enough $\mathbb{F}_p$-linearly independent solutions of $\phi_1(x, b)$, we can do the same trick with another $b' \in B_1$, and add as many (finitely) solution as we want, $\mathbb{F}_p$-linearly independent from one another and from the ones corresponding to b, in a finite extension of $\mathbb{F}_{p^n}$. Once we have done it for all elements of B_1, we do the same with every element $b \in B_2$, continuing to use Lemma 5.4.1 to get solutions of $\phi_2(x^2, b)$ $\mathbb{F}_p$-linearly independent from one another and from the previous ones. As every B_i is finite and they are in finite number, we can finish to add $\mathbb{F}_p$-linearly independent solutions of ϕ_i in a finite number of steps and the claim is proven. $\square$

From Proposition 5.1.1, the axioms for ACFG are given by the following scheme: for all quantifier free $\mathscr{L}_{\text{ring}}$-formula $\phi(x, y)$, for all $0 \leq k \leq |x|$ and $0 \leq k' \leq |y|$

$$\forall y \left((\theta_\phi(y) \wedge \langle y_1, \dots, y_{k'} \rangle \cap G = \{0\}) \to (\exists x \phi(x, y) \wedge \langle x, y_1, \dots, y_{k'} \rangle \cap G = \langle x_1, \dots, x_k \rangle) \right)$$

with the following convention: $a_1, \dots, a_0 = \emptyset$. We will denote the previous sentence by $\Gamma(\phi, k, k')$. Now we construct by induction a model of ACFG starting from $(\mathbb{F}_{p^n}, G_0)$. Let $(\phi_i(x^i, y^i))_{i<\omega}$ be an enumeration of all quantifier-free formula in $\mathscr{L}_{\text{ring}}$. We construct an increasing sequence $(n_j)_{j<\omega}$ starting with $n_0 = n$ and additive subgroups G_j of $\mathbb{F}_{p^{n_j}}$ such that for all $s < \omega$, for $\phi_1(x^1, y^1), \dots, \phi_s(x^s, y^s)$, for all $1 \leq l \leq s$, for all $0 \leq k \leq |x^l|$ and $0 \leq k' \leq |y^l|$ the following holds for all $|y^l|$-tuples b from $\mathbb{F}_{p^{n_s}}$

If $\mathbb{F}_{p^{n_s}} \models \theta_{\phi_l}(b) \wedge \langle b_1, \dots, b_{k'} \rangle \cap G_s = \{0\}$ then there exists $a^{l,k}$ an $|x^l|$-tuple from $\mathbb{F}_{p^{n_{s+1}}}$ such that $\mathbb{F}_{p^{n_{s+1}}} \models \phi(a^{l,k}, b) \wedge \langle a^{l,k}, b_1, \dots, b_{k'} \rangle \cap G_{s+1} = \langle a_1^{l,k}, \dots, a_k^{l,k} \rangle$. $(\star)$

Assume that for some $s < \omega$ we have $n_0, \dots, n_s$ and $G_0 \subseteq \mathbb{F}_{p^{n_0}}, \dots, G_s \subseteq \mathbb{F}_{p^{n_s}}$ constructed as above. For every $i \leq s$, we define as above $B_i = \left\{ b \in \mathbb{F}_{p^{n_s}}^{|y^i|} \mid b \models \theta_{\phi_i}(y) \right\}$, and we apply the claim with $k_i = |x^i| + 1$, to get some $n_{s+1} > n_s$. For each $1 \leq i \leq s$

and $b \in B_i$ we have $|x^i| + 1$ many $|x^i|$-tuples $a^{i,1}(b), \dots, a^{i,k_i}(b)$ from $\mathbb{F}_{p^{n_{s+1}}}$ all $\mathbb{F}_p$-independents over $\mathbb{F}_{p^{n_s}}$ and such that for all j, we have $\mathbb{F}_{p^{n_{s+1}}} \models \phi_i(a^{i,j}(b), b)$. Now define G_{s+1} to be

$$G_s \oplus \bigoplus_{1\leq i\leq s} \bigoplus_{b\in B_i} \langle a_1^{i,2}(b)\rangle \oplus \langle a_1^{i,3}(b), a_2^{i,3}(b)\rangle \oplus \dots \oplus \langle a_1^{i,k_i}(b), \dots, \oplus a_{k_i}^{i,k_i}(b)\rangle.$$

We extend G_s by the low triangle of the $(|x_i|+1)\times|x_i|$ matrix $(a_k^{i,j}(b))_{1\leq j\leq k_i, 1\leq k\leq |x^i|}$ for each $i < s$ and $b \in B_i$:

$$\begin{pmatrix} a_1^{i,1} & a_2^{i,1} & \dots & a_{|x^i|}^{i,1} \\ a_1^{i,2} & a_2^{i,2} & \dots & a_{|x^i|}^{i,2} \\ a_1^{i,3} & a_2^{i,3} & \dots & a_{|x^i|}^{i,2} \\ \vdots & & \ddots & \\ a_1^{i,k_i} & a_2^{i,k_i} & \dots & a_{|x^i|}^{i,k_i} \end{pmatrix}.$$

Now we have for each $1 \leq i \leq s$ and any $0 \leq k \leq |x^i|$ and $0 \leq k' \leq |y^i|$, if $b \in B_i$, then there exists $a^{i,k}(b) \in \mathbb{F}_{p^{n_{s+1}}}^{|x^i|}$ such that $\mathbb{F}_{p^{n_{s+1}}} \models \phi_i(a^{i,k}(b), b)$. By construction if $\langle b_1, \dots, b_{k'}\rangle \cap G_s = \{0\}$, and by $\mathbb{F}_p$-linear independence of all the $a^{i,k}$, we have $\langle a^{i,k}, b_1, \dots, b_{k'}\rangle \cap G_{s+1} = \langle a_1^{i,k}, \dots, a_k^{i,k}\rangle$. By induction we construct a familly $(\mathbb{F}_{p^{n_i}}, G_i)$ satisfying $(\star)$. Now let

$$G = \bigcup_{i<\omega} G_i \subseteq \overline{\mathbb{F}_p}.$$

By construction, we have that $(\overline{\mathbb{F}_p}, G)$ is a model of ACFG. □

Recall from Section 1.6 that $\mathrm{Sg}(\overline{\mathbb{F}_p})$ endowed with the Chabauty topology is a Cantor space. Let

$$\mathcal{C} = \left\{G \in Sg(\overline{\mathbb{F}_p}) \mid (\overline{\mathbb{F}_p}, G) \models \text{ACFG}\right\}.$$

Recall that a set is G_δ if it is a countable intersection of open sets.

Proposition 5.4.3. *$\mathcal{C}$ is a dense G_δ of $Sg(\overline{\mathbb{F}_p})$.*

Proof. We first show that it is dense. By Lemma 1.6.1, the topology on $\mathrm{Sg}(\overline{\mathbb{F}_p})$ is generated by balls of the form $\mathcal{B}(G_0, \mathbb{F}_{p^n}) = \left\{H \in \mathrm{Sg}(\overline{\mathbb{F}_p}) \mid H \cap \mathbb{F}_{p^n} = G_0\right\}$ where G_0 is a subgroup of $\mathbb{F}_{p^n}$. By Proposition 5.4.2, evey such ball contains an element of $\mathcal{C}$, hence $\mathcal{C}$ is dense. We show that it is a G_δ. First, from Proposition 5.1.1, ACFG is axiomatised by adding to the theory ACF_G the following $\mathscr{L}_G$-sentences, for all tuples of variable $x' \subset x$, $y' \subset y$ and $\mathscr{L}_{\text{ring}}$-formula $\phi(x,y)$

$$\forall y(\langle y'\rangle \cap G = \{0\} \wedge \theta_\phi(y)) \to (\exists x \phi(x,y) \wedge \langle xy'\rangle \cap G = \langle x'\rangle)$$

which is equivalent to

$$\forall y \exists x \left[\neg\theta_\phi(y)) \vee \langle y'\rangle \cap G \neq \{0\} \vee (\phi(x,y) \wedge \langle xy'\rangle \cap G = \langle x'\rangle)\right].$$

Let $\phi(x,y)$, $x' \subseteq x$ and $y' \subseteq y$ be given. Let b be a $|y|$-tuple, and consider the set

$$\mathscr{O}_b = \bigcup_{a \in \overline{\mathbb{F}_p}^{|x|}, \overline{\mathbb{F}_p} \models \phi(a,b)} \left\{ H \mid \langle b' \rangle \cap H \neq \{0\} \right\} \cup \left\{ H \mid \langle ab' \rangle \cap H = \langle a' \rangle \right\}.$$

The set $\{H \mid \langle b' \rangle \cap H \neq \{0\}\}$ is equal to $\bigcup_{u \in \langle b' \rangle \setminus \{0\}} \{H \mid u \in H\}$ which is clearly open. From Lemma 1.6.1, $\{H \mid \langle ab' \rangle \cap H = \langle a' \rangle\}$ is also open, so $\mathscr{O}_b$ is open. Now it is an easy checking that

$$\mathcal{C} = \bigcap_{\phi(x,y), x' \subseteq x, y' \subseteq y} \; \bigcap_{b \in \overline{\mathbb{F}_p}^{|y|}, \overline{\mathbb{F}_p} \models \theta_\phi(b)} \mathscr{O}_b.$$

Hence $\mathcal{C}$ is G_δ. □

Remark 5.4.4 (Ultraproduct model of ACFG). From the proof of Proposition 5.4.2, starting from $G_0 \subseteq \mathbb{F}_{p^{n_0}}$, there exists a strictly increasing sequence $(n_i)_{i<\omega}$ of integers and an increasing sequence of groups $G_i \subseteq \mathbb{F}_{p^{n_i}}$ satisfying $(\star)$. Let $\mathscr{U}$ be a nonprincipal ultrafilter on ω, it does not take long to see that the ultraproduct $\prod_{\mathscr{U}}(\overline{\mathbb{F}_p}, G_i))$ is a model of ACFG, in which the group is pseudo-finite. The construction of the $G_i's$ in the proof of Proposition 5.4.2 is rather artificial. Is there more "natural" generic subgroups of $\overline{\mathbb{F}_p}$? Given an arbitrary set $\{G_i \mid i < \omega\}$ of subgroups of $\overline{\mathbb{F}_p}$ and a non principal ultrafilter $\mathscr{U}$ on ω, how likely is it that $\prod_{\mathscr{U}}(\overline{\mathbb{F}_p}, G_i))$ is a model of ACFG?

Remark 5.4.5 (Characteristic 0). Let $\mathscr{P}$ be the set of prime numbers and $\mathscr{U}$ a non-principal ultrafilter on $\mathscr{P}$. For each $q \in \mathscr{P}$ let G_q be a subgroup of $\overline{\mathbb{F}_q}$ such that $(\overline{\mathbb{F}_q}, G_q)$ is a model of ACFG (here we mean $\mathrm{ACF_qG}$). Recall that $\mathbb{C} \cong \prod_{q \in \mathscr{P}} \overline{\mathbb{F}_q}/\mathscr{U}$. Consider the ultraproduct

$$(\mathbb{C}, V) \cong \prod_{q \in \mathscr{P}} (\overline{\mathbb{F}_q}, G_q)/\mathscr{U}.$$

It is clear that V is a subgroup of $\mathbb{C}$. For each $q \in \mathscr{P}$,

$$Stab_{\overline{\mathbb{F}_q}}(G_q) := \left\{ a \in \overline{\mathbb{F}_q} \mid aG_q \subseteq G_q \right\} = \mathbb{F}_q,$$

this follows from Proposition 5.3.4 *(3)*. Hence $F = Stab_{\mathbb{C}}(V)$ is a pseudo-finite subfield of $\mathbb{C}$, and V is an F-vector space. It follows from Proposition 3.2.7 that $(\mathbb{C}, V)$ is *not* existentially closed in the class of $\mathscr{L}_G$-structures consisting of a field of characteristic 0 in which G is an additive subgroup. Nonetheless, some properties such as the ones in Proposition 5.3.4 will be satisfied by $(\mathbb{C}, V)$ (replacing $\mathbb{F}_p$ by F).

Remark 5.4.6. Observe that the proof of Lemma 5.4.1 gives the following: if F is an infinite *locally finite* field[3], and that for some universal $\mathscr{L}_{\mathrm{ring}}$-formula $\phi(x,y)$ there exists an existential formula $\theta_\phi(y)$ such that for all tuple b, we have $F \models \theta_\phi(b)$ if and only if there exists a realisation a of $\phi(x,b)$ in an elementary extension of F such that a is $\mathbb{F}_p$-linearly independent over F; then for all finite subfields $F_0 \subset F_1$ of F, if $F_0 \models \theta_\phi(b)$

[3]A *locally finite* field is a field such that every finitely generated subfield is finite. Equivalently it is embeddable in $\overline{\mathbb{F}_p}$.

there exists a finite subfield F_2 of F and a tuple a from F_2 realizing $\phi(x, b)$ which is $\mathbb{F}_p$-linearly independent over F_1. By the same method as in the proof of Theorem 5.4.2, we may construct an increasing sequence of finite fields $(F_i)_{i<\omega}$ and finite subgroups $G_i \subseteq F_i$ such that for an enumeration of universal formula $\phi(x, y)$ and existential formula $\theta_\phi(y)$, if (F_i, G_i) satisfies the premise of the axiom, then the conclusion is satisfied in (F_{i+1}, G_{i+1}). Now consider the theory Psf_c (see Subsection 1.5.2), it is model-complete, hence every formula is equivalent to an existential formula and a universal formula, with some constants. It is then possible to choose constants $c(i)$ in F_i such that $X^n + c_{n-1,n}(i)X^{n-1} + \cdots + c_{0,n}(i)$ is irreducible over F_i, for all n. Then one can check that a non principal ultraproduct of $(F_i, c(i))_{i<\omega}$ is a model of Psf_c, hence the ultraproduct $\prod_{\mathscr{U}}(F_i, c_i, G_i)$ is a model of $\mathrm{Psf}_c G$ (see Example 3.2.3).

CHAPTER 6

Imaginaries in ACFG

Let (K, G) be a saturated model of ACFG. It is easy to see that for all $a \in K \setminus G$, there exists $b \in K \setminus G$ algebraically independent from a over $\mathbb{F}_p$ such that $a - b \in G$ (see Lemma 6.1.1). Let $\alpha = a/G = b/G \in (K, G)^{eq}$. If it exists, a canonical parameter for α in K would be definable over both a and b, hence it would be definable over an element of $\overline{\mathbb{F}_p}$. This would give an embedding of K/G into the countable set $\mathrm{dcl}^{eq}(\emptyset)$ which is absurd in a saturated model (K, G) for cardinality reasons.

Let (K, G) be a model of ACFG, there is a canonical projection

$$\pi : K \to K/G.$$

Consider the 2-sorted structure, $(K, K/G)$ with the $\mathscr{L}_{\mathrm{ring}}$-structure on K, the group structure on K/G (in the language of abelian groups) and the group epimorphism $\pi : K \to K/G$. We forget about the predicate G as it is 0-definable in $(K, K/G)$. The structure $(K, K/G)$ is bi-interpretable with (K, G). We fix (K, G) and $(K, K/G)$ for the rest of this chapter.

In this chapter we show that $(K, K/G)$ has weak elimination of imaginaries, hence imaginaries of (K, G) can be weakly eliminated up to the quotient K/G.

Some definable imaginaries in (K, G) can be easily eliminated in the structure $(K, K/G)$.

Example 6.0.1. Let $\zeta : K \to K$ be a $\mathscr{L}_{\mathrm{ring}}$-definable group endomorphism. Then in $(K, K/G)^{eq}$, every element in $K/\zeta^{-1}(G)$ is interdefinable with an element in K/G. Indeed, for any element $a \in K$ and any automorphism σ of $(K, K/G)$, $\sigma(a) - a \in \zeta^{-1}(G)$ if and only if σ fixes $\pi(\zeta(a))$, hence $\pi(\zeta(a))$ is a canonical parameter for the class of a modulo $\zeta^{-1}(G)$.

Let $\zeta_1, \cdots, \zeta_n$ be $\mathbb{F}_p$-linearly independent $\emptyset$-$\mathscr{L}_{\mathrm{ring}}$-definable group endomorphisms $K \to K$. Let $\pi_\zeta : K \to K/\zeta_1^{-1}(G) \cap \cdots \cap \zeta_n^{-1}(G)$ and consider an element α of the sort $K/\zeta_1^{-1}(G) \cap \cdots \cap \zeta_n^{-1}(G)$ in $(K, K/G)^{eq}$. From Proposition 5.3.5 the natural map

$$K/\zeta_1^{-1}(G) \cap \cdots \cap \zeta_n^{-1}(G) \to K/\zeta_1^{-1}(G) \times \cdots \times K/\zeta_n^{-1}(G)$$

is an isomorphism. Let a be such that $\pi_\zeta(a) = \alpha$. For each $1 \leq i \leq n$ let $\alpha_i = \pi(\zeta_i^{-1}(a)) \in K/G$. Then the tuple $\alpha_1, \cdots, \alpha_n$ is a canonical parameter for α.

If quotients of the form $K/\zeta^{-1}(G)$ can be fully eliminated, what about quotients of the form $K/\zeta(G)$? In that case the kernel of ζ is a finite vector space, hence a canonical parameter for $\alpha \in K/\zeta(G)$ is a finite set of the form $\pi(a+\ker(\zeta))$ which is not necessarily eliminable in $(K, K/G)$ as shows Example 6.3.5. We even show in Example 6.3.6 that adding canonical parameters for the sort K/G is not sufficient to eliminate all finite imaginaries of the structure $(K, K/G)$.

In this chapter, greek letters Γ, α denote subsets or tuples (which might be infinite) from K/G. Any tuple in the structure $(K, K/G)$ will be denoted by $a\gamma$, with a a tuple from K, γ a tuple from K/G. We also extend π for (finite or infinite) tuples by $\pi(a) := (\pi(a_i))_i$.

Contents

6.1 First steps with imaginaries

Let σ be a field automorphism of K. It is clear that the following are equivalent:

- σ is an $\mathscr{L}^G$-automorphism of K;
- there exists $\tilde{\sigma} : K/G \to K/G$ such that $\pi \circ \sigma = \tilde{\sigma} \circ \pi$.

$$\begin{array}{ccc} (K,G) & \xrightarrow{\sigma} & (K,G) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ K/G & \xrightarrow{\tilde{\sigma}} & K/G \end{array}$$

An automorphism of the structure $(K, K/G)$ is a pair $(\sigma, \tilde{\sigma})$ as above. It follows that for a, b, C from K, we have

$$a \equiv_C^{(K,G)} b \iff a \equiv_C^{(K,K/G)} b.$$

In this chapter, the relation $\equiv$ means having the same type in the structure $(K, K/G)$.

Lemma 6.1.1. *Let a, b be two tuples of the same length from K. Let $C, D \subseteq K$ and assume that*

- *$\pi(a)$ is an $\mathbb{F}_p$-independent tuple over $\pi(\overline{C})$*
- *$\pi(b)$ is an $\mathbb{F}_p$-independent tuple over $\pi(\overline{C})$*

Then there exists $a' \equiv_C a$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}_C^{\mathrm{ACF}} D$ and $\pi(a') = \pi(b)$.

Proof. Let $x \mathop{\smile\!\!\!\!\!\!\mid}_C^{\mathrm{ACF}} K$ such that $x \equiv_C^{\mathrm{ACF}} a$, and $f : \overline{Cx} \to \overline{Ca}$ a field isomorphism over C sending x to a. Let $G_{Cx} = f^{-1}(G(\overline{Ca}))$. Consider now the subgroup of $\overline{CDbx}$ defined by

$$H = G_{Cx} + G(\overline{CbD}) + \langle x_i - b_i \mid i \leq |x| \rangle.$$

We show that the type in the sense of ACFG defined by the pair $(\overline{CDbx}, H)$ is consistent. As $x \mathop{\smile\!\!\!\!\!\!\mid}_C^{\mathrm{ACF}} K$ we have $\overline{CDbx} \cap K = \overline{CDb}$. In order to prove that $H \cap \overline{CDb} = G(\overline{CDb})$, it suffices to show that

$$\overline{CDb} \cap (G_{Cx} + \langle x_i - b_i \mid i \leq |x| \rangle) \subseteq G(\overline{C}).$$

Assume that $g_{Cx} + \sum_i \lambda_i (x_i - b_i) \in \overline{CDb}$, where $g_{Cx} \in G_{Cx}$. It follows that $g_{Cx} + \sum_i \lambda_i x_i \in \overline{CDb}$. On the other hand $g_{Cx} + \sum_i \lambda_i x_i \in \overline{Cx}$. As $x \mathop{\smile\!\!\!\!\!\!\mid}_C^{\mathrm{ACF}} bD$ we have $\overline{Cx} \cap \overline{CDb} = \overline{C}$ hence $g_{Cx} + \sum_i \lambda_i x_i \in \overline{C}$. Apply $\pi \circ f$ to get that $\sum_i \lambda_i \pi(a_i) \in \pi(\overline{C})$ hence by hypothesis $\lambda_i = 0$ for all $i \leq |x|$. It follows that $g_{Cx} \in \overline{C}$ and so $g_{Cx} \in G(\overline{C})$. We have showed that $\overline{CDb} \cap (G_{Cx} + \langle x_i - b_i \mid i \leq |x| \rangle) \subseteq G(\overline{C})$. The type is consistent by Proposition 2.3.6, so realised by say a'. As $x \mathop{\smile\!\!\!\!\!\!\mid}_C^{\mathrm{ACF}} D$ we have $a' \mathop{\smile\!\!\!\!\!\!\mid}_C^{\mathrm{ACF}} D$. In order to show that $a' \equiv_C a$ we have to check that $H \cap \overline{Cx} = G_{Cx}$, this is similar to the argument above, using this time that $\pi(b)$ is $\mathbb{F}_p$-independent over $\pi(\overline{C})$. We have $a'_i - b_i \in G$ hence $\pi(a'_i) = \pi(b_i)$, for all $i \leq |x|$. $\square$

Lemma 6.1.2 (Minimal representative)**.** *Let a, C be in K such that $\pi(a)$ is an $\mathbb{F}_p$-independent tuple over $\pi(\overline{C})$. Then there exists a' of same length as a, algebraically independent over Cb such that*

- $\pi(a) = \pi(a')$
- $\pi(\overline{Ca'}) = \langle \pi(\overline{C})\pi(a)\rangle$
- $a' \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ACF}}_{C} b$.

Proof. It is again a type to realize. Consider x of same length as a and algebraically independent over Cba. Let V be a $\mathbb{F}_p$-vector space complement to $\overline{C}\oplus\langle x\rangle$ in $\overline{Cx}$ and set

$$H = G(\overline{Cab}) + \langle x-a\rangle + V.$$

We check that the pair $(\overline{Cabx}, H)$ defines a consistent type over Cab. First $H\cap\overline{Cab} = G(\overline{Cab}) + (\langle x-a\rangle + V)\cap\overline{Cab}$. For $v\in V$, if $\sum_i \lambda_i(x_i-a_i)+v\in\overline{Cab}$ then $\sum_i\lambda_i x_i + v\in\overline{Cab}$. As $\overline{Cab}\cap\overline{Cx} = \overline{C}$, $\sum_i \lambda_i x_i + v\in\overline{C}$ hence $v=0$ and, as x is $\mathbb{F}_p$-independent over $\overline{C}$, $\lambda_i = 0$ for all $i\leq |x|$. The type is consistent by Proposition 2.3.6. We show that $H\cap\overline{Cx} = G(\overline{C})+V$. First $H\cap\overline{Cx} = V + \overline{Cx}\cap(G(\overline{Cab})+\langle x-a\rangle)$. Let $g+\sum_i\lambda_i(x_i-a_i)\in(G(\overline{Cab})+\langle x-a\rangle)\cap\overline{Cx}$, then $g+\sum_i\lambda_i a_i\in\overline{Cab}\cap\overline{Cx} = \overline{C}$ and so applying π gives $\sum_i\lambda_i\pi(a_i)\in\pi(\overline{C})$ hence $\lambda_i = 0$ for all $i\leq|x|$. It follows that $\overline{Cx}\cap(G(\overline{Cab})+\langle x-a\rangle) = G(\overline{C})$ hence $H\cap\overline{Cx} = G(\overline{C})+V$. Assume that a' realises this type, it is clear that $\pi(a) = \pi(a')$ and $a' \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ACF}}_{C} b$. By construction there exists $V'\subseteq\overline{Ca'}$ such that $\overline{Ca'} = \overline{C}\oplus\langle a\rangle\oplus V'$ and $G(\overline{Ca'}) = G(\overline{C})\oplus V'$, so it follows that $\pi(\overline{Ca'}) = \pi(C)\oplus\langle\pi(a')\rangle$. □

In particular if α is an $\mathbb{F}_p$-independent tuple over $\pi(\overline{C})$ then there exists some algebraically independent tuple a over C such that $\pi(a) = \alpha$ and $\pi(\overline{Ca}) = \langle\pi(\overline{C})\alpha\rangle$. We call such a tuple a *minimal representative* of α over C. Lemma 6.1.2 states that minimal representatives always exists and that they can be taken independent in the sense of fields from any parameters.

Corollary 6.1.3. *Let α and β be tuples in K/G of the same length, γ tuple from K/G and $C\subseteq K$. If α and β are $\mathbb{F}_p$-independent tuples over $\langle\pi(\overline{C})\gamma\rangle$ then $\alpha\equiv_{C\gamma}\beta$.*

Proof. We may assume that γ is linearly independent over $\pi(\overline{C})$ and let r_γ be a minimal representative of γ over C. Let a and b be representatives of α and β over $\overline{Cr_\gamma}$. Using Lemma 6.1.1, there exists $a'\equiv_{Cr_\gamma} a$ such that $\pi(a') = \pi(b) = \beta$. Let σ be an automorphism of $(K, K/G)$ over Cr_γ sending a on a'. It is clear that σ fixes γ and sends α to β hence $\alpha\equiv_{C\gamma}\beta$. □

Remark 6.1.4. A consequence of Corollary 6.1.3 is that the induced structure on K/G is the one of a pure $\mathbb{F}_p$-vector space.

We will describe the algebraic closure acl in the structure $(K, K/G)$. It is classical that every formula in the language of $(K, K/G)$ (or of $(K,G)^{eq}$) without parameters and with free variables in the home sort K is equivalent to an $\mathscr{L}^G$-formula. In particular $\mathrm{acl}(C)\cap K = \overline{C}$ for all $C\subseteq K$.

Corollary 6.1.5. *Let $C \subseteq K$ and $\gamma \subseteq K/G$, then*

- $\mathrm{acl}(C\gamma) \cap K = \overline{C}$
- $\mathrm{acl}(C\gamma) \cap K/G = \langle \pi(\overline{C})\gamma \rangle$.

Proof. For the first assertion, we may assume that γ is an $\mathbb{F}_p$-independent tuple over $\pi(\overline{C})$. Let u be in $\mathrm{acl}(C\gamma) \cap K$ witnessed by an algebraic formula $\phi(x, c, \gamma)$ with $c \in C$. Using twice Lemma 6.1.2, let r_γ be a minimal representative of γ over C, and r'_γ a minimal representative of γ over C such that $r'_\gamma \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_C r_\gamma$. As u satisfies $\phi(x, c, \pi(r_\gamma))$ and $\phi(x, c, \pi(r'_\gamma))$, u belongs to $\overline{Cr_\gamma} \cap \overline{Cr'_\gamma} = \overline{C}$ (note that we don't use the minimality here). The reverse inclusion being trivial, it follows that $\mathrm{acl}(C\gamma) \cap K = \overline{C}$.

For the second assertion, assume that $\alpha \notin \langle \pi(\overline{C})\gamma \rangle$. By Corollary 6.1.3, any element in $K/G \setminus \langle \pi(\overline{C})\gamma \rangle$ has the same type as α over $C\gamma$ hence $\alpha \notin \mathrm{acl}(C\gamma)$. The reverse inclusion being trivial, it follows that $\mathrm{acl}(C\gamma) \cap K/G = \langle \pi(\overline{C})\gamma \rangle$. $\square$

6.2 Independence in $(K, K/G)$

Recall the weak independence in (K, G):

$$a \mathop{\smile\!\!\!\!\!\!\mid}^{w}_{C} b \iff a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_{C} b \text{ and } G(\overline{Ca} + \overline{Cb}) = G(\overline{Ca}) + G(\overline{Cb})$$

It is an easy checking that under the assumption that $\overline{Ca} \cap \overline{Cb} = \overline{C}$ the following two assertions are equivalent:

- $G(\overline{Ca} + \overline{Cb}) = G(\overline{Ca}) + G(\overline{Cb})$
- $\pi(\overline{Ca}) \cap \pi(\overline{Cb}) = \pi(\overline{C})$

We define the following relation in $(K, K/G)$:

$$a\alpha \mathop{\smile\!\!\!\!\!\!\mid}^{w}_{C\gamma} b\beta \iff a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_{C} b \text{ and } \langle \pi(\overline{Ca})\alpha\gamma \rangle \cap \langle \pi(\overline{Cb})\beta\gamma \rangle = \langle \pi(\overline{C})\gamma \rangle$$

It is the right candidate for Kim-independence in $(K, K/G)$. We study only the restriction of this relation to sets $a\alpha, b\beta, C\gamma$ with $\alpha\beta\gamma \subseteq \pi(\overline{Ca}) \cap \pi(\overline{Cb})$. This restriction can be described only in terms of the structure (K, G) as we will see now.

An infinite tuple λ of elements of $\mathbb{F}_p$ is *almost trivial* if $\lambda_i = 0$ for cofinitely many i's. If γ is an infinite tuple, an element $u \in \langle \gamma \rangle$ is an almost trivial linear combination of γ_i's, i.e. there exists λ almost trivial such that $u = \sum_i \lambda_i \gamma_i$. Given two tuples a and b, the tuple consisting of the coordinates $a_i - b_i$ is denoted by $a - b$.

Lemma 6.2.1. *Let a, b be tuples such that γ is a (finite or infinite) tuple from $\pi(\overline{a}) \cap \pi(\overline{b})$. Assume that $\overline{a} \cap \overline{b} = \overline{C}$, then the following are equivalent:*

(1) $\pi(\overline{a}) \cap \pi(\overline{b}) = \langle \pi(\overline{C})\gamma \rangle$

(2) $G(\overline{a}+\overline{b}) = G(\overline{a}) + G(\overline{b}) + \langle r^a - r^b \rangle$ *for some (all) representatives* r^a, r^b *of* γ *in* $\overline{a}$ *and* $\overline{b}$ *respectively.*

Proof. (1) implies *(2)*. Let $u_a \in \overline{a}$ and $u_b \in \overline{b}$ such that $u_a - u_b \in G$. Then $\pi(u_a) = \pi(u_b) \in \pi(\overline{C}) + \langle \gamma \rangle$ so there exists $u_c \in \overline{C}$ and $\lambda \in \mathbb{F}_p^{|\gamma|}$ such that for some (any) representatives r^a and r^b of γ in $\overline{a}$ and $\overline{b}$ respectively, there exists $g_a \in G(\overline{a})$, $g_b \in G(\overline{b})$ and an almost trivial sequence $\lambda \in \mathbb{F}_p^{|\gamma|}$ with

$$u_a = g_a + u_c + \sum_i \lambda_i r_i^a$$
$$u_b = g_b + u_c + \sum_i \lambda_i r_i^b.$$

It follows that $u_a - u_b \in G(\overline{a}) + G(\overline{b}) + \langle r^a - r^b \rangle$.
(2) implies *(1)*. If $u_a \in \overline{a}$ and $u_b \in \overline{b}$ are such that $\pi(u_a) = \pi(u_b)$, then $u_a - u_b \in G(\overline{a}+\overline{b})$ hence $u_a - u_b = g_a + g_b + \sum_i \lambda_i (r_i^a - r_i^b)$ (for an almost trivial sequence $\lambda \in \mathbb{F}_p^{|\gamma|}$). It follows that $u_a - g_a - \sum_i \lambda_i r_i^a \in \overline{a} \cap \overline{b} = \overline{c}$, so $\pi(u_a) \in \pi(\overline{C}) + \langle \gamma \rangle$. □

Lemma 6.2.2 (Maximal representative)**.** *Let* γ *be a tuple* $\mathbb{F}_p$*-independent over* $\pi(\overline{C})$ *and* d *a tuple from* K *such that* $\pi(d) = \gamma$. *Then there exists* $(K', G') \succ (K, G)$ *and a tuple* r_γ *of length* $|\gamma|$ *in* K'*, algebraically independent over* K *such that*

$$G(\overline{Kr_\gamma}) = G(K) + \langle r_\gamma - d \rangle.$$

Furthermore the following hold for all tuples a, b *from* K *containing* C *such that* $\gamma \in \pi(\overline{a}) \cap \pi(\overline{b})$*:*

(1) if $C = \overline{C}$ *then* $a \equiv_{C\gamma} b$ *if and only if* $a \equiv_{\overline{Cr_\gamma}} b$*;*

(2) $a \mathop{\smile\!\!\!\!\!\mid}^w_{C\gamma} b$ *if and only if* $a \mathop{\smile\!\!\!\!\!\mid}^w_{Cr_\gamma} b$.

Proof. Let x be an algebraically independent tuple over K of size $|d|$, and define H on $K(x)$ to be $G(K) + \langle x - d \rangle$. It is easy to see that $(\overline{K(x)}, H)$ defines a consistent type over K so let r_γ be a realization of this type in an elementary extension (K', G') of (K, G). We may assume that (K', G') is κ-saturated and κ-homogeneous for some big enough κ.
Claim. if $C = \overline{C}$ and $r'_\gamma \equiv_{C\gamma} r_\gamma$ with $r'_\gamma \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ACF}}_C b$ and $G(\overline{Cbr'_\gamma}) = G(\overline{b}) + \langle r'_\gamma - r^b \rangle$ for some $r^b \in \pi^{-1}(\gamma) \cap \overline{b}^{|\gamma|}$, then any $\mathscr{L}^G$-isomorphism over $C\gamma$ that sends an enumeration R'_γ of $\overline{Cr'_\gamma}$ to an enumeration R_γ of $\overline{Cr_\gamma}$ (and sends r'_γ to r_γ) extends to an $\mathscr{L}^G$-isomorphism between $\overline{R'_\gamma b}$ and $\overline{R_\gamma b}$ which fixes b.

Proof of the Claim. . Let σ be an automorphism of $(K', K'/G')$ over $C\gamma$ sending r'_γ to r_γ. Then it sends any enumeration R'_γ of $\overline{Cr'_\gamma}$ to an enumeration R_γ of $\overline{Cr_\gamma}$. We may assume that $b = \overline{b}$. By stationarity of the type $tp^{\mathrm{ACF}}(b/C)$, the field isomorphism $\sigma \restriction \overline{CR'_\gamma}$

extends to $\tilde{\sigma} : \overline{bR'_\gamma} \to \overline{bR_\gamma}$ with $\tilde{\sigma}$ fixing b. We show that $\tilde{\sigma}$ is an $\mathscr{L}^G$-isomorphism. First observe that since $G(\overline{Kr_\gamma}) = G(K) + \langle r_\gamma - r^b\rangle$ then $G(\overline{br_\gamma}) = G(b) + \langle r_\gamma - r^b\rangle$. As $\tilde{\sigma}$ fixes b and sends r'_γ to r_γ it is clear that $\tilde{\sigma}$ send $G(\overline{br'_\gamma})$ to $G(\overline{br_\gamma})$ so $\tilde{\sigma}$ is an $\mathscr{L}^G$-isomorphism. Now this isomorphism extends to an automorphism of (K', G') and an automorphism of $(K', K'/G')$ that fixes γ as it send r'_γ to r_γ. □

(1). Assume that $a \equiv_{C\gamma} b$ and let σ be an automorphism of $(K', K'/G')$ over $C\gamma$ sending a on b. As before, we have that $G(\overline{ar_\gamma}) = G(\overline{a}) + \langle r_\gamma - r^a\rangle$ and $G(\overline{br_\gamma}) = G(\overline{b}) + \langle r_\gamma - r^b\rangle$, for some (any) representatives r^a, r^b of γ in $\overline{a}$, $\overline{b}$ respectively. Let R_γ be an enumeration of $\overline{Cr_\gamma}$ and $R'_\gamma = \sigma(R_\gamma)$, $r'_\gamma = \sigma(r_\gamma)$. As $r_\gamma \mathop{\smile\!\!\!\!\!\!\mid\,}^{\mathrm{ACF}}_{C} a$, we have $r'_\gamma \mathop{\smile\!\!\!\!\!\!\mid\,}^{\mathrm{ACF}}_{C} b$. Furthermore $G(\overline{ar_\gamma}) = G(\overline{a}) + \langle r_\gamma - r^a\rangle$ and $aR_\gamma \equiv_{C\gamma} bR'_\gamma$, then $G(\overline{Cbr'_\gamma}) = G(\overline{b}) + \langle r'_\gamma - r^b\rangle$. By the claim, $\sigma^{-1} \restriction \overline{Cr'_\gamma}$ extends $\overline{Cr'_\gamma b}$ with the identity on b hence $R_\gamma \equiv_{Cb\gamma} R'_\gamma$. It follows that $aR_\gamma \equiv_{C\gamma} bR_\gamma$. The other direction is trivial.

(2). From left to right. It is clear that $a \mathop{\smile\!\!\!\!\!\!\mid\,}^{\mathrm{ACF}}_{Cr_\gamma} b$. We want to show that $G(\overline{ar_\gamma} + \overline{br_\gamma}) = G(\overline{ar_\gamma}) + G(\overline{br_\gamma})$. Observe that $G(\overline{abr_\gamma}) = G(\overline{ab}) + \langle r^a - r_\gamma\rangle$ for any tuple r^a from $\overline{a}$ with $\pi(r^a)$. Let $u \in \overline{ar_\gamma}$ and $v \in \overline{br_\gamma}$. If $u + v \in G$ there exists $g_{ab} \in G(\overline{ab})$ and $\lambda \in \mathbb{F}_p^{|x|}$ such that $u + v = g_{ab} + \sum_i \lambda_i(r^a_i - r_{\gamma i})$ for an almost trivial tuple λ. It follows that $g_{ab} \in (\overline{ar_\gamma} + \overline{br_\gamma}) \cap \overline{ab} = \overline{a} + \overline{b}$ by Lemma 1.5.11. As $a \mathop{\smile\!\!\!\!\!\!\mid\,}^{w}_{C\gamma} b$ and using Lemma 6.2.1, we have that $G(\overline{a} + \overline{b}) = G(\overline{a}) + G(\overline{b}) + \langle r^a - r^b\rangle$. We deduce that $g_{ab} = g_a + g_b + \sum_i \mu_i(r^a_i - r^b_i)$, for an almost trivial tuple μ. For all i, $r^a_i - r_{\gamma i} \in G(\overline{ar_\gamma})$ and $r_{\gamma i} - r^b_i \in G(\overline{br_\gamma})$ hence $g_{ab} = g_a + g_b + \sum_i \mu_i(r^a_i - r_{\gamma i}) + \sum_i \mu_i(r_{\gamma i} - r^b_i) \in G(\overline{ar_\gamma}) + G(\overline{br_\gamma})$. It follows that $u + v \in G(\overline{ar_\gamma}) + G(\overline{br_\gamma})$. The other inclusion being trivial we have $G(\overline{ar_\gamma} + \overline{br_\gamma}) = G(\overline{ar_\gamma}) + G(\overline{br_\gamma})$.

From right to left. First, $r_\gamma \mathop{\smile\!\!\!\!\!\!\mid\,}^{\mathrm{ACF}}_{C} b$ hence by Transitivity and Monotonicity $a \mathop{\smile\!\!\!\!\!\!\mid\,}^{\mathrm{ACF}}_{C} b$. By hypothesis, $G(\overline{ar_\gamma} + \overline{br_\gamma}) = G(\overline{ar_\gamma}) + G(\overline{br_\gamma})$. Furthermore $G(\overline{ar_\gamma}) = G(\overline{a}) + \langle r_\gamma - r^a\rangle$ and $G(\overline{br_\gamma}) = G(\overline{b}) + \langle r_\gamma - r^b\rangle$. It is easy to see that

$$(G(\overline{a}) + G(\overline{b}) + \langle r_\gamma - r^a\rangle + \langle r_\gamma - r^b\rangle) \cap (\overline{a} + \overline{b}) = G(\overline{a}) + G(\overline{b}) + \langle r^b - r^a\rangle.$$

It follows that $a \mathop{\smile\!\!\!\!\!\!\mid\,}^{w}_{C\gamma} b$. □

Remark 6.2.3. Let $\mathop{\smile\!\!\!\!\!\!\mid\,}^{ST}$ be the following relation, defined for $\gamma \in \pi(\overline{Ca}) \cap \pi(\overline{Cb})$:

$$a \mathop{\smile\!\!\!\!\!\!\mid\,}^{ST}_{C\gamma} b \iff a \mathop{\smile\!\!\!\!\!\!\mid\,}^{\mathrm{ACF}}_{C} b \text{ and } G(\overline{Cab}) = G(\overline{Ca}) + G(\overline{Cb}) + \langle r^a_\gamma - r^b_\gamma\rangle$$
$$\text{for some (any) representatives } r^a_\gamma,\ r^b_\gamma \text{ of } \gamma \text{ in } \overline{Ca},\ \overline{Cb} \text{ respectively.}$$

A *maximal representative of* γ *over* C *with respect to* b is a representative r_γ such that $r_\gamma \mathop{\smile\!\!\!\!\!\!\mid\,}^{ST}_{C\gamma} b$. The previous result implies that this relation satisfies Full Existence and Stationnarity over algebraically closed sets. This relation clearly extends the strong independence in (K, G).

Theorem 6.2.4. *The relation* $\mathop{\smile\!\!\!\!\!\!\mid\,}^{w}$ *satisfies the following properties.*

(1) *(Full Existence) Let $a, b, C = \overline{C}$ in K and $\gamma \in K/G$ such that $\gamma \in \pi(\overline{Ca}) \cap \pi(\overline{Cb})$ and γ $\mathbb{F}_p$-independent over $\pi(\overline{C})$. Then there exists $a' \equiv_{C\gamma} a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} b$.*

(2) *(Transitivity) If $a\alpha \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} b\beta$ and $a\alpha \mathop{\smile\!\!\!\!\!\!|}^{w}_{Cb\gamma\beta} d\delta$ then $a\alpha \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} bd\beta\delta$*

(3) *(Independence theorem) Let $c_1, c_2, a, b, C = \overline{C}$ in K and $\gamma \in K/G$ such that $\gamma \in \pi(\overline{Ca}) \cap \pi(\overline{Cb}) \cap \pi(\overline{Cc_1}) \cap \pi(\overline{Cc_2})$ and γ $\mathbb{F}_p$-independent over $\pi(\overline{C})$.*

If $c_1 \equiv_{C\gamma} c_2$ and $c_1 \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} a, c_2 \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} b, a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} b$, then there exists c such that $c \equiv_{Ca\gamma} c_1$, $c \equiv_{Cb\gamma} c_2$ and $c \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} a, b$.

Proof. Transitivity is just checking from the definition of $\mathop{\smile\!\!\!\!\!\!|}^{w}$. For *Full Existence*, assume the hypothesis and let r_γ be a maximal representative as in Lemma 6.2.2. By Full Existence of $\mathop{\smile\!\!\!\!\!\!|}^{w}$ in (K, G) there exists $a' \equiv_{Cr_\gamma} a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^{w}_{Cr_\gamma} b$. Using again Lemma 6.2.2, $a' \equiv_{C\gamma} a$ and $a' \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} b$. For *Independence theorem*, we use the same strategy. Assume the hypothesis and let r_γ be a maximal representative of γ as in Lemma 6.2.2. From Lemma 6.2.2, we have that $c_1 \equiv_{\overline{Cr_\gamma}} c_2$ and $c_1 \mathop{\smile\!\!\!\!\!\!|}^{w}_{\overline{Cr_\gamma}} a$, $c_2 \mathop{\smile\!\!\!\!\!\!|}^{w}_{\overline{Cr_\gamma}} b$, $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{\overline{Cr_\gamma}} b$. As $\mathop{\smile\!\!\!\!\!\!|}^{w}$ in (K, G) satisfies $\mathop{\smile\!\!\!\!\!\!|}^{a}$-amalgamation over algebraically closed sets there exists c such that $c \equiv_{\overline{Cr_\gamma a}} c_1$, $c \equiv_{\overline{Cr_\gamma b}} c_2$ and $c \mathop{\smile\!\!\!\!\!\!|}^{w}_{\overline{Cr_\gamma}} a, b$. It follows that $c \equiv_{Ca\gamma} c_1$, $c \equiv_{Cb\gamma} c_2$, and by Lemma 6.2.2, $c \mathop{\smile\!\!\!\!\!\!|}^{w}_{C\gamma} a, b$. ☐

Remark 6.2.5. Notice that $\mathop{\smile\!\!\!\!\!\!|}^{w}$ satisfies $\mathop{\smile\!\!\!\!\!\!|}^{a}$-amalgamation over algebraically closed fields in (K, G). In Theorem 6.2.4, we can weaken the hypothesis $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} b$ to $a \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} b$ because if $a \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} b$ and $r \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} ab$, then $a \mathop{\smile\!\!\!\!\!\!|}^{a}_{Cr} b$ (this result is contained in the proof of Lemma 7.2.2).

6.3 Weak elimination of imaginaries in $(K, K/G)$

The following Lemma is a rewriting of the classical argument for the proof of elimination of imaginaries that appears for instance in [CP98] and [KR18]. It is similar to [CK17, Proposition 4.25], the only difference being that in our case, $\mathop{\smile\!\!\!\!\!\!|}$ is defined only on some subsets, and the base set might contain imaginaries, but the proof is the same.

Lemma 6.3.1. *Let $\mathscr{M}$ be a κ-homogeneous and κ-saturated structure. Let $E \subset \mathscr{M}^{eq}$. Assume that there exists a binary relation $\mathop{\smile\!\!\!\!\!\!|}_E$ on some tuples from $\mathscr{M}$ such that*

- *(Invariance) If $a \mathop{\smile\!\!\!\!\!\!|}_E b$ and $ab \equiv_E a'b'$ then $a' \mathop{\smile\!\!\!\!\!\!|}_E b'$*
- *(Extension) If $a \mathop{\smile\!\!\!\!\!\!|}_E b$ and d tuple from $\mathscr{M}$ then there exists $a' \equiv_{Eb} a$ and $a' \mathop{\smile\!\!\!\!\!\!|}_E bd$*
- *(Independent consistency) If $a_1 \mathop{\smile\!\!\!\!\!\!|}_E a_2$, $b \mathop{\smile\!\!\!\!\!\!|}_E a_2$ and $a_2 \equiv_E b$, then there exists a such that $a \equiv_{Ea_1} a_2$, $a \equiv_{Ea_2} b$.*

Let $e \in \mathscr{M}^{eq}$. If there exists a 0-definable function f in $\mathscr{M}^{eq}$ and a_1, a_2 in $\mathscr{M}$ such that $f(a_1) = f(a_2) = e$ and $a_1 \mathop{\smile\!\!\!\!\!\!|}_E a_2$ then $e \in \mathrm{dcl}^{eq}(E)$.

Proof. If e is not in $\mathrm{dcl}^{eq}(E)$, then there exists $e' \neq e$ such that $e' \equiv_E e$. Let σ be an automorphism of $\mathscr{M}^{eq}$ over E sending e on e'. Let $b_1 b_2 = \sigma(a_1 a_2)$. By Invariance, $b_1 \mathop{\smile\!\!\!\!\!\!|}_E b_2$ and $f(b_1) = f(b_2) = e'$. By Extension there exists $b \equiv_{Eb_1} b_2$ such that $b \mathop{\smile\!\!\!\!\!\!|}_E a_2$. By Independent Consistency, there exists a such that $a \equiv_{Ea_1} a_2$, $a \equiv_{Ea_2} b$. From $a \equiv_{Ea_1} a_2$ follows that $f(a) = f(a_1) = e$ and from $a \equiv_{Ea_2} b$ follows that $f(a) \neq e$, a contradiction. $\square$

Remark 6.3.2. Recall that Extension follows from Full Existence, Symmetry and Transitivity. Independent consistency is a consequence of the independence theorem. It follows from Theorem 6.2.4 that for all $C = \overline{C}$ and γ $\mathbb{F}_p$-independent over $\pi(\overline{C})$, the restriction of $\mathop{\smile\!\!\!\!\!\!|}^w_{C\gamma}$ to tuples a such that $\gamma \in \pi(\overline{Ca})$ satisfies the hypothesis of the previous Lemma.

The following classical fact follows from a group theoretic Lemma due to P.M. Neumann ([Neu76]). It appears first in [EH93, Lemma 1.4].

Fact 6.3.3. *Let $\mathscr{M}$ be a saturated model, X a 0-definable set, $e \in \mathscr{M}$, $E = \mathrm{acl}(e) \cap X$ and a tuple a from X. Then there is a tuple b from X such that*

$$a \equiv_{Ee} b \text{ and } \mathrm{acl}(Ea) \cap \mathrm{acl}(Eb) \cap X = E.$$

Theorem 6.3.4. *Let $e \in (K, G)^{eq}$ then there exists a tuple $c\gamma$ from $(K, K/G)$ such that $c\gamma \in \mathrm{acl}^{eq}(e)$ and $e \in \mathrm{dcl}^{eq}(c\gamma)$. It follows that $(K, K/G)$ has weak elimination of imaginaries.*

Proof. We work in $(K, G)^{eq}$, seeing $(K, K/G)$ as a 0-definable subset. Suppose that e is an imaginary element, there is a tuple a from K and a 0-definable function f such that $e = f(a)$. We set $C\langle\pi(C)\gamma\rangle = \mathrm{acl}^{eq}(e) \cap (K, K/G)$. We may assume that γ is $\mathbb{F}_p$-linearly independent over $\pi(C)$. As $\gamma \subseteq \mathrm{acl}^{eq}(e) \cap K/G \subseteq \mathrm{acl}^{eq}(a) \cap K/G$ we have that $\mathrm{acl}^{eq}(Ca\gamma) \cap (K, K/G) = \overline{Ca}\pi(\overline{Ca})$ and $\gamma \subseteq \pi(\overline{Ca})$. By Fact 6.3.3 there exists $b \equiv_{C\gamma e} a$ such that

$$\mathrm{acl}^{eq}(Ca\gamma) \cap \mathrm{acl}^{eq}(Cb\gamma) \cap (K, K/G) = C\langle\pi(C)\gamma\rangle.$$

Again, $\mathrm{acl}^{eq}(Cb\gamma) \cap (K, K/G) = \overline{Cb}\pi(\overline{Cb})$ and $\gamma \subseteq \pi(\overline{Cb})$. Furthermore $f(b) = e$ and

$$(\overline{Ca}\pi(\overline{Ca})) \cap (\overline{Cb}\pi(\overline{Cb})) = C\langle\pi(C)\gamma\rangle.$$

We construct a sequence $(a_i)_{i<\omega}$ such that

$$a_{n+1} \mathop{\smile\!\!\!\!\!\!|}^w_{C\gamma a_n} a_1, \ldots, a_{n-1} \text{ and } a_n a_{n+1} \equiv_{C\gamma} ab.$$

Start by $a_1 = a$ and $a_2 = b$. Assume that $a_1, \ldots, a_n$ has already been constructed. We have that $a_{n-1} \equiv_{C\gamma} a_n$ so let σ be a $c\gamma$-automorphism of the monster such that $\sigma(a_{n-1}) = a_n$. By Full Existence (Theorem 6.2.4) there exists $a_{n+1} \equiv_{Ca_n\gamma} \sigma(a_n)$ such that $a_{n+1} \mathop{\smile\!\!\!\!\!\!|}^w_{Ca_n\gamma} a_1, \ldots, a_{n-1}$. It follows that

$$a_n a_{n+1} \equiv_{C\gamma} a_n \sigma(a_n) \equiv_{C\gamma} a_{n-1} a_n.$$

Let $(a_i)_{i<\omega}$ be such a sequence. In particular the following holds for all $i<j<k$

$$a_k \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_{Ca_j} a_i,\ \overline{Ca_i}\cap\overline{Ca_j}=\overline{C} \text{ and } \pi(\overline{Ca_i})\cap\pi(\overline{Ca_j})=\langle\pi(C)\gamma\rangle.$$

By Ramsey and compactness we may assume that $(a_i)_{i<\omega}$ is indiscernible over $C\gamma$. As the three properties above holds for the whole sequence, it is in the Erenfeucht-Mostowski type of the sequence, and hence is still true for the indiscernible sequence. Note that $f(a_i)=e$. We have that $(a_i)_{i<\omega}$ is totally indiscernible over C in the sense of ACF hence $a_1a_2a_3\equiv^{\mathrm{ACF}}_C a_1a_3a_2$. Furthermore we have $a_1 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_{Ca_2} a_3$, hence by INVARIANCE $a_1 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_{Ca_3} a_2$. By elimination of imaginaries in ACF it follows that $a_1 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_{C} a_2$, since $\overline{Ca}\cap\overline{Cb}=\overline{C}$. As $\pi(\overline{Ca_1})\cap\pi(\overline{Ca_2})=\langle\pi(C)\gamma\rangle$, we have that

$$a_1 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{w}_{C\gamma} a_2.$$

As $f(a_1)=f(a_2)=e$, we deduce from Lemma 6.3.1 that $e\in\mathrm{dcl}^{eq}(C\gamma)$. $\square$

Example 6.3.5 ($(K,K/G)$ does not eliminate finite imaginaries)**.** The structure on K/G is the one of an $\mathbb{F}_p$-vector space (with twisted algebraic and definable closures, $\mathrm{acl}(\alpha)=\langle\pi(\overline{\mathbb{F}_p})\alpha\rangle$ and $\mathrm{dcl}(\alpha)=\langle\pi(\mathrm{dcl}(\mathbb{F}_p))\alpha\rangle$). This follows from Corollaries 6.1.3 and 6.1.5. Consider the unordered pair $\{\alpha,\beta\}$ for two singletons $\alpha,\beta\in K/G$, linearly independent over $\pi(\overline{\mathbb{F}_p})$. Assume that there exists a tuple $d\gamma$ such that for all automorphism σ of $(K,K/G)$

$$\sigma(d\gamma)=d\gamma \iff \sigma(\{\alpha,\beta\})=\{\alpha,\beta\}\,.$$

As $d\gamma$ and $\alpha\beta$ are interalgebraic, we have first that $d\subset\overline{\mathbb{F}_p}$ and hence $\alpha,\beta\in\mathrm{acl}(\gamma)\cap K/G=\mathrm{dcl}(\gamma)\cap K/G=\langle\gamma\rangle$. As α,β are linearly independent over $\mathrm{acl}(\emptyset)$, we have $\alpha\beta\equiv_\emptyset\beta\alpha$ so let σ be an automorphism of $(K,K/G)$ sending $\alpha\beta$ on $\beta\alpha$. As σ fixes γ, it fixes $\langle\gamma\rangle$ hence $\alpha=\beta$, a contradiction.

Example 6.3.6 ($K\times(K/G)^{eq}$ does not eliminate finite imaginaries)**.** Let t be a transcendental element over $\mathbb{F}_p$. We assume that $G(\overline{\mathbb{F}_p(t)})=\overline{\mathbb{F}_p(t)}$ (in a model (K,G) of ACFG such that $G(\overline{\mathbb{F}_p})=\overline{\mathbb{F}_p}$). Let $\alpha,\beta\in K/G$ be $\mathbb{F}_p$-independent, and let e be the unordered pair $\left\{\sqrt{t}\alpha,-\sqrt{t}\beta\right\}$. We have the following:

(1) $\mathrm{dcl}^{eq}(e)\cap K=\mathrm{dcl}(t)$

(2) $\mathrm{dcl}^{eq}(e)\cap(K/G)^{eq}=\mathrm{dcl}^{eq}(\{\alpha,\beta\})\cap(K/G)^{eq}$

(1) The right to left inclusion is clear. Let $u\in\mathrm{dcl}^{eq}(e)\cap K$, in particular $u\in\mathrm{dcl}^{eq}(t,\alpha\beta)\cap K\subseteq\mathrm{acl}^{eq}(t,\alpha\beta)\cap K=\overline{\mathbb{F}(t)}$. Assume that $u\notin\mathrm{dcl}(t)$. There exists $u'\neq u$ with $u'\equiv_t u$. Let α',β' such that $u'\alpha'\beta'\equiv_t u\alpha\beta$. As α,β and α',β' are $\mathbb{F}_p$-lineary independent over $\pi(\overline{\mathbb{F}(t,u)})=\pi(\overline{\mathbb{F}(t)})=\{0\}$, we have that $\alpha\beta\equiv_{\overline{\mathbb{F}_p(t)}}\alpha'\beta'$ (Corollary 6.1.3). It follows that $u'\equiv_{t,\alpha,\beta}u$ hence $u'\equiv_e u$ so $u\notin\mathrm{dcl}^{eq}(e)$, a contradiction.
(2) The right to left inclusion is clear. Let $\{\gamma_1,\dots,\gamma_n\}$ be an element of $\mathrm{dcl}^{eq}(e)\cap$

$(K/G)^{eq}$. For all i, γ_i is algebraic over $t\alpha\beta$, by Corollary 6.1.5 $\gamma_i \in \langle \pi(\overline{\mathbb{F}_p(t)}), \alpha, \beta \rangle = \langle \alpha, \beta \rangle$. It follows that permutations of the set $\left\{\sqrt{t}\alpha, -\sqrt{t}\beta\right\}$ that permutes $\{\gamma_1, \ldots, \gamma_n\}$ are exactly permutations of the set $\{\alpha, \beta\}$ that permutes $\{\gamma_1, \ldots, \gamma_n\}$ hence $\{\gamma_1, \ldots, \gamma_n\} \in \mathrm{dcl}^{eq}(\{\alpha, \beta\})$. In fact, such a set $\{\gamma_1, \ldots, \gamma_n\}$ is the union of two sets of the same cardinal (possibly intersecting), every element in one set is of the form $\lambda\alpha + \mu\beta$ and has a "dual" element $\mu\alpha + \lambda\beta$ in the other set.

If e is interdefinable with an element from $K \times (K/G)^{eq}$, by *(1)* and *(2)*, we may assume that $e \in \mathrm{dcl}^{eq}(t\,\{\alpha, \beta\})$. By hypothesis $\alpha\beta \equiv_{\overline{\mathbb{F}_p(t)}} \beta\alpha$, hence an automorphism sending $\sqrt{t}, -\sqrt{t}\alpha\beta$ to $\sqrt{t}, -\sqrt{t}\beta\alpha$ fixes $t\,\{\alpha, \beta\}$ and moves e to $\left\{\sqrt{t}\beta, -\sqrt{t}\alpha\right\}$, hence $e \notin \mathrm{dcl}^{eq}(t\,\{\alpha, \beta\})$, a contradiction.

CHAPTER 7

Forking and thorn-forking in ACFG

In this chapter, we give a description of forking and thorn-forking in the theory ACFG. We also link these notions with other classical relations or other independence relations encountered in the previous chapters. The results of this chapter are summarized by the diagram Figure 7.1, in which all arrows are strict.

Contents

$$
\begin{tikzcd}
 & & & & \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}} \arrow[d, equal] & \\
 & & \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}} \restriction K \arrow[d, equal] & & \mathop{\smile\!\!\!\!\!\!\mid}^{a\,m} \arrow[dr] & \\
\mathop{\smile\!\!\!\!\!\!\mid}^{st} \arrow[r] & \mathop{\smile\!\!\!\!\!\!\mid}^{wsm} \arrow[r] & \mathop{\smile\!\!\!\!\!\!\mid}^{wm} \arrow[r] \arrow[d, equal] & \mathop{\smile\!\!\!\!\!\!\mid}^{w} \arrow[ur] \arrow[dr] \arrow[d, equal] & & \mathop{\smile\!\!\!\!\!\!\mid}^{a} \\
 & & \mathop{\smile\!\!\!\!\!\!\mid}^{d} \arrow[d, equal] & \mathop{\smile\!\!\!\!\!\!\mid}^{K} & \mathop{\smile\!\!\!\!\!\!\mid}^{aeq}\restriction K. \arrow[ur] & \\
 & & \mathop{\smile\!\!\!\!\!\!\mid}^{f} & & &
\end{tikzcd}
$$

Figure 7.1: Interactions of independence relations in ACFG.

7.1 Forcing base monotonicity and extension

In this subsection, given a ternary relation $\mathop{\smile\!\!\!\!\!|}$ in an arbitrary theory, we introduce the relations $\mathop{\smile\!\!\!\!\!|}^m$ and $\mathop{\smile\!\!\!\!\!|}^*$, following the work of Adler in [Adl09a].

Definition 7.1.1 (Monotonised)**.** Let $\mathop{\smile\!\!\!\!\!|}$ be any ternary relation, we define $\mathop{\smile\!\!\!\!\!|}^m$ to be the relation defined by

$$A \mathop{\smile\!\!\!\!\!|}^m_C B \iff \forall D \subseteq CB \;\; A \mathop{\smile\!\!\!\!\!|}_{CD} BC.$$

We call $\mathop{\smile\!\!\!\!\!|}^m$ the *monotonised* of $\mathop{\smile\!\!\!\!\!|}$.

Note that the relation $\mathop{\smile\!\!\!\!\!|}^M$ in [Adl09a, Section 4] is the relation $\mathop{\smile\!\!\!\!\!|}^{a\,m}$ in our context.

Lemma 7.1.2. *The relation $\mathop{\smile\!\!\!\!\!|}^m$ satisfies* Base Monotonicity*. Furthermore, for each of the following point*

- Invariance
- Monotonicity
- Transitivity

if $\mathop{\smile\!\!\!\!\!|}$ satisfies it then so does $\mathop{\smile\!\!\!\!\!|}^m$.

Proof. Let A, B, C, D such that $A \mathop{\smile\!\!\!\!\!|}^m_C BD$. Then for all $D' \subseteq \operatorname{acl}(BCD)$ we have that $A \mathop{\smile\!\!\!\!\!|}_{CD'} B$ so in particular for all $D' \subseteq \operatorname{acl}(BCD)$ containing D we have $A \mathop{\smile\!\!\!\!\!|}_{CD'} B$ hence for all $D'' \subseteq \operatorname{acl}(BCD)$ we have $A \mathop{\smile\!\!\!\!\!|}_{CDD''} B$ hence $A \mathop{\smile\!\!\!\!\!|}^m_{CD} B$. To prove that Invariance is preserved, note that there exists an isomorphism $\sigma : ABC \to A'B'C'$ which extends to $\operatorname{acl}(ABC) \to \operatorname{acl}(A'B'C')$ and so induces an isomorphism $ABCD \to A'B'C'\sigma(D)$ for all $D \subseteq \operatorname{acl}(BC)$. For Monotonicity, it is an easy checking. For Transitivity Assume that $B \mathop{\smile\!\!\!\!\!|}^m_C A$ and $A' \mathop{\smile\!\!\!\!\!|}^m_{CB} A$, and take $D \subseteq \operatorname{acl}(AC)$. We have in particular that $B \mathop{\smile\!\!\!\!\!|}_{CD} A$ and $A' \mathop{\smile\!\!\!\!\!|}_{CBD} A$ hence using Transitivity of $\mathop{\smile\!\!\!\!\!|}$ we have $A'B \mathop{\smile\!\!\!\!\!|}_{CD} A$. This holds for any $D \subseteq \operatorname{acl}(AC)$ hence $A'B \mathop{\smile\!\!\!\!\!|}^m_C A$. $\square$

Let $\mathop{\smile\!\!\!\!\!|}$, $\mathop{\smile\!\!\!\!\!|}'$ be two ternary relations, such that $\mathop{\smile\!\!\!\!\!|}'$ is stronger than $\mathop{\smile\!\!\!\!\!|}$. If $\mathop{\smile\!\!\!\!\!|}'$ satisfies Base Monotonicity then $\mathop{\smile\!\!\!\!\!|}'$ is stronger than $\mathop{\smile\!\!\!\!\!|}^m$. Note that $\mathop{\smile\!\!\!\!\!|}$ may be symmetric and $\mathop{\smile\!\!\!\!\!|}^m$ not (see Corollary 7.2.3). However in some cases, the monotonised is symmetric, as shows the following example.

Example 7.1.3. We work here in ACF. We have

$$A \mathop{\smile\!\!\!\!\!|}^{a\,m}_C B \iff A \mathop{\smile\!\!\!\!\!|}^{\mathrm{ACF}}_C B.$$

Indeed the right to left implication follows from $\mathop{\smile\!\!\!\!\!|}^{\mathrm{ACF}} \to \mathop{\smile\!\!\!\!\!|}^a$ and the fact that $\mathop{\smile\!\!\!\!\!|}^{\mathrm{ACF}}$ satisfies Base Monotonicity. From left to right, assume that $A \not\mathop{\smile\!\!\!\!\!|}^{\mathrm{ACF}}_C B$, we may assume that A, B, C are algebraically closed, and $C = A \cap B$. There exists $b_1, \dots, b_s \in B$ algebraically independent over C such that for $D = \{b_2, \dots, b_s\}$, then we have $b_1 \in (\overline{AD} \cap B) \setminus \overline{CD}$ so $A \not\downarrow^{am}_C B$.

This result translates as follows: in ACF, $\downarrow^f = \downarrow^{am}$. It raises the following question: when do we recover forking independence from the monotonised of the relation $\downarrow^a$? Does the SYMMETRY of the monotonised of a symmetric relation imply nice features on the theory? Observe that the proof above shows that in any pregeometry (S, cl), the independence relation associated with the pregeometry is obtained by forcing BASE MONOTONICITY on the relation $A \downarrow_C B \iff \mathrm{cl}(AC) \cap \mathrm{cl}(BC) = \mathrm{cl}(C)$.

The following example shows that the monotonised does not preserve LOCAL CHARACTER. Also it implies that $\downarrow^{st}$ doesn't satisfy LOCAL CHARACTER since $\downarrow^{st} \to \downarrow^{wm}$.

Example 7.1.4. In ACFG, the relation $\downarrow^{wm}$ does not satisfy LOCAL CHARACTER.
Let κ be any uncountable cardinal and consider the set $A = \{t_i, t'_i \mid i < \kappa\}$ and an element t such that $t(t_i, t'_i)_{i<\kappa}$ are algebraically independent over K. Let $F = \overline{\mathbb{F}_p(t, A)}$ and define H over F as $G(\overline{\mathbb{F}_p}) + \langle t \cdot t_i + t'_i \mid i < \kappa \rangle$. The pair (F, H) defines a consistent type over $\emptyset$, as $\overline{\mathbb{F}_p} \cap H = G(\overline{\mathbb{F}_p})$ and $F \cap K = \overline{\mathbb{F}_p}$, so we assume that t, A are realisation of the type in K. By contradiction suppose that there exists $A_0 \subset A$ with $|A_0| \leq \aleph_0$ such that $t \downarrow^{wm}_{A_0} A$. By definition, for all $D \subseteq A$ we have $t \downarrow^{w}_{A_0 D} A$. Let $D = \{t_i \mid i < \kappa\} \setminus A_0$. We have that

$$G(\overline{tDA_0} + \overline{A}) = G(\overline{tDA_0}) + G(\overline{A}).$$

We compute the $\mathbb{F}_p$-dimension over $G(\overline{\mathbb{F}_p})$ on each side of the previous equation. On one hand, we have $t \cdot t_i + t'_i \in G(\overline{tDA_0} + \overline{A})$ for all $i < \kappa$, as they are $\mathbb{F}_p$-linearly independent over $\overline{\mathbb{F}_p}$ we have $\mathbb{F}_p\text{-}\dim(G(\overline{tDA_0} + \overline{A})/G(\overline{\mathbb{F}_p})) \geq \kappa$. For all $i < \kappa$, $t \cdot t_i + t'_i \in G(\overline{tDA_0})$ if and only if $t'_i \in \overline{tDA_0}$ if and only if $t'_i \in A_0$, because if t'_i is algebraic over $t, A_0, t_1, \dots, t_k$ then t is in A_0 otherwise this contradicts that t, A are algebraically independent. We conclude that $\mathbb{F}_p\text{-}\dim(G(\overline{tDA_0})/G(\overline{\mathbb{F}_p})) \leq |A_0| \leq \aleph_0$. As $G(\overline{A}) = G(\overline{\mathbb{F}_p})$ we have that $\mathbb{F}_p\text{-}\dim([G(\overline{tDA_0}) + G(\overline{A})]/G(\overline{\mathbb{F}_p})) \leq \aleph_0$ so the equality cannot hold.

Definition 7.1.5 (Adler, [Adl09a] Section 3)**.** For $\downarrow$ any ternary relation, $\downarrow^*$ is defined as follows:

$$A \underset{C}{\downarrow^*} B \iff \forall \hat{B} \supseteq B \ \exists A' \equiv_{BC} A \ A' \underset{C}{\downarrow} \hat{B}.$$

Fact 7.1.6 ([Adl09a] Lemma 3.1)**.** *If $\downarrow$ satisfies* INVARIANCE *and* MONOTONICITY *then $\downarrow^*$ satisfies* INVARIANCE*,* MONOTONICITY *and* EXTENSION*. Furthermore, for each of the following point*

- BASE MONOTONICITY
- TRANSITIVITY
- FULL EXISTENCE

if $\downarrow$ satisfies it then so does $\downarrow^*$.

Recall from Section 1.2 that $a \mathop{\smile\!\!\!\!\!\mid}^u_C b$ if and only if $tp(a/Cb)$ is finitely satisfiable in C.

Remark 7.1.7. Let $(b_i)_{i<\kappa}$ be a C-indiscernible infinite sequence with $\kappa > \omega$. Then for all $\geq \alpha \geq \omega$

$$b_{<\beta} \mathop{\smile\!\!\!\!\!\mid}^u_{Cb_{<\alpha}} b_\beta.$$

Furthermore, for κ big enough, the sequence $(b_i)_{i<\kappa}$ is indiscernible over $\mathrm{acl}(C)$ (see [Cas11, Corollary 1.7, 2.]).

Remark 7.1.8. By Lemma 1.2.5 and Fact 7.1.6, if $\mathop{\smile\!\!\!\!\!\mid}$ satisfies Invariance, Monotonicity, then $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfies Invariance, Monotonicity, Extension and Closure over algebraically closed sets. If $\mathop{\smile\!\!\!\!\!\mid}$ satisfies also Base Monotonicity, then so does $\mathop{\smile\!\!\!\!\!\mid}^*$ hence $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfies Closure over any sets. In particular, by Lemma 7.1.2 if $\mathop{\smile\!\!\!\!\!\mid}$ satisfies Invariance and Monotonicity, then $\mathop{\smile\!\!\!\!\!\mid}^{m*}$ satisfies Invariance, Monotonicity, Closure, Base Monotonicity, Extension. Assume that $\mathop{\smile\!\!\!\!\!\mid}$ satisfies Full Existence and Transitivity, then $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfies the following $a \mathop{\smile\!\!\!\!\!\mid}^*_C b \to \mathrm{acl}(Ca) \mathop{\smile\!\!\!\!\!\mid}^*_C b$. Indeed, assume that $a \mathop{\smile\!\!\!\!\!\mid}^*_C b$, then by Fact 7.1.6, $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfies Full Existence so we have $\mathrm{acl}(Ca) \mathop{\smile\!\!\!\!\!\mid}^*_{Ca} b$. By Fact 7.1.6, $\mathop{\smile\!\!\!\!\!\mid}^*$ also satisfies Transitivity, hence $\mathrm{acl}(Ca) \mathop{\smile\!\!\!\!\!\mid}^*_C b$. By Lemma 7.1.2 and Fact 7.1.6, if $\mathop{\smile\!\!\!\!\!\mid}$ satisfies Transitivity then so does $\mathop{\smile\!\!\!\!\!\mid}^{m*}$. It follows that if $\mathop{\smile\!\!\!\!\!\mid}$ satisfies Invariance, Monotonicity, Transitivity and if $\mathop{\smile\!\!\!\!\!\mid}^m$ satisfies Full Existence, then

$$a \mathop{\smile\!\!\!\!\!\mid}_C b \to \mathrm{acl}(Ca) \mathop{\smile\!\!\!\!\!\mid}^{m*}_{\mathrm{acl}(C)} \mathrm{acl}(Cb).$$

Lemma 7.1.9. *Let $\mathop{\smile\!\!\!\!\!\mid}$ be a ternary relation, which satisfies*

- Invariance, Monotonicity*;*
- $\mathop{\smile\!\!\!\!\!\mid}^u$-amalgamation *over algebraically closed sets.*

Then $\mathop{\smile\!\!\!\!\!\mid}^{m*} \to \mathop{\smile\!\!\!\!\!\mid}^f$.

Proof. We show that $\mathop{\smile\!\!\!\!\!\mid}^{m*} \to \mathop{\smile\!\!\!\!\!\mid}^d$, the result follows from the fact that $\mathop{\smile\!\!\!\!\!\mid}^f = \mathop{\smile\!\!\!\!\!\mid}^{d*}$ (Section 1.2). By Lemma 7.1.2, Fact 7.1.6, Remark 7.1.8, and the hypothesis on $\mathop{\smile\!\!\!\!\!\mid}$, $\mathop{\smile\!\!\!\!\!\mid}^{m*}$ satisfies Invariance, Monotonicity, Base Monotonicity, Extension and Closure. Assume $a \mathop{\smile\!\!\!\!\!\mid}^{m*}_C b$, for any a, b, C. Let $(b_i)_{i<\kappa}$ be a C-indiscernible sequence with $b = b_0$, for a big enough κ. By Remark 7.1.7, $b_{<i} \mathop{\smile\!\!\!\!\!\mid}^u_{Cb_{<\omega}} b_i$ for all $i \geq \omega$. By Fact 1.2.3, and Lemma 1.2.5, $\mathop{\smile\!\!\!\!\!\mid}^u$ satisfies Closure and Monotonicity, hence $b_{<i} \mathop{\smile\!\!\!\!\!\mid}^u_{\mathrm{acl}(Cb_{<\omega})} b_i$. Also $(b_i)_{i\geq\omega}$ is $Cb_{<\omega}$-indiscernible, so if κ is big enough, by Remark 7.1.7 we have that $b_i \equiv_{\mathrm{acl}(Cb_{<\omega})} b_\omega$. There exists a C-automorphisme sending b to b_ω hence there exists some a_ω such that $a_\omega b_\omega \equiv_C ab$. By Invariance, we have $a_\omega \mathop{\smile\!\!\!\!\!\mid}^{m*}_C b_\omega$, so by Closure we have $a_\omega \mathop{\smile\!\!\!\!\!\mid}^{m*}_{\mathrm{acl}(C)} \mathrm{acl}(Cb_\omega)$, hence by Extension there exists a'_ω such that $a'_\omega \equiv_{\mathrm{acl}(Cb_\omega)} a_\omega$ and $a'_\omega \mathop{\smile\!\!\!\!\!\mid}^{m*}_{\mathrm{acl}(C)} b_\omega b_{<\omega}$. It follows from Closure and Base Monotonicity that

$$a'_\omega \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(Cb_{<\omega})} b_\omega.$$

We also have

$$a'_\omega b_\omega \equiv_C a_\omega b_\omega \equiv_C ab.$$

For each $i \geq \omega$ there exists an $\mathrm{acl}(Cb_{<\omega})$-automorphism σ_i sending b_ω to b_i, so setting $a'_i = \sigma_i(a'_\omega)$ we have:

$$\forall i \geq \omega \quad a'_i b_i \equiv_{\mathrm{acl}(Cb_{<\omega})} a'_\omega b_\omega \text{ and } a'_i \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(Cb_{<\omega})} b_i.$$

We show that there exists a'' such that $a'' b_i \equiv_{\mathrm{acl}(Cb_{<\omega})} a_\omega b_\omega$ for all $\omega \leq i < \omega + \omega$. By induction and compactness, it is sufficient to show that for all $\omega \leq i < \omega + \omega$, there exists a''_i such that for all $\omega \leq k \leq i$ we have $a''_i b_k \equiv_{\mathrm{acl}(Cb_{<\omega})} a_\omega b_\omega$ and $a''_i \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(Cb_{<\omega})} b_{\leq i}$. For the case $i = \omega$ take $a''_\omega = a'_\omega$. Assume that a''_i has been constructed, we have

$$a'_{i+1} \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(Cb_{<\omega})} b_{i+1} \text{ and } b_{\leq i} \mathop{\smile\!\!\!\!\!\mid}^{u}_{\mathrm{acl}(Cb_{<\omega})} b_{i+1} \text{ and } a''_i \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(Cb_{<\omega})} b_{\leq i}.$$

As $a'_{i+1} \equiv_{\mathrm{acl}(Cb_{<\omega})} a''_i$, by $\mathop{\smile\!\!\!\!\!\mid}^u$-AMALGAMATION over algebraically closed sets, there exists a''_{i+1} such that

(1) $a''_{i+1} b_{i+1} \equiv_{\mathrm{acl}(Cb_{<\omega})} a'_{i+1} b_{i+1}$

(2) $a''_{i+1} b_{\leq i} \equiv_{\mathrm{acl}(Cb_{<\omega})} a''_i b_{\leq i}$

(3) $a''_{i+1} \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(Cb_{<\omega})} b_{\leq i+1}$.

By induction and compactness there exists a'' be such that $a'' b_i \equiv_{\mathrm{acl}(Cb_{<\omega})} a_\omega b_\omega$ for all $\omega \leq i < \omega + \omega$. By indiscernibility of $(b_i)_{i<\kappa}$ there exists a''' such that for all $i < \kappa$ $a''' b_i \equiv_C ab$, hence $a \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} b$. $\square$

Remark 7.1.10. It is important to observe that since $\mathop{\smile\!\!\!\!\!\mid}^u$ is not in general a symmetric relation, the parameters a and b in the statement of $\mathop{\smile\!\!\!\!\!\mid}^u$-AMALGAMATION do not play a symmetrical role. If a relation satisfies $\mathop{\smile\!\!\!\!\!\mid}^u$-amalgamation, we mean that $tp(c_1/Ca)$ and $tp(c_2/Cb)$ can be amalgamated whenever $a \mathop{\smile\!\!\!\!\!\mid}^{u}_{C} b$ or $b \mathop{\smile\!\!\!\!\!\mid}^{u}_{C} a$.

Proposition 7.1.11. *Let* $\mathop{\smile\!\!\!\!\!\mid}$ *be a relation such that*

(1) $\mathop{\smile\!\!\!\!\!\mid}$ *is weaker than* $\mathop{\smile\!\!\!\!\!\mid}^d$*;*

(2) $\mathop{\smile\!\!\!\!\!\mid}$ *satisfies* INVARIANCE, MONOTONICITY, $\mathop{\smile\!\!\!\!\!\mid}^u$*-*AMALGAMATION *over algebraically closed sets;*

(3) $\mathop{\smile\!\!\!\!\!\mid}^m$ *satisfies* EXTENSION *over algebraically closed sets;*

Then $\mathop{\smile\!\!\!\!\!\mid}^m = \mathop{\smile\!\!\!\!\!\mid}^f = \mathop{\smile\!\!\!\!\!\mid}^d$.

Proof. The relation $\mathop{\smile\!\!\!\!\!\mid}^d$ satisfies BASE MONOTONICITY by Fact 1.2.3 hence from *(1)* we have $\mathop{\smile\!\!\!\!\!\mid}^d \rightarrow \mathop{\smile\!\!\!\!\!\mid}^m$. By hypothesis *(3)*, $\mathop{\smile\!\!\!\!\!\mid}^m = \mathop{\smile\!\!\!\!\!\mid}^{m*}$, hence by *(2)* and Lemma 7.1.9 we have $\mathop{\smile\!\!\!\!\!\mid}^d = \mathop{\smile\!\!\!\!\!\mid}^m = \mathop{\smile\!\!\!\!\!\mid}^f$. $\square$

7.2 Forking in ACFG

We show that forking in ACFG is obtained by forcing the property Base Monotonicity on Kim-independence.

We work in a big model (K, G) of ACFG.

Lemma 7.2.1. *Let A, B, C be three additive subgroups of K, then $A \cap (B + C) = A \cap [B + C \cap (A + B)]$.*

Proof. Let $a \in A \cap (B + C)$. There exist $b \in B$ and $c \in C$, such that $a = b + c$. Then $c = a - b \in C \cap (A + B)$ hence $a \in A \cap [B + C \cap (A + B)]$. The other inclusion is trivial. $\square$

Lemma 7.2.2 (Mixed Transitivity on the left)**.** *Let A, B, C, D be algebraically closed sets, with A, B, D containing C and $B \subseteq D$. If $A \mathop{\smile\!\!\!\!\!\!|}^{wm}_{C} B$ and $A \mathop{\smile\!\!\!\!\!\!|}^{st}_{B} D$ then $A \mathop{\smile\!\!\!\!\!\!|}^{wm}_{C} D$.*

Proof. Let A, B, C, D be as in the hypothesis. Let $E \subseteq D$ containing C, we want to show that $A \mathop{\smile\!\!\!\!\!\!|}^{w}_{E} D$. We may assume that E is algebraically closed. We clearly have $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{E} D$, so we have to show that

$$G(\overline{AE} + D) = G(\overline{AE}) + G(D).$$

From $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} E, B$ we have $\overline{AE} \cap \overline{AB} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{E} E, B$ and $\overline{AE} \cap \overline{AB} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{B} E, B$. By elimination of imaginaries in ACF, $\overline{AE} \cap \overline{AB} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{E \cap B} E, B$. By Lemma 1.5.11, it follows that $\overline{AE} \cap \overline{AB} = \overline{A(E \cap B)}$.

Claim. $(\overline{AE} + D) \cap (\overline{AB} + D) = \overline{A(E \cap B)} + D$.

Proof of the claim. By modularity, we have that $(\overline{AE} + D) \cap (\overline{AB} + D) = D + \overline{AE} \cap (\overline{AB} + D)$. By Lemma 7.2.1 we have that

$$\overline{AE} \cap (\overline{AB} + D) = \overline{AE} \cap \left(\overline{AB} + (\overline{AE} + \overline{AB}) \cap D\right).$$

By Lemma 1.5.11, we have $(\overline{AE} + \overline{AB}) \cap D = E + B$, hence

$$\begin{aligned} \overline{AE} \cap (\overline{AB} + D) &= \overline{AE} \cap (\overline{AB} + E + B) \\ &= \overline{AE} \cap (\overline{AB} + E) \\ &= \overline{AE} \cap \overline{AB} + E \text{ by modularity} \\ &= \overline{A(E \cap B)} + E. \end{aligned}$$

It follows that $(\overline{AE} + D) \cap (\overline{AB} + D) = \overline{A(E \cap B)} + D + E = \overline{A(E \cap B)} + D$. $\square$

By hypothesis, $G(\overline{AD}) = G(\overline{AB}) + G(D)$, so, by the claim

$$G(\overline{AE} + D) = G(\overline{AE} + D) \cap (G(\overline{AB}) + G(D)) = G\left(\overline{A(E \cap B)} + D\right) \cap G(\overline{AB}) + G(D).$$

Furthermore $G\left(\overline{A(E\cap B)}+D\right)\cap G(\overline{AB}) = G\left(\overline{A(E\cap B)}+D\cap \overline{AB}\right) = G\left(\overline{A(E\cap B)}+B\right)$. As $A \mathop{\smile\hspace{-0.9em}\mid}^{wm}_{C} B$ we have $G(\overline{A(E\cap B)}+B) = G(\overline{A(E\cap B)}) + G(B)$. We conclude that

$$G(\overline{AE}+D) = G(\overline{A(E\cap B)}) + G(B) + G(D) = G(\overline{A(E\cap B)}) + G(D).$$

$\square$

Corollary 7.2.3. *In* ACFG, $\mathop{\smile\hspace{-0.9em}\mid}^{wm}$ *satisfies* EXTENSION. *In particular, in* $\mathop{\smile\hspace{-0.9em}\mid}^{wm} = \mathop{\smile\hspace{-0.9em}\mid}^{f} = \mathop{\smile\hspace{-0.9em}\mid}^{d}$.

Proof. Assume that $a \mathop{\smile\hspace{-0.9em}\mid}^{wm}_{C} b$ and d is given. By FULL EXISTENCE of $\mathop{\smile\hspace{-0.9em}\mid}^{st}$ there exists $a' \equiv_{Cb} a$ such that $a' \mathop{\smile\hspace{-0.9em}\mid}^{st}_{Cb} d$. Also $a' \mathop{\smile\hspace{-0.9em}\mid}^{wm}_{C} b$ hence by Lemma 7.2.2 $a' \mathop{\smile\hspace{-0.9em}\mid}^{wm}_{C} b, d$, which shows EXTENSION for $\mathop{\smile\hspace{-0.9em}\mid}^{wm}$. In particular $\mathop{\smile\hspace{-0.9em}\mid}^{w}$ satisfies hypothesis *(3)* of Proposition 7.1.11. We check that it satisfies the rest of the hypotheses of Proposition 7.1.11. *(1)* follows from Corollary 5.2.4. From Theorem 5.2.2, $\mathop{\smile\hspace{-0.9em}\mid}^{w}$ satisfies the properties INVARIANCE, MONOTONICITY and $\mathop{\smile\hspace{-0.9em}\mid}^{u}$-AMALGAMATION over algebraically closed sets (since $\mathop{\smile\hspace{-0.9em}\mid}^{u} \to \mathop{\smile\hspace{-0.9em}\mid}^{a}$, by Fact 1.2.3), so $\mathop{\smile\hspace{-0.9em}\mid}^{w}$ satisfies *(2)*. $\square$

7.3 Thorn-Forking in ACFG

Let (K, G) be a monster model of ACFG. Let $\mathop{\smile\hspace{-0.9em}\mid}^{aeq}$ be the relation $\mathop{\smile\hspace{-0.9em}\mid}^{a}$ in the sense of $(K, G)^{eq}$ (Section 1.2). The thorn-forking independence relation $\mathop{\smile\hspace{-0.9em}\mid}^{þ}$ is the relation defined over subsets of $(K, G)^{eq}$ by $\mathop{\smile\hspace{-0.9em}\mid}^{þ} = (\mathop{\smile\hspace{-0.9em}\mid}^{aeq})^{m*}$. We will only consider the restrictions of $\mathop{\smile\hspace{-0.9em}\mid}^{aeq}$and $\mathop{\smile\hspace{-0.9em}\mid}^{þ}$ to the home sort, which we denote respectively by $\mathop{\smile\hspace{-0.9em}\mid}^{aeq}\restriction K$ and $\mathop{\smile\hspace{-0.9em}\mid}^{þ}\restriction K$. By Corollary 6.1.5 and Theorem 6.3.4, for $a, b, C \subset K$

$$a \mathop{\smile\hspace{-0.9em}\mid}^{aeq}_{C} b \iff \overline{Ca}\cap\overline{Cb} = \overline{C} \text{ and } \pi(\overline{Ca})\cap\pi(\overline{Cb}) = \pi(\overline{C}).$$

Fact 7.3.1 ([Adl09a] Theorem 4.3)**.** *The following are equivalent.*

- *T is rosy*
- *$\mathop{\smile\hspace{-0.9em}\mid}^{þ}$ in T^{eq} satisfies* LOCAL CHARACTER.

Proposition 7.3.2. *Let (K, G) be a model of* ACFG*. Then* $\mathop{\smile\hspace{-0.9em}\mid}^{þ}\restriction K = \mathop{\smile\hspace{-0.9em}\mid}^{wm} = \mathop{\smile\hspace{-0.9em}\mid}^{f} = \mathop{\smile\hspace{-0.9em}\mid}^{d}$. *In particular* ACFG *is not rosy.*

Proof. Assume that $a \mathop{\smile\hspace{-0.9em}\mid}^{þ}_{C} b$. In particular $a \mathop{\smile\hspace{-0.9em}\mid}^{aeq\ m}_{C} b$ so for all $C \subseteq D \subseteq \overline{Cb}$ we have $\overline{Da}\cap\overline{Cb} = \overline{D}$ hence by Example 7.1.3 we have

$$a \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{ACF}}_{C} b.$$

On the other hand, we have $\pi(\overline{Ca})\cap\pi(\overline{Cb}) = \pi(\overline{C})$, hence by Section 6.1

$$a \mathop{\smile\hspace{-0.9em}\mid}^{w}_{C} b.$$

It follows that $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\text{þ}} \restriction K \to \mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{wm}$. By Fact 1.2.3, $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{d} \to \mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{aeq}\restriction K$, hence as $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{f}$ satisfies Base Monotonicity and Extension it follows that $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{f} \to \mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\text{þ}} \restriction K$. Hence by Corollary 7.2.3 we conclude that $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\text{þ}} \restriction K = \mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{wm} = \mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{f} = \mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{d}$. As ACFG is not simple, $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{f}$ does not satisfy Local Character, so $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\text{þ}} \restriction K$ does not satisfy Local Character hence neither does $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\text{þ}}$. By Fact 7.3.1, ACFG is not rosy. $\square$

Remark 7.3.3. There is another way of proving that ACFG is not rosy which does not use the description of forking in ACFG but only the fact that $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\text{þ}} \restriction K \to \mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{wm}$. Indeed $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{wm}$ does not satisfy Local Character from Example 7.1.4 hence neither does $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\text{þ}} \restriction K$ and hence neither does $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\text{þ}}$.

Remark 7.3.4. It is worth mentioning that in the definition of $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\text{þ}}$, the relation $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{aeq}$ cannot be replaced by $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{a}$. Indeed, in the structure (K, G), by Example 7.1.3 ${\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{a}}^{m} = \mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\mathrm{ACF}}$ and then as Extension clearly holds for $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\mathrm{ACF}}$, we have ${\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{a}}^{M*} = \mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{\mathrm{ACF}}$. This relation satisfies Local Character. This means that ${\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{a}}^{M*}$ is not the restriction of ${\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{aeq}}^{m*}$ to the home sort. This is what Adler mention in [Adl09a, Example 4.5].

7.4 Forking and thorn-forking in other generic constructions

Forking and dividing. In the three following examples:

(1) *Generic $\mathscr{L}$-structure $T^{\emptyset}_{\mathscr{L}}$ [KR17, Proposition 3.18];*

(2) *Generic $K_{n,m}$-free bipartite graph [CK17, Corollary 4.12];*

(3) *omega-free PAC fields [Cha02, Theorem 3.3];*

we also have that forking and dividing coincides for types, and coincides with the monotonised of Kim-independence. In *(1)* and *(2)* the strategy is the following: first prove that $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{d} = \mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{Km}$ and then show that $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{d}$ satisfies Extension. The latter is obtained using Full Existence of the *strong* independence relation and a similar mixed transitivity result. This is discussed in [KR18, Subsection 3.3]. We followed a close strategy: using Lemma 7.1.9 (based on the approach of *(3)*), have that ${\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{w}}^{M*}$ strengthens $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{d}$. Then we use a mixed transitivity result and Full Existence of the strong independence to show that $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.7em}}^{Km}$ satisfies Extension. These results suggest that Proposition 7.1.11 can be used to show that in other examples of NSOP_1 theories, forking and dividing agrees on types, for instance in Steiner triple system [BC18], or bilinear form over an infinite dimensional vector space over an algebraically closed field [Gra99] [CR16].

Strong independence and Mixed Transitivity. There is also a notion of *strong independence* in the three previous examples which is symmetric and stationary over algebraically closed sets. Concerning *(3)* the strong independence satisfies also the other axioms for mock stability [KK11, Example 0.1 (3)]. In *(2)*, it also satisfies Full Existence, Monotonicity and Transitivity [CK17, Proposition 4.20]. In *(1)*, it is defined in [KR18, Remark 3.19], as a remark, to state a mixed transitivity result, but nothing about it is proven. It is likely that *(1)* and *(2)*, are also mock stable, witnessed by the strong independence. Informally, the strong independence is in general defined to hold between two sets when they are the most unrelated to each other with respect to the ambient theory. Another way of seeing this relation is by saying that the two sets can be somehow "*freely amalgamated*". The definition given in [KR17, Remark 3.19] make this precise, for $C \subseteq A \cap B$, we have $A \mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}_C B$ if and only if the substructure spanned by ABC is isomorphic to the fibered coproduct of the structures spanned by A and B over the substructure spanned by C. This definition coincides with our definition of strong independence in ACFG.

Question 1. *Is there a model-theoretic definition of the strong independence that encompasses the strong independence in the three examples above and in* ACFG*?*

The mixed transitivity result (Lemma 7.2.2) is starting to be reccurent in NSOP_1 examples. It holds in example *(1)* ([KR18, Remark 3.19]) and in *(2)* ([CK17, Lemma 4.23]). Note that a similar mixed transitivity appears in a SOP_3 (hence SOP_1) example: the generic K_n-free graph ([Con17a]), this was observed in [KR18, Remark 3.19].

The mixed transitivity result holds as well in omega-free PAC fields. Let $\mathop{\smile\!\!\!\!\!\!\mid}^{w}$ be the *weak* independence and $\mathop{\smile\!\!\!\!\!\!\mid}^{st}$ the *strong* independence in the sens of [Cha02, (1.2)]. Then for all A, B, C, D acl-closed in an omega free PAC field, with $C \subseteq A \cap B$ and $B \subseteq D$ we have:

$$\text{If } A \mathop{\smile\!\!\!\!\!\!\mid}^{wm}_C B \text{ and } A \mathop{\smile\!\!\!\!\!\!\mid}^{st}_B D \text{ then } A \mathop{\smile\!\!\!\!\!\!\mid}^{wm}_C D.$$

This is contained in the proof[1] of [Cha02, (3.1) Proposition].

Thorn-forking. The three other examples are also not rosy. For *(1)*, it is [KR17, Subsection 3.3], for *(2)*, it is [CK17, Proposition 4.28] and for *(3)*, it is [Cha08, Subsection 3.5]. Also, for both *(1)* and *(2)* we have $\mathop{\smile\!\!\!\!\!\!\mid}^{f} = \mathop{\smile\!\!\!\!\!\!\mid}^{d} = \mathop{\smile\!\!\!\!\!\!\mid}^{þ}$, and they both weakly eliminate imaginaries.

The following questions have been asked for the last two or three years by specialists in regards to the observations above.

Question 2. (Q_1) *Does forking equals dividing for types in every* NSOP_1 *theory?*

(Q_2) *Does the mixed transitivity result holds in every* NSOP_1 *theory?*

(Q_3) *Is there an* NSOP_1 *not simple rosy theory?*

[1]In the proof of [Cha02, (3.1) Proposition], D contains B, ψ is over C and $F \cap (C\psi(D))^s = C\psi(D)$, hence $\psi(D)$ and C satisfies condition *(I3)* over B, so $A_1 = \psi(A_0)$ and C satisfies condition *(I3)* over E. As A_1 and C satisfies condition *(I1)* over E, A_1 and C are strongly independent over E. Also A_1 and B satisfy condition *(I1)* and *(I2)* over E. The rest of the proof consist in proving that A_1 and C satisfy condition *(I2)* over E.

$$\begin{tikzcd}
 & & & & \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{ACF}} \arrow[d, equal] & \\
 & & \mathop{\smile\hspace{-0.9em}\mid}^{\flat} \restriction K \arrow[d, equal] & & \mathop{\smile\hspace{-0.9em}\mid}^{a\,m} \arrow[dr] & \\
\mathop{\smile\hspace{-0.9em}\mid}^{st} \arrow[r] & \mathop{\smile\hspace{-0.9em}\mid}^{wsm} \arrow[r] & \mathop{\smile\hspace{-0.9em}\mid}^{wm} \arrow[r] \arrow[d, equal] & \mathop{\smile\hspace{-0.9em}\mid}^{w} \arrow[ur] \arrow[dr] \arrow[d, equal] & & \mathop{\smile\hspace{-0.9em}\mid}^{a} \\
 & & \mathop{\smile\hspace{-0.9em}\mid}^{d} \arrow[d, equal] & \mathop{\smile\hspace{-0.9em}\mid}^{K} & \mathop{\smile\hspace{-0.9em}\mid}^{aeq} \restriction K. \arrow[ur] & \\
 & & \mathop{\smile\hspace{-0.9em}\mid}^{f} & & &
\end{tikzcd}$$

Figure 7.2: Interactions of independence relations in ACFG.

Remark 7.4.1. In omega-free PAC fields [Cha02], the strong independence $\mathop{\smile\hspace{-0.9em}\mid}^{st}$ and the weak independence $\mathop{\smile\hspace{-0.9em}\mid}^{w}$ are linked by the following relation for A, B, C acl-closed, $A \cap B = C$:

$$A \mathop{\smile\hspace{-0.9em}\mid}^{st}_{C} B \iff \text{for all } C \subseteq D \subseteq A \text{ and } C \subseteq D' \subseteq B\ A \mathop{\smile\hspace{-0.9em}\mid}^{w}_{DD'} B.$$

In ACFG this is not the case. Let (K, G) be a model of ACFG and for conveniance assume that $G(\overline{\mathbb{F}_p}) = \{0\}$. Let t and t' be algebraically independent over $\mathbb{F}_p$, let $u = t \cdot t'$. Assume that $G(\overline{\mathbb{F}_p(t, t')}) = \langle u \rangle$. Then by Lemma 1.5.8, $u \notin \overline{\mathbb{F}_p(t)} + \overline{\mathbb{F}_p(t')}$, so $G(\overline{\mathbb{F}_p(t)}) + G(\overline{\mathbb{F}_p(t')}) = \{0\}$ so $t \mathop{\smile\hspace{-0.9em}\not\mid}^{st} t'$. We show that for all $D \subseteq \overline{\mathbb{F}_p(t)}$ and $D' \subseteq \overline{\mathbb{F}_p(t')}$ we have $t \mathop{\smile\hspace{-0.9em}\mid}^{w}_{DD'} t'$. Let D and D' be as such. There are three cases to consider (the middle case is symmetric):

$$\begin{array}{llll}
t \cdot t' \in \overline{D't} \text{ and } t \cdot t' \in \overline{Dt'} & G(\overline{D't}) = \langle u \rangle & G(\overline{Dt'}) = \langle u \rangle & G(\overline{D't} + \overline{Dt'}) = \langle u \rangle \\
t \cdot t' \in \overline{D't} \text{ and } t \cdot t' \notin \overline{Dt'} & G(\overline{D't}) = \langle u \rangle & G(\overline{Dt'}) = \{0\} & G(\overline{D't} + \overline{Dt'}) = \langle u \rangle \\
t \cdot t' \notin \overline{D't} \text{ and } t \cdot t' \notin \overline{Dt'} & G(\overline{D't}) = \{0\} & G(\overline{Dt'}) = \{0\} & G(\overline{D't} + \overline{Dt'}) = \{0\}
\end{array}$$

In every cases we have $G(\overline{D't} + \overline{Dt'}) = G(\overline{D't}) + G(\overline{Dt'})$. As $t \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{ACF}}_{DD'} t'$ is clear we have $t \mathop{\smile\hspace{-0.9em}\mid}^{w}_{DD'} t'$.

Summary on independence relations in ACFG. Every arrow in Figure 7.2 is strict, from that point of view, ACFG is different from *(1)*, *(2)* and *(3)*.

Denote by $A \mathop{\smile\hspace{-0.9em}\mid}^{wsM}_{C} B$ the relation for all $C \subseteq D \subseteq \overline{AC}$ and $C \subseteq D' \subseteq \overline{BC}$ $A \mathop{\smile\hspace{-0.9em}\mid}^{w}_{DD'} B$. Remark 7.4.1 states that $\mathop{\smile\hspace{-0.9em}\mid}^{st}$ is strictly stronger than $\mathop{\smile\hspace{-0.9em}\mid}^{KsM}$, in *(3)*, this is not the case. In *(1)*, we have that $\mathop{\smile\hspace{-0.9em}\mid}^{a} = \mathop{\smile\hspace{-0.9em}\mid}^{aeq} = \mathop{\smile\hspace{-0.9em}\mid}^{K}$ is strictly weaker than $\mathop{\smile\hspace{-0.9em}\mid}^{a\,m} = \mathop{\smile\hspace{-0.9em}\mid}^{d} = \mathop{\smile\hspace{-0.9em}\mid}^{f} = \mathop{\smile\hspace{-0.9em}\mid}^{\flat}$. In *(2)*, $\mathop{\smile\hspace{-0.9em}\mid}^{a} = \mathop{\smile\hspace{-0.9em}\mid}^{aeq}$ is strictly weaker than $\mathop{\smile\hspace{-0.9em}\mid}^{K}$ and $\mathop{\smile\hspace{-0.9em}\mid}^{Km} = \mathop{\smile\hspace{-0.9em}\mid}^{d} = \mathop{\smile\hspace{-0.9em}\mid}^{f} = \mathop{\smile\hspace{-0.9em}\mid}^{\flat}$.

Part B

Expanding the integers by p-adic valuations

CHAPTER 8

Quantifier elimination and dp-rank

For a prime number p, let $v_p : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ be the p-adic valuation, namely, $v_p(a) = \sup\{k \in \mathbb{N} : p^k | a\}$. Let $\emptyset \neq P \subseteq \mathbb{N}$ be a (possibly infinite) set of primes, and let L_P be the language $\{+, 0\} \cup \{|_p : p \in P\}$, where each $|_p$ is a binary relation. We expand $(\mathbb{Z}, +, 0)$ to an L_P-structure $\mathcal{Z}_P$ by interpreting $a|_p b$ as $v_p(a) \leq v_p(b)$ for each $p \in P$. We denote by $T_P := Th(\mathcal{Z}_P)$. For convenience, we enumerate P by $P = \{p_\alpha : \alpha < |P|\}$, and p without a subscript usually denotes some $p \in P$. If $P = \{p\}$ we write T_p instead of $T_{\{p\}}$, etc.

In this chapter, we prove (see Theorem 8.2.1) that T_P eliminates quantifiers in a natural definitional expansion: $L_P^E = L_P \cup \{-, 1\} \cup \{D_n : n \geq 1\}$ where $-$ and 1 are interpreted in the obvious way, and for each $n \geq 1$, D_n is an unary relation symbol interpreted as $\{na : a \in \mathbb{Z}\}$.

Using quantifier elimination, we are able to determine the dp-rank of T_P, and we prove (Theorem 8.3.2) that for $P \neq \emptyset$, dp-rk$(T_P) = |P|$. In particular, for a single prime p we have that T_p is dp-minimal, i.e. dp-rank$(T_p) = 1$.

Contents

8.1 Axioms and basic sentences of T_P

For convenience, in this section and in section 8.2 we work with the valuation functions v_p instead of the relations $|_p$. Let us define a multi-sorted language L_P^M for the valuations v_p on $(\mathbb{Z}, +, 0)$ for $p \in P$ as follows: let Z be the main sort with a function symbol $+$ and a constant symbol 0, interpreted as in $(\mathbb{Z}, +, 0)$. For each $p \in P$ we add a distinct sort Γ_p together with the symbols $<_p$, 0_p, S_p and ∞_p, interpreted as a distinct copy of $(\mathbb{N} \cup \{\infty\}, <, 0, S, \infty)$ where S is the successor function. Finally, we add a function symbol $v_p : Z \to \Gamma_p$, interpreted as the p-adic valuation[1]. When confusion is possible, we denote by $\mathbf{v}_p$ the usual valuation in the metatheory, to distinguish it from the function symbol v_p. We omit the subscript p in $<_p$, 0_p, S_p, ∞_p and Γ_p when no confusion is possible.

We use the following standard notation. Let $k \in \mathbb{N}$ be a nonnegative integer.

- In the Z sort, $\underline{k}$ denotes $\underbrace{1 + 1 + \cdots + 1}_{k \text{ times}}$ if $k > 0$ and 0 if $k = 0$. Also, $\underline{-k}$ denotes $-\underline{k}$.
- For an element a from Z, ka denotes $\underbrace{a + a + \cdots + a}_{k \text{ times}}$ if $k > 0$ and 0 if $k = 0$, $(-k)a$ denotes $-(ka)$, similarly for a variable x in place of a.
- For an element γ from Γ_p, $\gamma + \underline{k}$ denotes $\underbrace{S(S(\dots(\gamma)\dots))}_{k \text{ times}}$, similarly for a variable u in place of γ, and $\underline{k}$ is an abbreviation for $0 + \underline{k}$.

The group $(\mathbb{Z}, +, 0)$ with valuations v_p for $p \in P$ can be seen as an L_P-structure and an L_P^M-structure which are interdefinable (with imaginaries) so they essentially define the same sets. We will therefore not distinguish between the L_P-structure and the L_P^M-structure on $(\mathbb{Z}, +, 0)$, except when dealing with dp-rank, where we always refer to the one-sorted language L_P.

For quantifier elimination we define $L_P^{M,E} = L_P^M \cup \{-, 1\} \cup \{D_n : n \geq 1\}$ as before. Quantifier elimination in L_P^E follows from quantifier elimination in $L_P^{M,E}$. We will, therefore, prove quantifier elimination for the theory T_P considered as an $L_P^{M,E}$-theory.

For $a \in \mathbb{Z}$ and $p \in P$, let $(a_i)_{i \in \mathbb{N}}$ be the p-adic representation of a, i.e. $a = \sum_{i \in \mathbb{N}} a_i p^i$ and each a_i is in $\{0, \dots, p-1\}$. For $\gamma \in \mathbb{N}$, the *prefix of a of length γ* is the sequence sequence $(a_i)_{i<\gamma}$. The *ball* of radius γ and center a is the set of all integers with same prefix of length γ as a.

Proposition 8.1.1. *The following sentences are true in $\mathcal{Z}_P$ and therefore are in T_P:*

(1) Any axiomatization for $Th(\mathbb{Z}, +, -, 0, 1, \{D_n\}_{n \geq 1})$ in the Z sort.

[1] It could be interesting to consider the language with just one sort $(N, <, 0, S, \infty)$ for valuation, instead of one for each $p \in P$. Since different valuations are allowed to interact with each other, the resulting structures might be much more complicated.

(2) *For each p, any axiomatization of $Th(\mathbb{N} \cup \{\infty\}, <, 0, S, \infty)$ in the sort $(\Gamma_p, <_p, 0_p, S_p, \infty_p)$.*

(3) *For each p : $\forall x(v_p(x) \geq 0 \wedge (v_p(x) = \infty \leftrightarrow x = 0))$.*

(4) *For each p : $\forall x, y(v_p(x+y) \geq min(v_p(x), v_p(y)))$.*

(5) *For each p : $\forall x, y(v_p(x) \neq v_p(y) \rightarrow v_p(x+y) = min(v_p(x), v_p(y)))$.*

(6) *For each p and $0 \neq n \in \mathbb{Z}$: $\forall x(v_p(nx) = v_p(x) + \underline{\mathbf{v}_p(n)})$.*

(7) *For each p : $v_p(\underline{p}) = 1$.*

(8) *For each p and $k \in \mathbb{N}$: Every ball in v_p of radius γ consists of exactly p^k disjoint balls of radius $\gamma + k$.*

Proof. (1)-(7) are obvious. For (8), let $a \in \mathbb{Z}$ and $\gamma \in \mathbb{N}$. The ball in v_p of radius γ around a is the set of integers such that, in p-adic representation, their prefix of length γ is the same as the prefix of a of length γ. There are p possibilities for each digit, so p^k possibilities for the k digits with indices $\gamma, \ldots, \gamma + k - 1$, which exactly correspond to the balls of radius $\gamma + k$ contained in the original ball. $\square$

Let T'_P be the theory implied by the axioms (1)-(8). All of the following propositions are first order, and we prove them using only T'_P. Let $\mathcal{M}$ be some fixed model of T'_P, with $\mathcal{Z}$ the Z-sort and Γ_p the Γ_p-sort.

Lemma 8.1.2. *For each p:*

(1) $\forall x, y(v_p(x-y) \geq min(v_p(x), v_p(y)))$.

(2) $\forall u \forall y \exists x(v_p(x-y) = u)$. *In particular, v_p is surjective.*

(3) *For each $n \neq 0$, $v_p(\underline{n}) = \underline{\mathbf{v}_p(n)}$.*

(4) *For each $k \geq 1$: $\forall x(v_p(x) \geq \underline{k} \leftrightarrow D_{p^k}(x))$.*

Proof. We only prove item (2), the others are easy to check. By Axiom (8) with $k = 1$, there are x_1, x_2 such that $v_p(x_1 - y) \geq u$, $v_p(x_2 - y) \geq u$, and $v_p(x_1 - x_2) < u + \underline{1}$. Hence by (1) above, $u + \underline{1} > v_p(x_1 - x_2) = v_p((x_1 - y) - (x_2 - y)) \geq \min(v_p(x_1 - y), v_p(x_2 - y)) \geq u$. So either $v_p(x_1 - y) = u$ or $v_p(x_2 - y) = u$. $\square$

The following lemmas are easy exercises.

Lemma 8.1.3.

(1) *Let $n_1, \ldots, n_l \in \mathbb{N}$, and let $N \in \mathbb{N}$ be such that $n_i | N$ for all $1 \leq i \leq l$. Let $b_1, \ldots, b_n$ be element of $\mathcal{Z}$. Then every boolean combination of formulas of the form $D_{n_i}(k_i x - b_i)$ is equivalent to a disjunction (possibly empty, i.e. a contradiction) of formulas of the form $D_N(x - \underline{r}_j)$, where for each j, $r_j \in \{0, 1, \ldots, N-1\}$.*

(2) Let $m \in \mathbb{N}$ and let $m', k \in \mathbb{N}$ be such that $m = p^k \cdot m'$ and $gcd(m', p) = 1$. Let $r \in \mathbb{Z}$, and let $r_1 = r \bmod m'$, $r_2 = r \bmod p^k$. Then the formula $D_m(x - \underline{r})$ is equivalent to $D_{m'}(x - \underline{r}_1) \wedge (v_p(x - \underline{r}_2) \geq k)$.

Lemma 8.1.4. *For a_1 and a_2 in $\mathcal{Z}$.*

(1) For every $k \geq 1$, the formula $v_p(x - a_1) < v_p(x - a_2) + \underline{k}$ is equivalent to

$$v_p(x - a_2) < v_p(a_2 - a_1) \vee v_p(x - a_2) > v_p(a_2 - a_1) \vee v_p(x - a_1) < v_p(a_2 - a_1) + \underline{k}.$$

(2) For every $k \geq 0$, the formula $v_p(x - a_1) + \underline{k} < v_p(x - a_2)$ is equivalent to $v_p(x - a_2) > v_p(a_2 - a_1) + \underline{k}$.

Lemma 8.1.5. *For a fixed $p \in P$, a_0, a_1 in $\mathcal{Z}$ and $\gamma_0, \gamma_1 \in \Gamma_p$.*

(1) Every formula of the form $v_p(x - a_0) \geq \gamma_0 \wedge v_p(x - a_1) < \gamma_1$ where $\gamma_0 \geq \gamma_1$, is either inconsistent (if $v_p(a_0 - a_1) \geq \gamma_1$) or equivalent to just $v_p(x - a_0) \geq \gamma_0$ (if $v_p(a_0 - a_1) < \gamma_1$).

(2) Every formula of the form $v_p(x - a_0) \geq \gamma_0 \wedge v_p(x - a_1) < \gamma_1$ where $\gamma_0 < \gamma_1$ and $v_p(a_0 - a_1) < \gamma_0$ is equivalent to just $v_p(x - a_0) \geq \gamma_0$.

Lemma 8.1.6. *Every two balls in Γ_p are either disjoint, or one is contained in the other. More generally, for $(a_i)_i \in \mathcal{Z}$, $(\gamma_i)_i \in \Gamma_p$, every conjunction of formulas of the form $v_p(x - a_i) \geq \gamma_i$ is either inconsistent, or equivalent to a single formula $v_p(x - a_{i_0}) \geq \gamma_{i_0}$, where $\gamma_{i_0} = max\{\gamma_i\}$.*

Definition 8.1.7. For $a, b \in Z$, $\gamma, \delta \in \Gamma_p$, define $(a, \gamma) \leq_p (b, \delta)$ if $\gamma \leq \delta$ and $v_p(a - b) \geq \gamma$. Define $(a, \gamma) \sim_p (b, \delta)$ if $(a, \gamma) \leq_p (b, \delta)$ and $(a, \gamma) \geq_p (b, \delta)$.

$(a, \gamma) \leq_p (b, \delta)$ means that $\gamma \leq \delta$ and, in p-adic representation, the prefix of a of length γ is contained in the prefix of b of length δ. This is equivalent to saying that the ball of radius γ around a (namely, $\{x : v_p(x - a) \geq \gamma\}$) contains the ball of radius δ around b.

Note that $\leq_p$ and $\sim_p$ are defined by quantifier-free formulas, and so do not depend on the model containing the elements under consideration.

Lemma 8.1.8. *The parameters a_i are in $\mathcal{Z}$ and γ_i are in Γ_p for some $p \in P$.*

(1) Every formula of the form $v_p(x - a_0) \geq \gamma_0 \wedge \bigwedge_{m=1}^{n} v_p(x - a_m) < \gamma_m$ is equivalent to the formula $v_p(x - a_0) \geq \gamma_0 \wedge \bigwedge_{m \in C} v_p(x - a_m) < \gamma_m$, for every $C \subseteq \{1, \ldots, n\}$ such that $\{(a_m, \gamma_m) : m \in C\}$ contains at least one element from each $\sim_p$-equivalence class of $\leq_p$-minimal elements among $\{(a_m, \gamma_m) : 1 \leq m \leq n\}$ (i.e. representatives for all the maximal balls). In particular, this is true for C consisting of one element from each such class, i.e. for C an antichain.

(2) *Assume that* $(a_0,\gamma_0),\dots,(a_n,\gamma_n)$ *are such that for all* $1\le m\le n$ *we have* $\gamma_m>\gamma_0$, $v_p(a_m-a_0)\ge\gamma_0$, *and* $k_m:=\gamma_m-\gamma_0$ *is a standard integer. Assume further that* $\{(a_m,\gamma_m)\,:\,1\le m\le n\}$ *is an antichain with respect to* $\le_p$. *Then every formula of the form* $v_p(x-a_0)\ge\gamma_0\wedge\bigwedge_{m=1}^n v_p(x-a_m)<\gamma_m$ *is equivalent to a formula of the form* $\bigvee_{i=1}^l v_p(x-b_i)\ge\gamma_N$ *with* N *such that* $\gamma_N=\max\{\gamma_m\,:\,1\le m\le n\}$, *where for all* i, $v_p(b_i-a_0)\ge\gamma_0$ *and for* $i\ne j$, $v_p(b_i-b_j)<\gamma_N$, *and where* $l=p^{k_N}-\sum_m p^{k_N-k_m}\ge 0$ *(it may be that* $l=0$, *i.e. a contradiction). In particular,* l *does not depend on the model* $\mathcal{M}$ *of* T_P' *containing the* a_i*'s and* γ_i*'s.*

Proof. We prove *(1)*. Let C be such. For each $1\le m\le n$ there is an m' such that $(a_{m'},\gamma_{m'})\le(a_m,\gamma_m)$ and $(a_{m'},\gamma_{m'})$ is minimal among the (a_i,γ_i)'s. So $\forall x(v_p(x-a_{m'})<\gamma_{m'}\to v_p(x-a_m)<\gamma_m)$. As $\{(a_i,\gamma_i)\,:\,i\in C\}$ contains one element from each $\sim$-equivalence class of $\le$-minimal elements, we may assume $m'\in C$.
We prove *(2)*. Assume without loss of generality that $\gamma_1\le\gamma_2\le\dots\le\gamma_n$. Let $b_0,\dots,b_{p^{k_n}-1}$ be the $x_0,\dots,x_{p^k-1}$ from Axiom 8 for k_n, γ_0, a_0. Then $v_p(x-a_0)\ge\gamma_0$ is equivalent to $\bigvee_{i=0}^{p^{k_n}-1}(v_p(x-b_i)\ge\gamma_n)$. For every $m\ge1$, let $c_{m,0},\dots,c_{m,p^{k_n-k_m}-1}$ be the $x_0,\dots,x_{p^k-1}$ from Axiom 8 for k_n-k_m, γ_m, a_m. Then $v_p(x-a_m)\ge\gamma_m$ is equivalent to $\bigvee_{i=0}^{p^{k_n-k_m}-1}(v_p(x-c_{m,i})\ge\gamma_n)$. For every m, $v_p(a_0-a_m)\ge\gamma_0$, so for every $0\le i\le p^{k_n-k_m}-1$, $v_p(c_{m,i}-a_0)\ge\gamma_0$. Hence by the choice of $\{b_j\}_j$, there is a unique $s_{m,i}<p^{k_n}$ such that $v_p(c_{m,i}-b_{s_{m,i}})\ge\gamma_n$. So $v_p(x-a_m)\ge\gamma_m$ is equivalent to $\bigvee_{i=0}^{p^{k_n-k_m}-1}(v_p(x-b_{s_{m,i}})\ge\gamma_n)$.

By the choice of $\{c_{m,i}\}_i$, $\bigwedge_{i\ne j}(v_p(c_{m,i}-c_{m,j})<\gamma_n)$, so also $\bigwedge_{i\ne j}(v_p(b_{s_{m,i}}-b_{s_{m,j}})<\gamma_n)$. In particular, $i\mapsto s_{m,i}$ is injective for a fixed m, hence $F_m:=\{s_{m,i}\,:\,0\le i\le p^{k_n-k_m}-1\}$ is of size $p^{k_n-k_m}$.

The sets $\{F_m\}_{m=1}^n$ must be mutually disjoint. Otherwise, there are $m_1<m_2$ and i,j such that $s_{m_1,i}=s_{m_2,j}$. Since $v_p(c_{m_1,i}-b_{s_{m_1,i}})\ge\gamma_n$ and $v_p(c_{m_2,j}-b_{s_{m_2,j}})\ge\gamma_n$ we get $v_p(c_{m_1,i}-c_{m_2,j})\ge\gamma_n\ge\gamma_{m_1}$. Since $v_p(c_{m_1,i}-a_{m_1})\ge\gamma_{m_1}$ and $v_p(c_{m_2,j}-a_{m_2})\ge\gamma_{m_2}\ge\gamma_{m_1}$, we get $v_p(a_{m_1}-a_{m_2})\ge\gamma_{m_1}$, a contradiction to the antichain assumption.

Let $F:=\bigcup_{m=1}^n F_m$. By the above, $\mid F\mid=\sum_m p^{k_n-k_m}$ and

$$\forall x\left((v_p(x-a_0)\ge\gamma_0\wedge\bigwedge_{m=1}^n v_p(x-a_m)<\gamma_m)\leftrightarrow(\bigvee_{i\notin F}v_p(x-b_i)\ge\gamma_n))\right).$$

□

Lemma 8.1.9. *For all elements* $a_i,a_{i,j}$ *in* $\mathcal{Z}$ *and* γ_i *in* Γ_p *for some* $p\in P$, *we have the following.*

(1) *If* b *is a solution to* $v_p(x-a_0)\ge\gamma_0\wedge\bigwedge_{i=1}^n v_p(x-a_i)<\gamma_i$ *and* $v_p(b'-b)\ge\gamma:=$ *max*$\{\gamma_0,\dots,\gamma_n\}$ *then* b' *is also a solution.*

(2) *Every formula of the form* $v_p(x-a_0)\ge\gamma_0\wedge\bigwedge_{m=1}^n v_p(x-a_m)<\gamma_m$ *where for each* $1\le m\le n$, $\gamma_m\ge\gamma_0+\underline{n}$, *has a solution.*

(3) *If $p_1, \dots, p_l \in P$ are different primes not dividing m and $\gamma_i \in \Gamma_{p_i}$, then every formula of the form* $(\bigwedge_{k=1}^{l} v_{p_k}(x - a_k) \geq \gamma_k) \wedge D_m(x - r)$ *has an infinite number of solutions.*

(4) *If $p_1, \dots, p_l \in P$ are different primes not dividing m and $\gamma_{k,j} \in \Gamma_{p_k}$, then every formula of the form*

$$\bigwedge_{k=1}^{l} \left(v_{p_k}(x - a_{k,0}) \geq \gamma_{k,0} \wedge \bigwedge_{i=1}^{n_k} v_{p_k}(x - a_{k,i}) < \gamma_{k,i} \right) \wedge D_m(x - r)$$

where for each $1 \leq k \leq l$ and $1 \leq i \leq n_k$, $\gamma_{k,i} \geq \gamma_{k,0} + \underline{n}_k$, has an infinite number of solutions. In particular, this holds if each $\gamma_{k,i} - \gamma_{k,0}$ is a nonstandard integer.

Proof. The proofs of *(1)* and *(3)* are left as an easy exercice. We prove *(2)*. By Axiom 8 for $k = n$, there are $b_0, \dots, b_{p^n-1}$ such that for all i, $v_p(b_i - a_0) \geq \gamma_0$, and for all $i \neq j$, $v_p(b_i - b_j) < \gamma_0 + \underline{n}$. Then some b_i must satisfy $\bigwedge_{m=1}^{n} v_p(x - a_m) < \gamma_m$, otherwise, since $p^n > n$, by the Pigeonhole Principle there are $i \neq j$ and m such that $v_p(b_i - a_m) \geq \gamma_m$ and $v_p(b_j - a_m) \geq \gamma_m$, and therefore also $v_p(b_i - b_j) \geq \gamma_m \geq \gamma_0 + \underline{n}$, a contradiction.
We prove *(4)*. For each $1 \leq k \leq l$, by *(2)* the formula $v_{p_k}(x - a_{k,0}) \geq \gamma_{k,0} \wedge (\bigwedge_{i=1}^{n_k} v_{p_k}(x - a_{k,i}) < \gamma_{k,i})$ has a solution b_k. Let $\gamma_k := \max\{\gamma_{k,0}, \dots, \gamma_{k,n_k}\}$. By *(3)* the formula $(\bigwedge_{k=1}^{l} v_{p_k}(x - b_k) \geq \gamma_k) \wedge D_m(x - r)$ has an infinite number of solutions $\{b'_j\}_{j \geq 1}$. By *(1)*, every b'_j is a solution to

$$\bigwedge_{k=1}^{l} \left(v_{p_k}(x - a_{k,0}) \geq \gamma_{k,0} \wedge \bigwedge_{i=1}^{n_k} v_{p_k}(x - a_{k,i}) < \gamma_{k,i} \right) \wedge D_m(x - r)$$

□

8.2 Quantifier elimination in T_P

Theorem 8.2.1. *For every nonempty set P of primes, the theory T_P eliminates quantifiers in the language L_P^E.*

Proof. As mentioned previously, we will in fact prove quantifier elimination for $T'_P \subseteq T_P$. It is enough to prove that for all models $\mathcal{M}_1$ and $\mathcal{M}_2$ of T'_P, with a common substructure A, and for all formulas $\phi(x)$ in a single variable x over A which are a conjunction of atomic or negated atomic formulas, we have $\mathcal{M}_1 \vDash \exists x \phi(x) \Rightarrow \mathcal{M}_2 \vDash \exists x \phi(x)$. Let $\mathcal{M}_1$, $\mathcal{M}_2$, A and $\phi(x)$ be such, and let $b \in \mathcal{M}_1$ be such that $\mathcal{M}_1 \vDash \phi(b)$.

As v_p is surjective for all $p \in P$, we may assume that x is of the Z sort. Since ϕ contains only finitely many symbols from L_P, we may assume for simplicity of notation that P is finite. So $\phi(x)$ is equivalent[2] to a conjunction of formulas of the forms:

(1) $n_i x = a_i$, for some $n_i \neq 0$.

(2) $n_i x \neq a_i$, for some $n_i \neq 0$.

(3) $D_{m_i}(n_i x - a_i)$, for some $n_i \neq 0$.

(4) $\neg D_{m_i}(n_i x - a_i)$, for some $n_i \neq 0$.

(5) $v_{p_\alpha}(n_{i,1}x - a_{i,1}) < v_{p_\alpha}(n_{i,2}x - a_{i,2}) + \underline{k}_i$, for some $p_\alpha \in P$, $n_{i,1} \neq 0$ or $n_{i,2} \neq 0$, and $k_i \in \mathbb{N}$.

(6) $v_{p_\alpha}(n_{i,1}x - a_{i,1}) + \underline{k}_i < v_{p_\alpha}(n_{i,2}x - a_{i,2})$, for some $p_\alpha \in P$, $n_{i,1} \neq 0$ or $n_{i,2} \neq 0$, and $k_i \in \mathbb{N}$.

(7) $v_{p_\alpha}(n_i x - a_i) \geq \gamma_i$, for some $p_\alpha \in P$ and $n_i \neq 0$.

(8) $v_{p_\alpha}(n_i x - a_i) < \gamma_i$, for some $p_\alpha \in P$ and $n_i \neq 0$.

By multiplicativity of the valuations we may assume that for all formulas of forms (5) or (6), either $n_{i,1} = n_{i,2}$, $n_{i,1} = 0$ or $n_{i,2} = 0$. Therefore, by Lemma 8.1.4, we may assume that every formula of form (5) or (6) is equivalent to a formula of form (7) or (8).

By Lemma 8.1.3, the conjunction of all the formulas of the forms (3) or (4) is equivalent to a formula of the form

$$\bigvee_j \left(D_{m_j}(x - r_j) \wedge \bigwedge_{\alpha < |P|} v_{p_\alpha}(x - s_{j,\alpha}) \geq \underline{k}_{j,\alpha} \right)$$

where for all j and α, $\gcd(m_j, p_\alpha) = 1$. As $\mathcal{M}_1 \vDash \phi(b)$, this disjunction is not empty. Let $D_m(x - r) \wedge \bigwedge_{\alpha<|P|} v_{p_\alpha}(x - s_\alpha) \geq \underline{k}_\alpha$ be one of the disjuncts which are satisfied by b. It is enough to find $b' \in \mathcal{M}_2$ which satisfies this disjunct, along with all the formulas of other forms. Note that $v_{p_\alpha}(x - s_\alpha) \geq \underline{k}_\alpha$ is of form (7), so altogether we want to find $b' \in \mathcal{M}_2$ which satisfies a conjunction of formulas of the forms:

(1) $n_i x = a_i$, $n_i \neq 0$.

(2) $n_i x \neq a_i$, $n_i \neq 0$.

[2] The negation of a formula of form (5) is $v_{p_\alpha}(n_{i,1}x - a_{i,1}) \geq v_{p_\alpha}(n_{i,2}x - a_{i,2}) + \underline{k}$, which is equivalent to $v_{p_\alpha}(n_{i,2}x - a_{i,2}) + \underline{k-1} < v_{p_\alpha}(n_{i,1}x - a_{i,1})$ if $k > 0$, which is of form (6), and to $v_{p_\alpha}(n_{i,2}x - a_{i,2}) < v_{p_\alpha}(n_{i,1}x - a_{i,1}) + 1$ if $k = 0$, which is of form (5). Similarly for the negation of a formula of form (6). Also, (7) and (8) are in essence special cases of (5) or (6), but they are required because in A the valuation may be not surjective.

(3) $D_m(x-r)$, where for all $\alpha < |P|$, $\gcd(m, p_\alpha) = 1$ (only a single such formula).

(4) $v_{p_\alpha}(n_i x - a_i) \geq \gamma_i$, $\alpha < |P|$, $n_i \neq 0$.

(5) $v_{p_\alpha}(n_i x - a_i) < \gamma_i$, $\alpha < |P|$, $n_i \neq 0$.

It is standard that we may assume that the conjunction does not contain formulas of the form (1). For each formula of the form (2), there is at most one element which does not satisfy it. So it is enough to prove that there are infinitely many elements in $\mathcal{M}_2$ which satisfy all the formulas of forms (3), (4) or (5).

Let $n := \prod_i n_i$. By multiplicativity of the valuations, the conjunction of formulas of forms (3), (4) or (5) is equivalent to the conjunction of:

(1) $v_{p_\alpha}(nx - \frac{n}{n_i} a_i) \geq \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.

(2) $v_{p_\alpha}(nx - \frac{n}{n_i} a_i) < \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.

(3) $D_{nm}(nx - nr)$.

By substituting $y = nx$, it is equivalent to satisfy:

(1) $v_{p_\alpha}(y - \frac{n}{n_i} a_i) \geq \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.

(2) $v_{p_\alpha}(y - \frac{n}{n_i} a_i) < \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.

(3) $D_{nm}(y - nr)$.

(4) $D_n(y)$.

Notice that formula (4) is already implied by formula (3). Again by Lemma 8.1.3, we may exchange $D_{nm}(y - nr)$ by a formula $D_{m'}(y - r')$, where for all $\alpha < |P|$, $\gcd(m', p_\alpha) = 1$. Also, by Lemma 8.1.6 we may assume that for each $\alpha < |P|$, there is only one formula of form (1). Altogether, it is enough to prove that in $\mathcal{M}_2$ there are infinitely many elements which satisfy the conjunction of the following formulas:

(1) $v_{p_\alpha}(x - a_{\alpha,0}) \geq \gamma_{\alpha,0}$ for all $\alpha < |P|$.

(2) $v_{p_\alpha}(x - a_{\alpha,i}) < \gamma_{\alpha,i}$ for all $\alpha < |P|$, $1 \leq i \leq n_\alpha$. $\circledast$

(3) $D_m(x-r)$, where for all $\alpha < |P|$, $\gcd(m, p_\alpha) = 1$ (only a single such formula).

By Lemma 8.1.5 (and since this formula is consistent in $\mathcal{M}_1$) we may assume that for all $\alpha < |P|$, $1 \leq i \leq n_\alpha$ we have $\gamma_{\alpha,0} < \gamma_{\alpha,i}$ and $v_{p_\alpha}(a_{\alpha,0} - a_{\alpha,i}) \geq \gamma_{\alpha,0}$. By Lemma 8.1.8 *(1)*, we may assume that for each $\alpha < |P|$, the set

$$\{(a_{\alpha,i}, \gamma_{\alpha,i}) \,:\, 1 \leq i \leq n_\alpha \,,\, \gamma_{\alpha,i} - \gamma_{\alpha,0} \text{ is a standard integer}\}$$

is an antichain with respect to $\leq_{p_\alpha}$ (Definition 8.1.7).

For each $\alpha < |P|$, let $S_\alpha = \{0 \le i \le n_\alpha : \gamma_{\alpha,i} - \gamma_{\alpha,0}$ is a standard integer$\}$ and $\gamma'_{\alpha,0} = max\{\gamma_{\alpha,i} : i \in S_\alpha\}$. For $s = 1, 2$ and for each $\alpha < |P|$, by Lemma 8.1.8 *(2)* the conjunction $v_{p_\alpha}(x - a_{\alpha,0}) \ge \gamma_{\alpha,0} \wedge \bigwedge_{i \in S_\alpha} v_{p_\alpha}(x - a_{\alpha,i}) < \gamma_{\alpha,i}$ is equivalent in $\mathcal{M}_s$ to a formula of the form $\bigvee_{i=1}^{l_\alpha} v_{p_\alpha}(x - a^s_{\alpha,0,i}) \ge \gamma'_{\alpha,0}$, where for all i, $a^s_{\alpha,0,i} \in \mathcal{M}_s$ and l_α does not depend on s. Note that $a^s_{\alpha,0,i}$ may not be in A. Furthermore, by Lemma 8.1.8 *(2)*, $v_{p_\alpha}(a^s_{\alpha,0,i} - a_{\alpha,0}) \ge \gamma_{\alpha,0}$ and for $i \ne j$, $v_{p_\alpha}(a^s_{\alpha,0,i} - a^s_{\alpha,0,j}) < \gamma'_{\alpha,0}$.

Together, the conjunction of the formulas in $\circledast$ is equivalent in $\mathcal{M}_s$ to the disjunction $\psi_s = \bigvee_{k=1}^{l} \psi_{s,k}$, where for each k, $\psi_{s,k}$ is the conjunction of the following formulas:

(1) $v_{p_\alpha}(x - a^s_{\alpha,0,k}) \ge \gamma'_{\alpha,0}$, for all $\alpha < |P|$.

(2) $v_{p_\alpha}(x - a_{\alpha,i}) < \gamma_{\alpha,i}$, for all $\alpha < |P|$, $i \notin S_\alpha$ (so $\gamma_{\alpha,0} < \gamma_{\alpha,i}$ and $\gamma_{\alpha,i} - \gamma_{\alpha,0}$ is not a standard integer).

(3) $D_m(x - r)$, where for all $\alpha < |P|$, $\gcd(m, p_\alpha) = 1$ (only a single such formula).

Furthermore, $l = \prod_{\alpha < |P|} l_\alpha$ does not depend on s.

Since ψ_1 is consistent in $\mathcal{M}_1$ (satisfied by nb), the disjunction for $s = 1$ is not empty, i.e., $l \ge 1$. And since l does not depend on s, the disjunction for $s = 2$ is also not empty. Consider one such disjunct, $\psi_{2,k}$. By Lemma 8.1.9 *(4)*, it has an infinite number of solutions. This completes the proof. □

Corollary 8.2.2. *T'_P is a complete theory. Hence $T'_P = T_P$.*

Proof. By quantifier elimination, it is enough to show that T'_P decides every atomic sentence. These are just the sentences equivalent to one of the forms: $\underline{n}_1 = \underline{n}_2$ in any sort, $\underline{k}_1 <_p \underline{k}_2$ in Γ_p, $D_m(\underline{n})$ in the Z sort and $v_p(\underline{n}_1) < v_p(\underline{n}_2)$ in the Z sort, all of which are clearly decided by T'_P. □

Remark 8.2.3. Suppose $\mathcal{M} \models T_P$ and $\phi(x)$ is a consistent formula in a single variable with parameters from $\mathcal{M}$. Then by quantifier elimination and Lemmas 8.1.3 and 8.1.4, $\phi(x)$ is equivalent to a disjunction of formulas, which are either of the form $x = a$ or of the form

$$D_m(x - r) \wedge \bigwedge_j nx \ne a_j \wedge \bigwedge_{p \in F} \left(v_p(n_p x - a_{p,0}) \ge \gamma_{p,0} \wedge \bigwedge_{i=1}^{l_p} v_p(n_p x - a_{p,i}) < \gamma_{p,i} \right),$$

where $F \subseteq P$ is finite. Moreover, one may assume $\gcd(n_p, p) = 1$.

For p a single prime number and $\mathcal{M} \models T_p$, the following lemma says that the definable subgroups of $(\mathcal{M}, +)$ are only those of the form $m\mathcal{M} \cap \{a \in \mathcal{M} : v(a) \ge \gamma\}$, for $m \in \mathbb{Z}$ and $\gamma \in \Gamma$ and for each such defining formula, there are only finitely many possible m's when varying the parameters of the formula.

Lemma 8.2.4 (Uniformly definable subgroups)**.** *For a single prime p, let $\phi(x, y)$ be any L_p^M-formula, and let $\theta(y)$ be the formula for "$(\phi(x, y)\, , +)$ is a subgroup". Then there are*

$n_1, \ldots, n_k \geq 1$, *having* $gcd(n_i, p) = 1$ *for each* i, *such that the following sentence is true in* T_p*:*

$$\forall y \left(\theta(y) \to \bigvee_{i=1}^{k} \exists w \forall x (\phi(x,y) \leftrightarrow (D_{n_i}(x) \wedge (v_p(x) \geq v_p(w))) \right).$$

Proof. It is enough to work in $\mathbb{Z}$. By quantifier elimination (and Lemma 8.1.3 (2)), $\phi(x,y)$ is equivalent to a formula of the form $\bigvee_i \bigwedge_j \phi_{i,j}(x,y)$, where for each i, j, $\phi_{i,j}(x,y)$ is one of the following:

(1) $t_{i,j}(x,y) = 0$, where $t_{i,j}(x,y)$ is a $\{+,-,1\}$-term, i.e., of the form $k_{i,j}x + l_{i,j}y + \underline{r}_{i,j}$ for $k_{i,j}, l_{i,j}, r_{i,j} \in \mathbb{Z}$.

(2) $t_{i,j}(x,y) \neq 0$, where $t_{i,j}(x,y)$ is a $\{+,-,1\}$-term.

(3) $v(t_{i,j}(x,y)) \geq v(s_{i,j}(x,y))$, where $t_{i,j}(x,y)$, $s_{i,j}(x,y)$ are $\{+,-,1\}$-terms (note that $v(t_{i,j}(x,y)) < v(s_{i,j}(x,y))$ is equivalent to $v(p \cdot t_{i,j}(x,y)) \leq v(s_{i,j}(x,y))$, which is of the same form).

(4) $D_{m_{i,j}}(t_{i,j}(x,y))$, where $t_{i,j}(x,y)$ is a $\{+,-,1\}$-term and $\gcd(m_{i,j}, p) = 1$.

For each i, let $J_i = \{j : \phi_{i,j}(x,y) \text{ is of the form } D_{m_{i,j}}(t_{i,j}(x,y))\}$, and let $m_i = \prod_{j \in J_i} m_{i,j}$. As in the proof of Lemma 8.1.3 (1), the satisfaction of the formula $D_{m_{i,j}}(t_{i,j}(x,y))$ depends only on the reminders of x and y mod $m_{i,j}$, which are determined by the reminders of x and y mod m_i. So there is a set $R_i \subseteq \{0, 1, \ldots, m_i - 1\}^2$ such that $\bigwedge_{j \in J_i} \phi_{i,j}(x,y)$ is equivalent to $\bigvee_{(r,s) \in R_i} (D_{m_i}(x - \underline{r}) \wedge D_{m_i}(y - \underline{s}))$. Therefore, $\phi(x,y)$ is equivalent to a formula of the form $\bigvee_i (D_{m_i}(x - \underline{r}_i) \wedge D_{m_i}(y - \underline{s}_i) \wedge \bigwedge_j \phi_{i,j}(x,y))$, where $\gcd(m_i, p) = 1$ and for each i, j, $\phi_{i,j}(x,y)$ is one of the following:

(1) $t_{i,j}(x,y) = 0$, where $t_{i,j}(x,y)$ is a $\{+,-,1\}$-term.

(2) $t_{i,j}(x,y) \neq 0$, where $t_{i,j}(x,y)$ is a $\{+,-,1\}$-term.

(3) $v(t_{i,j}(x,y)) \geq v(s_{i,j}(x,y))$, where $t_{i,j}(x,y)$, $s_{i,j}(x,y)$ are $\{+,-,1\}$-terms.

For each i, let $\phi_i(x,y)$ be the i'th disjunct, i.e., the formula $D_{m_i}(x - \underline{r}_i) \wedge D_{m_i}(y - \underline{s}_i) \wedge \bigwedge_j \phi_{i,j}(x,y)$.

Let $b \in \mathbb{Z}$ be such that $\phi(\mathbb{Z}, b)$ is a subgroup. If $\phi(\mathbb{Z}, b)$ is finite, it must be $\{0\}$. To account for this case, we may take $n_1 = 1$, and for $w = 0$ we have that $\phi(x,b)$ is equivalent to $D_{n_1}(x) \wedge (v_p(x) \geq v_p(0))$. If $\phi(\mathbb{Z}, b)$ is infinite, then $\phi(\mathbb{Z}, b) = n\mathbb{Z}$ for some $n \geq 1$. Moreover, there must be an i_0 such that $\phi_{i_0}(\mathbb{Z}, b)$ is infinite. So $D_{m_{i_0}}(b - \underline{s}_{i_0})$ holds, hence $\phi_{i_0}(x,b)$ is equivalent to just $D_{m_{i_0}}(x - \underline{r}_{i_0}) \wedge \bigwedge_j \phi_{i_0,j}(x,b)$. As $\phi(\mathbb{Z}, b)$ is infinite, it is clear that no formula $\phi_{i_0,j}(x,y)$ is of the form (1), hence $\phi_{i_0}(x,b)$ is equivalent to $D_{m_{i_0}}(x - \underline{r}_{i_0}) \wedge \bigwedge_j \phi_{i_0,j}(x,b)$, where for each j, $\phi_{i_0,j}(x,b)$ is one of the following:

(1) $k_{i_0,j}x \neq c_{i_0,j}$.

(2) $v(k'_{i_0,j}x - c'_{i_0,j}) \geq v(k''_{i_0,j}x - c''_{i_0,j})$.

Applying Lemma 8.1.4 to formulas as in (2), we may assume that $\phi_{i_0}(x,b)$ is equivalent to $D_{m_{i_0}}(x-\underline{r}_{i_0}) \wedge \bigwedge_j \phi_{i_0,j}(x,b)$, where for each j, $\phi_{i_0,j}(x,b)$ is one of the following:

(1) $k_{i_0,j}x \neq c_{i_0,j}$.

(2) $v(k_{i_0,j}x - c_{i_0,j}) \geq \gamma_{i_0,j}$.

(3) $v(k_{i_0,j}x - c_{i_0,j}) < \gamma_{i_0,j}$.

The formula $v(k_{i_0,j}x - c_{i_0,j}) \geq \gamma_{i_0,j}$ defines a coset of $p^{\gamma_{i_0,j}}\mathbb{Z}$, and the formula $v(k_{i_0,j}x - c_{i_0,j}) < \gamma_{i_0,j}$ defines a finite union of cosets of $p^{\gamma_{i_0,j}}\mathbb{Z}$. Let

$$J = \{j : \phi_{i_0,j}(x,b) \text{ is of form 2 or 3}\}$$

and let $\delta = \max\{\gamma_{i_0,j} : j \in J\}$. Then for every $j \in J$, every coset of $p^{\gamma_{i_0,j}}\mathbb{Z}$ is a finite union of cosets of $p^\delta\mathbb{Z}$. So $\bigcap_{j\in J} \phi_{i_0,j}(\mathbb{Z},b)$ is a finite intersection of finite unions of cosets of $p^\delta\mathbb{Z}$, and hence is itself just a finite union of cosets of $p^\delta\mathbb{Z}$ (since every two cosets are either equal or disjoint). Therefore, $\phi_{i_0}(\mathbb{Z},b)$ is a set of the form $U\backslash F$, where F is a finite set (the set of points excluded by the inequalities $k_{i_0,j}x \neq c_{i_0,j}$), and U is a finite union of the form $\bigcup_{j=1}^N((m_{i_0}\mathbb{Z}+r_{i_0})\cap(p^\delta\mathbb{Z}+c_j))$. For each j, $(m_{i_0}\mathbb{Z}+r_{i_0})\cap(p^\delta\mathbb{Z}+c_j)$ is a coset of $m_{i_0}p^\delta\mathbb{Z}$ (it is not empty, since $\gcd(m_{i_0},p)=1$), so U is of the form $\bigcup_{j=1}^N(m_{i_0}p^\delta\mathbb{Z}+d_j)$. As $\phi_{i_0}(\mathbb{Z},b)$ is infinite, this union is not empty.

Now, $(m_{i_0}p^\delta\mathbb{Z}+d_1)\backslash F \subseteq U\backslash F = \phi_{i_0}(\mathbb{Z},b) \subseteq \phi(\mathbb{Z},b) = n\mathbb{Z}$, so n divides $m_{i_0}p^\delta$ since F is finite. Write $n = n'p^\gamma$ with $\gcd(n',p)=1$. Then $n'|m_{i_0}$, and in particular, $n' \leq m_{i_0}$. So $\phi(x,b)$ is equivalent to $D_n(x)$, which is equivalent to $D_{n'}(x) \wedge v(x) \geq \gamma$, and $n' \leq m_{i_0}$. Recall that i_0 depends on b, but there are only finitely many i's, so $m = \max\{m_i\}$ exists, and hence, for any b such that $\phi(x,b)$ is a subgroup, there is an $n' \leq m$ with $\gcd(n',p)=1$, and there is a γ such that $\phi(x,b)$ is equivalent to $D_{n'}(x) \wedge v(x) \geq \gamma$, and we are done. $\square$

8.3 dp-rank of T_P

Quantifier elimination now enables us to determine the dp-rank of T_P. Recall from Section 1.4.3 the definition of dp-rk. In this section, we work in the one-sorted language L_P^E.

Proposition 8.3.1. *For any prime p, T_p is dp-minimal (in the one-sorted language L_P^E).*

Proof. We set $L = L_p^E$ and $T = T_p$. Let L^- contain the symbols of L, except for the divisibility relations $\{D_n\}_{n\geq 1}$. Let $\mathcal{Z}^-$ be the reduct of $\mathcal{Z}_p$ to L^-. Let $\mathbb{Q}_p^-$ be $\mathbb{Q}_p$ as an L^--structure. It is a reduct of the structure $(\mathbb{Q}_p,+,-,\cdot,0,1,|_p)$, which is dp-minimal (see [DGL11, Theorem 6.6]), and therefore is also dp-minimal. Note that $\mathcal{Z}^-$ is a substructure of $\mathbb{Q}_p^-$.

Let $L' = L \cup \{Z\}$. Interpret Z in $\mathbb{Q}_p$ as $\mathbb{Z}$, and interpret each D_n such that $D_n \cap \mathbb{Z}$ is the usual divisibility relation and $D_n \cap (\mathbb{Q}_p \backslash \mathbb{Z}) = \emptyset$, thus making it an L'-structure $\mathbb{Q}'_p$. Let $\mathcal{M}$ be an ω_1-saturated model of $Th(\mathbb{Q}'_p)$, and let $A = Z(\mathcal{M})$ be the interpretation of Z in it. Then A is an ω_1-saturated model of T.

Suppose that T is not dp-minimal. Then, by Fact 1.4.16, there is an ict-pattern of length 2, hence there are formulas $\phi(x, y)$, $\psi(x, z)$ in L with $|x| = 1$, and elements $(b_i : i < \omega)$, $(c_j : j < \omega)$, $(a_{i,j} : i, j < \omega)$ in A such that $\phi(a_{i,j}, b_{i'})$ if and only if $i = i'$ and $\psi(a_{i,j}, c_{j'})$ if and only if $j = j'$. By Theorem 8.2.1 we may assume that ϕ, ψ are quantifier-free and in disjunctive normal form. Let N be the largest n such that D_n appears in ϕ or ψ. Color each pair (i, j) such that $i > j$ by $a_{i,j} \bmod N!$. By Ramsey Theorem, we may assume that all the elements $a_{i,j}$ with $i > j$ have the same residue modulo $N!$, and so modulo all $n \leq N$.

Write ϕ as $\bigvee_k \bigwedge_l (\phi'_{k,l} \wedge \phi''_{k,l})$ and ψ as $\bigvee_k \bigwedge_l (\psi'_{k,l} \wedge \psi''_{k,l})$, where $\phi'_{k,l}$, $\psi'_{k,l}$ are atomic or negated atomic L^--formulas and $\phi''_{k,l}$, $\psi''_{k,l}$ are atomic or negated atomic formulas containing no relations other than $\{D_n\}_{n \geq 1}$. For each k, denote by ϕ_k, ψ_k the formulas $\bigwedge_l (\phi'_{k,l} \wedge \phi''_{k,l})$ and $\bigwedge_l (\psi'_{k,l} \wedge \psi''_{k,l})$ respectively.

For every $i > j$ we have $\phi(a_{i,j}, b_i)$, so there is a $k_{i,j}$ such that $\phi_{k_{i,j}}(a_{i,j}, b_i)$. Again by Ramsey Theorem, we may assume that all the $k_{i,j}$'s are equal to some k_0, so for every $i > j$ we have $\phi_{k_0}(a_{i,j}, b_i)$. For every $i' \neq i$ we have $\neg\phi(a_{i',j}, b_i)$, so in particular $\neg\phi_{k_0}(a_{i',j}, b_i)$. Similarly, we may assume that for some k_1, for every $i > j$ we have $\psi_{k_1}(a_{i,j'}, c_j)$ if and only if $j = j'$.

Let ϕ'_k, ψ'_k be the formulas obtained from ϕ_k, ψ_k respectively, by deleting all the formulas $\phi''_{k,l}$, $\psi''_{k,l}$. So ϕ'_k, ψ'_k are L^--formulas.

For every $m \in \mathbb{N}$, let $I_m = \{m + 1, \ldots, 2m\}$, $J_m = \{1, \ldots, m\}$. For every $(i, j) \in I_m \times J_m$, we have $\phi_{k_0}(a_{i,j}, b_i)$ and therefore also $\phi'_{k_0}(a_{i,j}, b_i)$. Let $i \neq i' \in I_m$, and suppose for a contradiction that $\phi'_{k_0}(a_{i',j}, b_i)$, i.e. $\bigwedge_l (\phi'_{k_0,l}(a_{i',j}, b_i))$. But we know that $\neg\phi_{k_0}(a_{i',j}, b_i)$, so for some l_0 we have $\neg\phi'_{k_0,l_0}(a_{i',j}, b_i) \vee \neg\phi''_{k_0,l_0}(a_{i',j}, b_i)$. Therefore, we get $\neg\phi''_{k_0,l_0}(a_{i',j}, b_i)$. But from $\phi_{k_0}(a_{i,j}, b_i)$ we also get $\phi''_{k_0,l_0}(a_{i,j}, b_i)$. Together, this contradicts the fact that all the elements $a_{i,j}$ with $i > j$ have the same residue modulo all $n \leq N$.

Altogether, in A, for every $(i, j) \in I_m \times J_m$ we have $\phi'_{k_0}(a_{i,j}, b_{i'})$ if and only if $i = i'$, and similarly also $\psi'_{k_1}(a_{i,j}, c_{j'})$ if and only if $j = j'$. Since ϕ'_{k_0}, ψ'_{k_1} are quantifier-free, and A is a substructure of $\mathcal{M}$, this holds also in $\mathcal{M}$. As m is arbitrary, this contradicts the dp-minimality of $Th(\mathbb{Q}_p^-)$. □

Theorem 8.3.2. *For every nonempty set P of primes, dp-rank$(T_P) = |P|$.*

Proof. dp-rank$(T_P) \leq |P|$ follows from Proposition 8.3.1 and Lemma 1.4.17 for $L_P^E = \bigcup_{\alpha < |P|} L_{p_\alpha}^E$. For $\alpha < |P|$ let $\phi_\alpha(x, y)$ be the formula $x|_{p_\alpha} y \wedge y|_{p_\alpha} x$ (i.e. $v_{p_\alpha}(x) = v_{p_\alpha}(y)$), and for $\alpha < |P|$, $i \in \mathbb{N}$ let $a_{\alpha,i}$ be such that $v_{p_\alpha}(a_{\alpha,i}) = i$. Let $F \subseteq |P|$ be finite. By Lemma 8.1.9 *(4)*, for every $\eta : F \to \mathbb{N}$ there is a b_η such that for every $\alpha \in F$, $v_{p_\alpha}(b_\eta) = v_{p_\alpha}(a_{\alpha,\eta(\alpha)})$. If P is finite, just take $F = |P|$. Otherwise, by compactness, there are such b_η for $F = |P|$ as well. These $\phi_\alpha(x, y)$, $a_{\alpha,i}$ and b_η form an ict-pattern of length $|P|$, hence, by Fact 1.4.16 dp-rank$(T_P) \geq |P|$. □

CHAPTER 9

Minimality phenomena

In this chapter, we show that there is no intermediate structures between $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, +, 0, <)$, and between $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, +, 0, |_p)$. Those are two *minimal* expansions of $(\mathbb{Z}, +, 0)$. We also introduce a fine notion of *reduct* which allows us to extend these minimality results to elementary extensions. We finish by some counter-examples of minimality in elementary extensions.

Contents

9.1 Minimality and Conant's example

Definition 9.1.1. Let L_1 and L_2 be two first-order languages, and let $\mathcal{M}_1$ be an L_1-structure and $\mathcal{M}_2$ an L_2-structure, both with the same underlying universe M. Let $A \subseteq M$ be a set of parameters.

(1) We say that $\mathcal{M}_1$ is an A-*reduct* of $\mathcal{M}_2$, and $\mathcal{M}_2$ is an A-*expansion* of $\mathcal{M}_1$, if for every $n \geq 1$, every subset of M^n which is L_1-definable over $\emptyset$ (equivalently, over A) is also L_2-definable over A. When $A = M$ we just say that $\mathcal{M}_1$ is a *reduct* of $\mathcal{M}_2$, and $\mathcal{M}_2$ is an *expansion* of $\mathcal{M}_1$. We will mostly use this with either $A = \emptyset$ or $A = M$.

(2) We say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are A-*interdefinable* if $\mathcal{M}_1$ is an A-reduct of $\mathcal{M}_2$ and $\mathcal{M}_2$ is an A-reduct of $\mathcal{M}_1$. When $A = M$ we just say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are *interdefinable.*

(3) Let $A \subseteq B \subseteq M$ be another set of parameters. We say that $\mathcal{M}_1$ is a B-*proper* A-*reduct* of $\mathcal{M}_2$, and $\mathcal{M}_2$ is a B-*proper* A-*expansion* of $\mathcal{M}_1$, if $\mathcal{M}_1$ is an A-reduct of $\mathcal{M}_2$, but $\mathcal{M}_2$ is not a B-reduct of $\mathcal{M}_1$. When $B = M$ we just say *proper* instead of B-proper. We will mostly use this with either $B = M$ or $B = \emptyset$.

Let $\mathcal{M}_1$ be an L_1-structure and $\mathcal{M}_2$ an L_2-structure, both with the same underlying universe M, and suppose that $\mathcal{M}_1$ is a $\emptyset$-reduct of $\mathcal{M}_2$. Then we can replace L_2 by $L_2 \cup L_1$, interpreting each L_1-symbol in $\mathcal{M}_2$ as it is interpreted in $\mathcal{M}_1$. As we have not added new $\emptyset$-definable sets, this new structure is $\emptyset$-interdefinable with the original $\mathcal{M}_2$. Therefore we may always assume for simplicity of notation that $L_1 \subseteq L_2$ and $\mathcal{M}_1 = \mathcal{M}_2|_{L_1}$.

A-reducts are preserved by elementary extensions and elementary substructures containing A, in the following sense:

Lemma 9.1.2. *Let $\mathcal{M} \prec \mathcal{N}$ be two L-structures with universes M and N respectively. Let $A \subseteq M$ and let $\mathcal{N}'$ be an A-reduct of $\mathcal{N}$ with language L'. Let $\mathcal{M}'$ be the structure obtained by restricting the relations and functions of $\mathcal{N}'$ to M. Then:*

(1) $\mathcal{M}'$ is well-defined, it is an A-reduct of $\mathcal{M}$, and $\mathcal{M}' \prec \mathcal{N}'$.

(2) $\mathcal{N}'$ is an A-proper A-reduct of $\mathcal{N}$ if and only if $\mathcal{M}'$ is an A-proper A-reduct of $\mathcal{M}$.

(3) $\mathcal{N}'$ is a proper A-reduct of $\mathcal{N}$ if and only if $\mathcal{M}'$ is a proper A-reduct of $\mathcal{M}$.

The proof of the previous Lemma is trivial.

Remark 9.1.3. Lemma 9.1.2 is not necessarily true if $A \not\subseteq M$. If $\mathcal{N}'$ contains a constant $c \notin M$, or a n-ary function f such that $f(M^n) \not\subseteq M$, then $\mathcal{M}'$ is not well-defined. Even when it is well-defined, the rest is still not necessarily true. For example, let $\mathcal{M} = (\mathbb{Z}, +, 0, 1, <)$, and let $\mathcal{N} = (N, +, 0, 1, <)$ be a nontrivial elementary extension of $\mathcal{M}$. Let $b \in N$ be a positive infinite element, and let $\mathcal{N}' = (N, +, 0, 1, [0, b])$. Then $\mathcal{M}' = (\mathbb{Z}, +, 0, 1, \mathbb{N}) \not\prec \mathcal{N}'$ (as $[0, b]$ contains an element $x = b$ such that $x \in [0, b]$ but $x + 1 \notin [0, b]$). Also, $\mathcal{M}'$ is interdefinable with $\mathcal{M}$, but we will see that $\mathcal{N}'$ is a proper reduct of $\mathcal{N}$.

Definition 9.1.4. Let $\mathcal{F}$ be a family of first-order structures, and let $\mathcal{M} \in \mathcal{F}$. We say that $\mathcal{M}$ is A-*minimal* in $\mathcal{F}$ if there are no A-proper A-reducts of $\mathcal{M}$ in $\mathcal{F}$. We say that $\mathcal{M}$ is A-*maximal* in $\mathcal{F}$ if there are no A-proper A-expansions of $\mathcal{M}$ in $\mathcal{F}$. When $A = M$ we just say that $\mathcal{M}$ is *minimal* or *maximal*, respectively.

Based on a result by Palacín and Sklinos [PS18], Conant and Pillay proved in [CP18] the following:

Fact 9.1.5 ([CP18] Theorem 1.2)**.** $(\mathbb{Z}, +, 0, 1)$ *has no proper stable expansions of finite dp-rank.*

In other words, $(\mathbb{Z}, +, 0, 1)$ is maximal among the stable structures of finite dp-rank.

Remark 9.1.6. This theorem is no longer true if we replace $(\mathbb{Z}, +, 0, 1)$ by an elementarily equivalent structure $(N, +, 0, 1)$. Let $(N, +, 0, 1, |_p)$ be a nontrivial elementary extension of $(\mathbb{Z}, +, 0, 1, |_p)$, let $b \in N$ be such that $\gamma := v_p(b)$ is nonstandard, and let $B = \{a \in N : b|_p a\} = \{a \in N : v_p(a) \geq \gamma\}$. Then $(N, +, 0, 1, B)$ is a proper expansion of $(N, +, 0, 1)$ of dp-rank 1, and in Proposition 9.3.2 we show that it is also stable.

In this section, we give another proof of the following fact, due to Conant.

Fact 9.1.7 ([Con18] Theorem 1.1)**.** $(\mathbb{Z}, +, 0, 1, <)$ *is minimal among the proper expansions of* $(\mathbb{Z}, +, 0, 1)$.

Remark 9.1.8. This is no longer true if we replace $(\mathbb{Z}, +, 0, 1, <)$ by an elementarily equivalent structure. Let $(N, +, 0, 1, <)$ be a nontrivial elementary extension of $(\mathbb{Z}, +, 0, 1, <)$, let $b \in N$ be a positive infinite element, and let $B = [0, b]$. Then $(N, +, 0, 1, B)$ is a proper expansion of $(N, +, 0, 1)$, and in Proposition 9.3.5 we show that it is also a proper reduct of $(N, +, 0, 1, <)$. Note that the formula $y - x \in B$ defines the ordering on B, so this structure is unstable. We will see (Remark 9.1.11) that every structure which is a proper expansion of $(N, +, 0, 1)$ and a reduct of $(N, +, 0, 1, <)$, and which has a definable one-dimensional set which is not definable in $(N, +, 0, 1)$, defines a set of the form $[0, b]$ for a positive infinite b. Hence a stable intermediate structure between $(N, +, 0, 1, <)$ and $(N, +, 0, 1)$, if such exists, cannot contain new definable sets of dimension one.

Nevertheless, a weaker version of Fact 9.1.7 does hold as well for elementarily equivalent structures.

Corollary 9.1.9. *Let* $(N, +, 0, 1, <)$ *be an elementary extension of* $(\mathbb{Z}, +, 0, 1, <)$*. Then* $(N, +, 0, 1, <)$ *is* $\emptyset$*-minimal among the* $\emptyset$*-proper* $\emptyset$*-expansions of* $(N, +, 0, 1)$.

Proof of Corollary 9.1.9 from Fact 9.1.7. As $(\mathbb{Z}, +, 0, 1, <)$ is a $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$, by Fact 9.1.7 it is obviously minimal among the proper $\emptyset$-expansions of $(\mathbb{Z}, +, 0, 1)$. In

$(\mathbb{Z}, +, 0, 1)$, every element is $\emptyset$-definable, so a proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$ is the same as a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$. Now if $\mathcal{N}$ is a $\emptyset$-proper $\emptyset$-reduct of $(\mathbb{Z}, +, 0, 1, <)$, and a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$, then also in $\mathcal{N}$ every element is $\emptyset$-definable, so $\mathcal{N}$ is a proper reduct of $(\mathbb{Z}, +, 0, 1, <)$. Hence $(\mathbb{Z}, +, 0, 1, <)$ is $\emptyset$-minimal among the $\emptyset$-proper $\emptyset$-expansions of $(\mathbb{Z}, +, 0, 1)$. By Lemma 9.1.2, we conclude. □

Lemma 1.4.3 allows us to give a simple proof for the unstable case of Corollary 9.1.9:

Theorem 9.1.10 (Conant, Unstable case of Corollary 9.1.9)**.** *Let* $(N, +, 0, 1, <)$ *be an elementary extension of* $(\mathbb{Z}, +, 0, 1, <)$*. Then* $(N, +, 0, 1, <)$ *is* $\emptyset$*-minimal among the* unstable $\emptyset$*-proper* $\emptyset$*-expansions of* $(N, +, 0, 1)$.

Proof. Let $\mathcal{N}$ be any unstable structure with universe N, which is a $\emptyset$-proper $\emptyset$-expansion of $(N, +, 0, 1)$ and a $\emptyset$-reduct of $(N, +, 0, 1, <)$. We show that $\mathcal{N}$ is $\emptyset$-interdefinable with $(N, +, 0, 1, <)$. It is enough to show that $x \geq 0$ is definable over $\emptyset$ in $\mathcal{N}$. Let L be the language of $\mathcal{N}$, $L^- = \{+, 0, 1\}$ and $L_< = \{+, 0, 1, <\}$. We may expand all these languages by adding the symbols $\{-\} \cup \{D_n : n \geq 1\}$, as all of them are already definable over $\emptyset$ in all three languages. As $\mathcal{N}$ is a $\emptyset$-expansion of $(N, +, 0, 1)$ and a $\emptyset$-reduct of $(N, +, 0, 1, <)$, we may replace L with $L \cup L^-$ and $L_<$ with $L_< \cup L \cup L^-$ without adding new $\emptyset$-definable sets to any structure. So we may assume that $L^- \subseteq L \subseteq L_<$.

Let $\mathcal{M}$ be a monster model for $Th(\mathbb{Z}, +, 0, 1, <)$, so $\mathcal{M}|_L$ is a monster for $Th(\mathcal{N})$. As $(N, +, 0, 1)$ is stable but $\mathcal{N}$ is not, by Lemma 1.4.3 there exist an L-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ and $b \in \mathcal{M}$ such that $\phi(x, b)$ is not L^--definable with parameters in $\mathcal{M}$. By quantifier elimination in $Th(\mathbb{Z}, +, 0, 1, <)$ and Lemma 8.1.3 (1) (which is a theorem of $Th(\mathbb{Z}, +, 0, 1)$), $\phi(x, b)$ is equivalent to a formula of the form

$$\bigvee_i (D_{m_i}(x - k_i) \wedge x \in [c_i, c_i'])$$

where $c_i, c_i' \in M \cup \{-\infty, +\infty\}$ and $[c_i, c_i']$ denotes the closed interval except if one of the bound is infinite, in which case it is open on the infinite side. Let $m = \prod_i m_i$. As each formula of the form $D_{m_i}(x - k)$ is equivalent to a disjunction of formulas of the form $D_m(x - k')$, we can rewrite this as

$$\bigvee_i (D_m(x - k_i) \wedge x \in [c_i, c_i'])$$

(with possibly different k_i's and numbering). By grouping together disjuncts with the same k_i, we can rewrite this as

$$\bigvee_i (D_m(x - k_i) \wedge \bigvee_j x \in [c_{i,j}, c_{i,j}'])$$

where for $i_1 \neq i_2$, $k_{i_1} \not\equiv k_{i_2} \mod m$. As this formula is equivalent to $\phi(x, b)$, which is not L^--definable with parameters in $\mathcal{M}$, there must be an i_0 such that $D_m(x - k_{i_0}) \wedge \bigvee_j x \in [c_{i_0,j}, c_{i_0,j}']$ is not L^--definable with parameters in $\mathcal{M}$. This latter formula, which we denote by $\phi_{i_0}(x,b)$, is equivalent to $\phi(x,b) \wedge D_m(x-k_{i_0})$, and so is L-definable. Let $\psi(x,b)$ be the formula $\phi_{i_0}(mx+k_{i_0},b)$. Then $\psi(x,b)$ is L-definable and equivalent to just $\bigvee_j mx+k_{i_0} \in [c_{i_0,j}, c'_{i_0,j}]$. This substitution is reversible as $\phi_{i_0}(x,b)$ is equivalent to $D_m(x-k_{i_0}) \wedge \psi(\frac{x-k_{i_0}}{m}, b)$, therefore also $\psi(x,b)$ is not L^--definable with parameters in $\mathcal{M}$. Each formula of the form $mx+k \in [c,c']$ is equivalent to the formula $x \in [\lfloor \frac{c-k}{m} \rfloor, \lceil \frac{c'-k}{m} \rceil]$, so we can rewrite $\psi(x,b)$ as $\bigvee_{i=1}^n x \in [c_i, c'_i]$. By reordering and combining intersecting intervals, we may assume that the intervals are disjoint and increasing, i.e., for all $i < n$, $c'_i \leq c_{i+1}$.

Now we show how from $\psi(x,b)$ we can get an L-definable formula equivalent to $[0,a]$, for a a positive nonstandard integer in $\mathcal{M}$. For each i, if $[c_i, c'_i]$ defines in $\mathcal{M}$ a finite set then it is L^--definable, and so $\psi(x,b) \wedge \neg x \in [c_i, c'_i]$ is also L-definable but not L^--definable (since $(\psi(x,b) \wedge \neg x \in [c_i,c'_i]) \vee x \in [c_i, c'_i]$ is again equivalent to $\psi(x,b)$). So we may assume that for all i, $[c_i, c'_i]$ is infinite. Note that as $\psi(x,b)$ is not L^--definable, it cannot be empty.

We want $\psi(x,b)$ to have a lower bound, i.e., $-\infty < c_1$. If $c_1 = -\infty$ but $c'_n \neq +\infty$, then we can just replace $\psi(x,b)$ with $\psi(-x,b)$. If both $c_1 = -\infty$ and $c'_n = +\infty$, we can replace $\psi(x,b)$ with $\neg\psi(x,b)$ and again remove all finite intervals. In both cases, $\psi(x,b)$ is still L-definable but not L^--definable, so it is still a nonempty disjunction of infinite disjoint intervals.

By replacing $\psi(x,b)$ with $\psi(x+c_1+1,b)$ we may assume that $c_1 = 0$, so the left-most interval is $[0,c'_1]$. If $c'_1 \neq +\infty$ let $a' = c'_1$, otherwise let $a' \in \mathcal{M}$ be any positive nonstandard integer. Let $\theta(x,b')$ denote the formula $\psi(x,b) \wedge \psi(a'-x,b)$. Then $\theta(x,b')$ is L-definable and equivalent to the infinite interval $[0,a']$. The proof of the following claim is an obvious consequence of quantifier elimination for Presburger arithmetic and is left to the reader.

Claim 1. For every $c \geq 0$ there exist $a > c$ and b such that $\theta(x,b)$ is equivalent to the interval $[0,a]$.

In particular, as N is a small subset of $\mathcal{M}$, there exists $c \in \mathcal{M}$ bigger than all elements of N. By the claim, there exist $\tilde{a} > c$ and $\tilde{b}$ such that $\theta(x,\tilde{b})$ is equivalent to the interval $[0,\tilde{a}]$, and so $\theta(N,\tilde{b}) = \{s \in N : s \geq 0\}$.

Let $\chi(y,z)$ be the formula $\chi_1(y,z) \wedge \chi_2(y,z) \wedge \chi_3(y,z)$ where:

- $\chi_1(y,z)$ is the formula $\theta(0,z) \wedge \theta(y,z) \wedge \neg\theta(-1,z) \wedge \neg\theta(y+1,z) \wedge \neg\theta(2y,z)$.
- $\chi_2(y,z)$ is the formula $\forall w((w \neq 0 \wedge \theta(w,z)) \rightarrow \theta(w-1,z))$.
- $\chi_3(y,z)$ is the formula $\forall w((w \neq y \wedge \theta(w,z)) \rightarrow \theta(w+1,z))$.

So $\chi(y,z)$ is L-definable over $\emptyset$.

Claim 2. For every $a,b \in \mathcal{M}$, $\mathcal{M} \vDash \chi(a,b)$ if and only if $a > 0$ and $\theta(\mathcal{M},b) = [0,a]$.

Proof of the claim. This can be formulated as a first order sentence in $L_<$ without parameters:

$$\mathcal{M} \vDash \forall y, z(\chi(y, z) \leftrightarrow (y > 0 \wedge \forall x(\theta(x, z) \leftrightarrow 0 \leq x \leq y))),$$

so it is enough to prove this for $\mathbb{Z}$. Let $a, b \in \mathbb{Z}$. If $a > 0$ and $\theta(\mathbb{Z}, b) = [0, a]$, then clearly $\mathbb{Z} \vDash \chi(a, b)$. Suppose $\mathbb{Z} \vDash \chi(a, b)$, and set $A := \theta(\mathbb{Z}, b)$. By χ_1, $0, a \in A$ and $-1, a+1, 2a \notin A$. Suppose towards contradiction that $a < 0$. Then from χ_2 it follows by induction that $(-\infty, a] \subseteq A$. But then $2a \in A$, a contradiction. So $a \geq 0$. If $a = 0$ then again $2a \in A$ is a contradiction. So $a > 0$. From χ_2 it follows by induction that $[0, a] \subseteq A$. Also, from $a+1 \notin A$ and χ_2 it follows by induction that $[a+1, \infty) \cap A = \emptyset$, and from $-1 \notin A$ and χ_3 it follows by induction that $(-\infty, -1] \cap A = \emptyset$. So $A = [0, a]$. $\square$

Now, let $\delta(x)$ be the formula

$$\exists y, z(\chi(y, z) \wedge \theta(x, z)).$$

Then $\delta(x)$ is L-definable over $\emptyset$, and we claim that it defines $x \geq 0$ in $\mathcal{N}$: For $s \in N$, if $\mathcal{N} \vDash \delta(s)$ then there are $a, b \in N$ such that $\mathcal{N} \vDash \chi(a, b) \wedge \theta(s, b)$, so by Claim 2, $s \in [0, a]$ hence $s \geq 0$. On the other hand, suppose $s \geq 0$. By the choice of $\tilde{a}, \tilde{b}$, $\mathcal{M} \vDash \chi(\tilde{a}, \tilde{b}) \wedge \theta(s, \tilde{b})$, so $\mathcal{M} \vDash \delta(s)$, and by elementarity, $\mathcal{N} \vDash \delta(s)$. Therefore, $x \geq 0$ is definable over $\emptyset$ in $\mathcal{N}$. $\square$

Remark 9.1.11. The part in the proof where we start with an L-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ and $b \in \mathcal{M}$ such that $\phi(x, b)$ is not L^--definable with parameters in $\mathcal{M}$, and show that there exists a formula $\theta(x, b')$ which is L-definable and equivalent to the infinite interval $[0, a']$, works the same for any structure $\mathcal{N}$ which is a proper expansion of $(N, +, 0, 1)$ and a reduct of $(N, +, 0, 1, <)$. $\mathcal{N}$ does not have to be a $\emptyset$-expansion of $(N, +, 0, 1)$ or a $\emptyset$-reduct of $(N, +, 0, 1, <)$, nor unstable, as long as such $\phi(x, y)$ and b exist (being a $\emptyset$-reduct is needed in the proof for $\phi(x, y)$ to also be $\emptyset$-definable in $L_<$). So in any structure $\mathcal{N}$ which is a proper expansion of $(N, +, 0, 1)$ and a reduct of $(N, +, 0, 1, <)$, and which has a definable one-dimensional set which is not definable in $(N, +, 0, 1)$, there exists a definable infinite interval, and hence it is unstable.

Combined with Fact 9.1.5, we recover Corollary 9.1.9 and Fact 9.1.7.

Proof of Corollary 9.1.9 from Theorem 9.1.10. Suppose for a contradiction that there exists a structure $\mathcal{N}$ with universe N, which is a $\emptyset$-proper $\emptyset$-expansion of $(N, +, 0, 1)$ and a $\emptyset$-proper $\emptyset$-reduct of $(N, +, 0, 1, <)$. So $\mathcal{N}$ is dp-minimal, and by Theorem 9.1.10, it must also be stable. By Lemma 9.1.2, relativization to $\mathbb{Z}$ gives us a structure $\mathcal{Z} \prec \mathcal{N}$ which is a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$ and a $\emptyset$-proper $\emptyset$-reduct of $(\mathbb{Z}, +, 0, 1, <)$. As every element of $(\mathbb{Z}, +, 0, 1)$ is $\emptyset$-definable, a reduct of $(\mathbb{Z}, +, 0, 1)$ is in fact a $\emptyset$-reduct, and so a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$ is in fact a proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$, which is of course a proper expansion. So $\mathcal{Z}$ is a stable dp-minimal proper expansion of $(\mathbb{Z}, +, 0, 1)$, a contradiction to Fact 9.1.5. $\square$

Proof of Fact 9.1.7 from Corollary 9.1.9. Suppose for a contradiction that there exists a structure $\mathcal{Z}$ with universe $\mathbb{Z}$, which is a proper expansion of $(\mathbb{Z},+,0,1)$ and a proper reduct of $(\mathbb{Z},+,0,1,<)$. In $\mathcal{Z}$, $+$, 0, and 1 are definable, but not necessarily $\emptyset$-definable. We expand $\mathcal{Z}$ to a structure $\mathcal{Z}'$ by adding $+$, 0, and 1 to the language. So $\mathcal{Z}'$ is a proper $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$, and still a proper reduct of $(\mathbb{Z},+,0,1,<)$. As every element of $(\mathbb{Z},+,0,1,<)$ is $\emptyset$-definable, a reduct of $(\mathbb{Z},+,0,1,<)$ is in fact a $\emptyset$-reduct. So $\mathcal{Z}'$ is a proper $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$, and a proper $\emptyset$-reduct of $(\mathbb{Z},+,0,1,<)$. As a proper $\emptyset$-expansion/reduct is obviously a $\emptyset$-proper $\emptyset$-expansion/reduct, this contradicts Corollary 9.1.9. □

9.2 The main result: $(\mathbb{Z},+,0,|_p)$ is a minimal expansion of $(\mathbb{Z},+,0)$

In this section, we focus on a single valuation. Let p be any prime. Unless stated otherwise, we work in a monster model $\mathcal{M}=(M,+,0,|_p)$ of T_p, and denote its value set by Γ. We may omit the subscript p when it is clear from the context. Recall that Γ is an elementary extension of $(\mathbb{N},<,0,S)$.

9.2.1 Preparatory lemmas

For $a\in M$, $\gamma\in\Gamma$, we denote by $B(a,\gamma)$ the definable set $\{x : v(x-a)\geq\gamma\}$ and call it the *ball* of *radius* γ around a. If $\gamma=\infty$ then $B(a,\gamma)$ is just $\{a\}$, and we call such balls *trivial*. Unless stated otherwise, balls are assumed to be nontrivial. Of course, $a\in B(a,\gamma)$, and if $b\in B(a,\gamma)$ then $B(b,\gamma)=B(a,\gamma)$. Also, by Lemma 8.1.2 (2), if $\delta\neq\gamma$ then $B(a,\delta)\neq B(a,\gamma)$. So the radius of a ball is well defined. We denote the radius of a ball B by $rad(B)$.

We call a *swiss cheese* any non-empty set F that can be written as $F=B_0\setminus\bigcup_{i=1}^n B_i$, where $\{B_i\}_{i=0}^n$ are balls. Note that this representation is not unique. As the intersection of any two balls is either empty or equals one of them, we may always assume that $\{B_i\}_{i=1}^n$ are nonempty, pairwise disjoint and contained in B_0.

Remark 9.2.1. Rephrasing Lemma 8.1.9 *(2)*, if B_0, $B_1,\ldots,B_n$ are balls such that for all $i\geq 1$, $rad(B_i)\geq rad(B_0)+\underline{n}$, then $B_0\setminus\bigcup_{i=1}^n B_i\neq\emptyset$. In particular, this holds if $|rad(B_i)-rad(B_0)|\notin\mathbb{N}$.

Proposition 9.2.2. *Let $\emptyset\neq F=B_0\setminus\bigcup_{i=1}^n B_i$ be a swiss cheese. Then there exists a unique ball B_0' such that $F\subseteq B_0'$ and B_0' is minimal with respect to this property. This B_0' satisfies $B_0'\subseteq B_0$, $|rad(B_0')-rad(B_0)|\in\mathbb{N}$, and it is also the unique ball $B\subseteq B_0$ such that there are at least two distinct balls B_1'' and B_2'', satisfying $rad(B_j'')=rad(B_0')+1$ and $B_j''\cap F\neq\emptyset$ for $j=1,2$.*

Proof. Let $I_1 = \{1 \leq i \leq n : |rad(B_i) - rad(B_0)| \in \mathbb{N}\}$, $I_2 = \{1, \ldots, n\} \backslash I_1$. By applying Lemma 8.1.8 *(2)* to $B_0 \backslash \bigcup_{i \in I_1} B_i \neq \emptyset$, we see that $B_0 \backslash \bigcup_{i \in I_1} B_i = \bigsqcup_{j=1}^{l} B_j''$, where $l \geq 1$ and for all j, $B_j'' \subseteq B_0$ and $rad(B_j'') = \max\{rad(B_i) : i \in I_1\}$. So $F = \bigsqcup_{j=1}^{l}(B_j'' \backslash \bigcup_{i \in I_2} B_i)$. By Remark 9.2.1, for each j, $B_j'' \backslash \bigcup_{i \in I_2} B_i \neq \emptyset$. If C is a ball such that $F \subseteq C$, then for each j, $B_j'' \backslash \bigcup_{i \in I_2} B_i \subseteq C$, and we claim that in fact $B_j'' \subseteq C$. Indeed, by Axiom 8, $B_j'' = \bigsqcup_{t=1}^{p} B_{j,t}''$ with $rad(B_{j,t}'') = rad(B_j'') + 1$, and again by Remark 9.2.1, for each t, $B_{j,t}'' \backslash \bigcup_{i \in I_2} B_i \neq \emptyset$. So $C \cap B_{j,t}'' \neq \emptyset$ but $C \not\subseteq B_{j,t}''$ (as also for $s \neq t$, $C \cap B_{j,s}'' \neq \emptyset$), therefore $B_{j,t}'' \subseteq C$. This holds for all t, hence $B_j'' \subseteq C$. In particular, $B_1'' \subseteq C$. As $|rad(B_1'') - rad(B_0)| \in \mathbb{N}$, there are only finitely many balls B such that $B_1'' \subseteq B \subseteq B_0$, so we may choose B_0' to be a minimal one (with respect to inclusion) among those that also satisfy $F \subseteq B$ (exists, since B_0 satisfies this). By this choice, $B_0' \subseteq B_0$ and $|rad(B_0') - rad(B_0)| \in \mathbb{N}$. If B is another ball such that $F \subseteq B$, then $F \subseteq B \cap B_0'$, and $B \cap B_0' \neq \emptyset$ is also a ball. Also, as we have shown, $B_1'' \subseteq B$, so $B_1'' \subseteq B \cap B_0' \subseteq B_0$. Hence by the choice of B_0', $B_0' = B \cap B_0' \subseteq B$. This shows that B_0' is the unique minimal ball containing F. Finally, let D be a ball and assume $F \subseteq D$. By Axiom 8 write $D = \bigsqcup_{t=1}^{p} D_t''$ with $rad(D_t'') = rad(D) + 1$. Then D is minimal if and only if for all t, $F \not\subseteq D_t''$, iff there are $t \neq s$ such that $F \cap D_t'' \neq \emptyset$ and $F \cap D_s'' \neq \emptyset$. □

Let F be a swiss cheese. By Proposition 9.2.2 we may write $F = B_0 \backslash \bigcup_{i=1}^{n} B_i$ where B_0 is the unique minimal ball containing F. We may also assume that $\{B_i\}_{i=1}^{n}$ are nonempty, pairwise disjoint and contained in B_0. Unless stated otherwise, all representations are assumed to satisfy these conditions. We call B_0 the *outer ball* of F, and define the *radius* of F to be $rad(F) := rad(B_0)$. We also call $\{B_i\}_{i=1}^{n}$ the *holes* of F. Note that this representation is still not unique (unless there are no holes at all), as each hole may always be split into p smaller holes, and sometimes there are sets of p holes which may each be combined into a single hole. There is a canonical representation for F, namely, the one with the minimal number of holes. But we will not use it. Rather, when dealing with holes without mentioning a specific representation, either the intended representation is clear from the context (e.g., when using Remark 9.2.3 (2) or (3) to split a swiss cheese with a given representation), or we may choose any representation and stick with it.

We say that B_i is a *proper hole* of F if $|rad(B_i) - rad(B_0)| \notin \mathbb{N}$. We call F a *proper cheese* if all of its holes are proper. Note that by Remark 9.2.1, being a proper cheese does not depend on the representation of the holes.

Remark 9.2.3.

(1) If $B_0, B_1, \ldots, B_n$ are balls such that for all $i \geq 1$, $B_i \subseteq B_0$ and $|rad(B_i) - rad(B_0)| \notin \mathbb{N}$, then B_0 is the outer ball of the swiss cheese $F = B_0 \backslash \bigcup_{i=1}^{n} B_i$, which is therefore proper.

(2) Let F be a swiss cheese, and let $k \geq 1$. Then F may be written as a disjoint union $F = \bigsqcup_{i=1}^{l} F_i$, where $1 \leq l \leq p^k$, and for each i, F_i is a swiss cheese such that

$rad(F_i) \geq rad(F) + k$ and $|rad(F_i) - rad(F)| \in \mathbb{N}$. Each hole of F_i is already a hole of F, and each hole of F is a hole of at most one of the $\{F_i\}_i$.
If F is proper, then $l = p^k$ and each F_i is a proper cheese of radius $rad(F_i) = rad(F) + k$. In this case, each hole of F is a hole of exactly one of the $\{F_i\}_i$.

(3) Let $F = B_0 \backslash \bigcup_{i=1}^n B_i$ be a swiss cheese, let $I_1 = \{1 \leq i \leq n : |rad(B_i) - rad(B_0)| \in \mathbb{N}\}$, and let $k_0 = \max\{rad(B_i) - rad(B_0) : i \in I_1\} \in \mathbb{N}$. Then for each $k \geq k_0$, F may be written as a disjoint union $F = \bigsqcup_{i=1}^l F_i$, where $1 \leq l \leq p^k$, and for each i, F_i is a *proper* swiss cheese of radius $rad(F_i) = rad(F) + k$. Each hole of F_i is already a proper hole of F, and each proper hole of F is a hole of exactly one of the $\{F_i\}_i$.

(4) Let F',F'' be two swiss cheeses of radiuses γ',γ'' respectively, and let $\gamma = \max\{\gamma', \gamma''\}$. Then $F' \cap F''$ is either empty, or also a swiss cheese of radius $rad(F' \cap F'') \geq \gamma$ such that $|rad(F' \cap F'') - \gamma| \in \mathbb{N}$.

(5) If both F',F'' are proper and $\gamma' = \gamma''$, and if $F' \cap F''$ is not empty, then F',F'' have the same outer ball, and $F' \cap F''$ is also a proper cheese of the same outer ball.

Lemma 9.2.4. *Let F,F' be two swiss cheeses of radiuses $\gamma \leq \gamma'$ respectively. If $F \cap F' \neq \emptyset$, then $F \cup F'$ is also a swiss cheese, of radius exactly γ. The set of holes of $F \cup F'$ is a subset of the union of the set of holes of F and the set of holes of F'.*

Proof. Write $F = B_0 \backslash \bigcup_{i=1}^n B_i$, $F' = B'_0 \backslash \bigcup_{j=1}^m B'_j$. If $F \cap F' \neq \emptyset$ then $B_0 \cap B'_0 \neq \emptyset$, hence $B_0 \supseteq B'_0$. Therefore,

$$F' \backslash F = F' \backslash \left(B_0 \backslash \bigcup_{i=1}^n B_i \right) = F' \backslash B_0 \cup \left(F' \cap \bigcup_{i=1}^n B_i \right) = \bigcup_{i=1}^n F' \cap B_i.$$

For each i: if $B'_0 \cap B_i = \emptyset$ then $F' \cap B_i = \emptyset$. Otherwise, as $B_0 \supseteq B'_0$, we also get $B_i \subseteq B'_0$ ($B_i \supseteq B'_0$ is impossible, as it implies $F \cap F' = \emptyset$), and in this case, $F' \cap B_i = B_i \backslash \bigcup_{j=1}^m (B_i \cap B'_j)$. Together, we get

$$F \cup F' = F \cup (F' \backslash F) = B_0 \backslash \left(\bigcup_{i \in I_1} B_i \cup \bigcup_{i \in I_2} \bigcup_{j=1}^m (B_i \cap B'_j) \right)$$

where I_1 is the set of i such that $B'_0 \cap B_i = \emptyset$ and I_2 is the set of i such that $B_i \subseteq B'_0$. This is a swiss cheese, and as $F \subseteq F \cup F' \subseteq B_0$ and $rad(F) = rad(B_0) = \gamma$, also $rad(F \cup F') = \gamma$ and B_0 is its outer ball. For each i such that $B_i \subseteq B'_0$ and each j, either $B_i \cap B'_j = \emptyset$ (in which case $B_i \cap B'_j$ does not appear as a hole of $F \cup F'$), or $B_i \cap B'_j = B_i$ or $B_i \cap B'_j = B'_j$, so the last part holds. □

Sometimes we want disjoint swiss cheeses to also have disjoint outer balls, but unfortunately, that is not always possible. An example for this is a union of two swiss cheeses,

$F_1 \cup F_2$, with $F_2 \subseteq B$ where B is one of the holes of F_1. If $|rad(B) - rad(F_1)| \in \mathbb{N}$, we may rewrite F_1 as a union of swiss cheeses of radius $rad(B)$, and, together with F_2, we have a union of swiss cheeses with disjoint outer balls. But if $|rad(B) - rad(F_1)| \notin \mathbb{N}$, we cannot do such a thing.

Definition 9.2.5. A *pseudo swiss cheese* is a definable set P such that there is a swiss cheese F with outer ball B such that $F \subseteq P \subseteq B$. By the following remark, we may call B the *outer ball* of P, and define the *radius* of P to be $rad(P) := rad(B)$. We also call P *pseudo proper cheese* if there is a proper cheese F with outer ball B such that $F \subseteq P \subseteq B$.

Remark 9.2.6. (1) In the previous definition, B is uniquely determined by P. Indeed, suppose F_1,F_2 are two swiss cheeses with outer balls B_1,B_2 respectively, such that $F_1 \subseteq P \subseteq B_1$ and $F_2 \subseteq P \subseteq B_2$. Then $rad(B_1) = rad(F_1) \geq rad(B_2)$ and $rad(B_2) = rad(F_2) \geq rad(B_1)$, so $rad(B_1) = rad(B_2)$. Also, $P \subseteq B_1 \cap B_2 \neq \emptyset$, so we must have $B_1 = B_2$.

(2) For every $k \geq 1$, every proper pseudo swiss cheese of radius γ can be written as a union of exactly p^k proper pseudo cheeses with disjoint outer balls of radius exactly $\gamma + k$.

(3) Note that the analogue to Remark 9.2.3 (2) is not true for pseudo swiss cheeses. For example, let B be a ball of radius γ, let $\{B_i\}_{i=0}^{p-1}$ be all the balls of radius $\gamma+1$ contained in B, let $\{B_{i,j}\}_{j=0}^{p-1}$ be all the balls of radius $\gamma+2$ contained in B_i, and let $C \subseteq B_{0,1}$ be a ball of radius $\delta > \gamma$ such that $|\delta - \gamma| \notin \mathbb{N}$. Then $P = C \sqcup \bigsqcup_{i=0}^{p-1} B_{i,0}$ is a pseudo swiss cheese of radius γ, but cannot be written as $\leq p$ pseudo swiss cheeses of radius $\geq \gamma + 1$, because $P \cap B_0$ is not a pseudo swiss cheese. Also, note that the intersection of two pseudo swiss cheeses is not necessarily a single pseudo swiss cheese. For example, take $P \cap B_0$ from above.

Lemma 9.2.7.

(1) Let P_1,P_2 be two pseudo swiss cheeses with outer balls B_1,B_2 respectively, such that $rad(B_1) \geq rad(B_2)$. If $B_1 \cap B_2 \neq \emptyset$ then $P_1 \cup P_2$ is also a pseudo swiss cheese, with outer ball B_2. If P_2 is proper, then $P_1 \cup P_2$ is also proper.

(2) Any finite union of pseudo swiss cheeses may be written as a union of pseudo swiss cheeses having disjoint outer balls. Also, any finite union of pseudo proper cheeses may be written as a union of pseudo proper cheeses having disjoint outer balls.

Proof. We prove *(1)*. $B_1 \cap B_2 \neq \emptyset$ and $rad(B_1) \geq rad(B_2)$, so $B_1 \subseteq B_2$ and therefore also $P_1 \subseteq B_2$. Let F_2 be a swiss cheese with outer ball B_2 such that $F_2 \subseteq P_2 \subseteq B_2$. Then $F_2 \subseteq P_1 \cup P_2 \subseteq B_2$. If P_2 is proper, then we may take F_2 to be proper, and so $P_1 \cup P_2$ is also proper.
We prove *(2)*. Let $A = \bigcup_{i=1}^n P_i$ such that for each i, P_i is a pseudo swiss cheese with outer ball B_i. Let $\{B'_j\}_{j=1}^m$ be the set of all the maximal balls (with respect to inclusion)

among $\{B_i\}_{i=1}^n$. Then $\{B'_j\}_{j=1}^m$ are pairwise disjoint. For each $1 \le j \le m$, let $I_j = \{i : B_i \cap B'_j \ne \emptyset\}$ and $P'_j = \bigcup_{i \in I_j} P_i$. So $\{1, \dots, n\} = \bigsqcup_{j=1}^m I_j$ and therefore $A = \bigcup_{j=1}^m P'_j$. By *(1)*, P'_j is a pseudo swiss cheese with outer ball B'_j. If for each i, P_i is proper, then by *(1)*, for each j, P'_j is also proper. □

Remark 9.2.8. The valuation v_p induces a topology on $\mathcal{M}$, generated by the balls. By Lemma 8.1.9 *(3)*, if $\gcd(m, p) = 1$, then the sets defined by $D_m(x - r)$ are dense in $\mathcal{M}$.

Lemma 9.2.9. *Let P be a pseudo swiss cheese with outer ball B and radius α, and assume $0 \in B$. Let G be a dense subgroup of $\mathcal{M}$, and let $A = P \cap G$. Then there exists $N \in \mathbb{N}$ and $a_1, \dots, a_N \in B \cap G$ such that $\bigcup_{i=1}^N (A + a_i) = B \cap G$.*

Proof. Observe that B is a subgroup of $\mathcal{M}$ since $0 \in B$. Let F be a swiss cheese with outer ball B such that $F \subseteq P \subseteq B$. By Remark 9.2.3 (3), for some finite k we may find a proper cheese $F' \subseteq F$ of radius $\alpha + k$. Let s be the number of holes in F'. By Remark 9.2.3 (2), we may write F' as a union of exactly p^s proper cheeses of radius $\alpha + k + s$. As $p^s > s$, at least one of these proper cheeses must have no holes, i.e., must be a ball, say D. Let $x \in D$ and $D_0 = D - x$. Then D_0 is a subgroup of B of index $N := p^{k+s}$. Let $x_1, \dots, x_N$ be representatives of the cosets, so $B = \bigcup_{i=1}^N x_i + D_0$. For each i, let $a_i \in x_i + D_0 \cap G$. As $a_i \in B \cap G$ and $A \subseteq B \cap G$, we have $(A + a_i) \subseteq B \cap G$, and therefore $\bigcup_{i=1}^N (A + a_i) \subseteq B \cap G$. On the other hand, as $A \supseteq D \cap G$, we also have $\bigcup_{i=1}^N (A + a_i) \supseteq B \cap G$. □

Lemma 9.2.10. *Let $A = G \cap \bigsqcup_{i=1}^n F_i$ where G is a dense subgroup of $\mathcal{M}$ and $\{F_i\}_{i=1}^n$ are disjoint proper cheeses with nonstandard radiuses. Then there are $N, m \in \mathbb{N}$ and $c_1, \dots, c_N \in G$ such that $\bigcap_{i=1}^N (A - c_i) = G \cap \bigsqcup_{i=1}^m P_i$ with P_i* pseudo *proper cheeses with disjoint outer balls, all of the same nonstandard radius, and $0 \in P_1$.*

Proof. It is of course enough to prove the lemma without the requirement $0 \in P_1$, as we may then arrange that by shifting by some $c \in G \cap P_1$.

Preparation step. By Remark 9.2.3 *(2)*, if F is a proper cheese of infinite radius γ then, for all $k \ge 0$, F can be written as a disjoint union of proper cheeses of radius $\gamma + k$. So there exists $\gamma_1, \dots, \gamma_n$, in distinct archimedean classes of Γ, such that we can write

$$\bigsqcup_{i=1}^n F_i = \bigsqcup_{i=1}^m \bigsqcup_{j=1}^{s_i} F_j^i,$$

where $s_1, \dots, s_m \ge 1$ and for all $1 \le i \le m$ and $1 \le j \le s_i$, $rad(F_j^i) = \gamma_i$ and F_j^i has a swiss cheese representation in which the radiuses of all the holes are in

$$R := \{\alpha \in \Gamma : \text{for all } 1 \le k \le m, \text{ if } |\alpha - \gamma_k| \in \mathbb{N} \text{ then } \alpha \le \gamma_k\}.$$

We call this representation of A a *good representation of A with respect to $\{\gamma_i\}_{i=1}^m$*.

If $m = 1$, we already have what we want, so we may assume that $m > 1$. For each i, j, let B_j^i be the outer ball of F_j^i. There are two cases:

Case 1: For every $1 < l \leq m$ and every $1 \leq u \leq s_l$ there is some $1 \leq v \leq s_1$ such that $B^1_v \cap B^l_u \neq \emptyset$.

This means that $\{B^1_j\}_{j=1}^{s_1}$ is the set of all the maximal balls with respect to inclusion among $\{B^i_j : 1 \leq i \leq m\,,\, 1 \leq j \leq s_i\}$. It follows that $\{B^1_j\}_{j=1}^{s_1}$ are outer balls of pseudo proper cheese containing all the F^i_j. Indeed, by the proof of Lemma 9.2.7 *(2)*, we may write

$$\bigsqcup_{i=1}^{m}\bigsqcup_{j=1}^{s_i} F^i_j = \bigsqcup_{j=1}^{s_1} P_j,$$

where for each j, P_j is a pseudo proper cheese such that $F^1_j \subseteq P_j \subseteq B^1_j$. So these are pseudo proper cheeses with disjoint outer balls, all of the same radius γ_1. So in this case we are done.

Case 2: There are $1 < l \leq m$ and $1 \leq v \leq s_l$ such that for every $1 \leq j \leq s_1$, $B^1_j \cap B^l_v = \emptyset$.

Let $a \in F^1_1 \cap G$ and $b \in F^l_v \cap G$ and set $A' = (A - a) \cap (A - b)$. Then $0 \in A' \neq \emptyset$. We show that A' has a good representation with respect to a subset of $\{\gamma_i\}_{i=1}^m$, of the form

$$A' = G \cap \bigsqcup_{i=1}^{m'}\bigsqcup_{j=1}^{s'_i} \tilde{F}^i_j$$

such that either there are no more proper cheeses of radius γ_1, or the number s'_1 of proper cheeses of radius γ_1 is strictly less than s_1. By reiterating this process, it will terminate either to the case in which every proper cheese is of the same radius or to *Case 1*, which proves the Lemma.

Write $A' = G \cap (\bigsqcup_{i=1}^{m}\bigsqcup_{j=1}^{s_i}\bigsqcup_{q=1}^{m}\bigsqcup_{r=1}^{s_i}(F^i_j - a)\cap(F^q_r - b))$. By the good representation, for each i, j we write $F^i_j = B^i_j \setminus \bigsqcup_t B^i_{j,t}$ with $rad(B^i_{j,t}) \in R$.

For every i and j, k, if $B^i_j - a \neq B^i_k - b$, then $(F^i_j - a) \cap (F^i_k - b) = \emptyset$, and if $B^i_j - a = B^i_k - b$, then $(F^i_j - a) \cap (F^i_k - b)$ is a proper cheese of radius $\gamma_i \geq \gamma_1$ such that all its holes can be written with radiuses in R.

For every $i < i'$ and j, k, if $(B^i_j - a) \cap (B^{i'}_k - b) = \emptyset$, then also $(F^i_j - a) \cap (F^{i'}_k - b) = \emptyset$. Otherwise, $(B^i_j - a) \supseteq (B^{i'}_k - b)$ and

$$(F^i_j - a) \cap (F^{i'}_k - b) = ((B^{i'}_k - b) \setminus \bigsqcup_{t'} (B^{i'}_{k,t'} - b)) \setminus \bigsqcup_{t} (B^i_{j,t} - a).$$

For each t such that $(B^i_{j,t} - a) \cap (B^{i'}_k - b) \neq \emptyset$ there are three cases:

(1) $rad(B^{i'}_k - b) > rad(B^i_{j,t} - a)$. Then $(B^{i'}_k - b)$ is included in the hole $(B^i_{j,t} - a)$ hence $(F^i_j - a) \cap (F^{i'}_k - b) = \emptyset$.

(2) $rad(B^{i'}_k - b) \le rad(B^i_{j,t} - a)$ and $rad(B^i_{j,t} - a)$ is at finite distance from $\gamma_{i'}$. As $rad(B^i_{j,t} - a) = rad(B^i_{j,t}) \in R$, we get

$$rad(B^{i'}_k - b) = rad(B^{i'}_k) = \gamma_{i'} \ge rad(B^i_{j,t} - a).$$

So $rad(B^{i'}_k - b) = rad(B^i_{j,t} - a)$, and so $(B^{i'}_k - b) = (B^i_{j,t} - a)$ and therefore $(F^i_j - a) \cap (F^{i'}_k - b) = \emptyset$.

(3) $rad(B^{i'}_k - b) \le rad(B^i_{j,t} - a)$ and $rad(B^i_{j,t} - a)$ is not at finite distance from $\gamma_{i'}$. Then $B^i_{j,t} - a$ is a proper hole of $(F^i_j - a) \cap (F^{i'}_k - b)$.

Therefore $(F^i_j - a) \cap (F^{i'}_k - b)$ is either empty or a proper cheese of radius $\gamma_{i'} > \gamma_i \ge \gamma_1$ such that all its holes can be written with radiuses in R.

So A' has a good representation that is the intersection of G with a (nonempty) disjoint union of proper cheeses, with radiuses among $\{\gamma_i\}_{i=1}^m$, such that all their holes have radiuses in R. Now either $s_1 = 1$, hence F^1_1 is the only cheese of radius γ_1 in the good representation of A and hence in the good representation of A' there are no more proper cheese of radius γ_1. Otherwise we have a good representation with respect to a subset of $\{\gamma_i\}_{i=1}^m$ of the form

$$A' = G \cap \bigsqcup_{i=1}^{m'} \bigsqcup_{j=1}^{s'_i} \tilde{F}^i_j$$

where $s'_1, \dots, s'_{m'} \ge 1$, and s'_1 is the number of cheese of radius γ_1. For every $1 \le l \le s'_1$, there must be j, k such that $\tilde{F}^1_l = (F^1_j - a) \cap (F^1_k - b)$. As $(F^1_j - a) \cap (F^1_k - b) \ne \emptyset \iff B^1_j - a = B^1_k - b$, for every j there is at most one k such that $(F^1_j - a) \cap (F^1_k - b) \ne \emptyset$, therefore $s'_1 \le s_1$. Suppose towards contradiction that $s'_1 = s_1$. Then for every j there is exactly one k such that $(F^1_j - a) \cap (F^1_k - b) \ne \emptyset$, in particular, for $j = 1$ there is exactly one l such that $(F^1_1 - a) \cap (F^1_l - b) \ne \emptyset$, and so also $B^1_1 - a = B^1_l - b$. By the choice of a, b, we have $0 \in (B^1_1 - a) \cap (B^l_v - b) = (B^1_l - b) \cap (B^l_v - b)$, so $b \in B^1_l \cap B^l_v \ne \emptyset$, a contradiction. Therefore $s'_1 < s_1$. □

Lemma 9.2.11. *Let $A = G \cap \bigsqcup_{i=1}^n P_i$ where G is a dense subgroup of $\mathcal{M}$ and $\{P_i\}_{i=1}^n$ are pseudo proper cheeses with disjoint outer balls, all of the same nonstandard radius α, such that $0 \in P_1$. Then there exists $N \in \mathbb{N}$ and $c_1, \dots, c_N \in G$ such that $\bigcap_{i=1}^N (A - c_i) = G \cap P$ for some pseudo proper cheese P of nonstandard radius such that $0 \in P$.*

Proof. It is of course enough to prove the lemma without the requirement $0 \in P$. We proceed by induction on n. For $n = 1$ we have nothing to prove. Suppose that the lemma holds for all $n' < n$. For each $1 \le i \le n$ let B_i be the outer ball of P_i, and let F_i be a proper cheese with outer ball B_i such that $F_i \subseteq P_i \subseteq B_i$. Let S be the set of all the balls of radius α, and let $S' = \{B_i : 1 \le i \le n\}$. Observe that $(S, +)$ is an infinite group with neutral element B_1 (since $0 \in P_1 \subseteq B_1$), and in particular, $S' \subsetneq S$. Let $C := \bigcup S' = \bigsqcup_{i=1}^n B_i$.

Claim. If for every $1 \leq i \leq n$ there is $a \in B_i$ such that $S' - a = S'$, then S' is a subgroup of S.

Proof of the claim. If $B, B' \in S$ then $rad(B) = rad(B')$, hence $(B - a) \cap B' \neq \emptyset \Rightarrow B - a = B'$. Also, for all $B'' \in S$ and $a, a' \in B''$, $a - a' \in B_1$ and therefore $B - a' = (B - a) + (a - a') = B - a$. From this and the hypothesis of the claim it follows that for each $1 \leq i \leq n$, $S' - B_i := \{B - B_i : B \in S'\} = S'$, which implies that S' is a subgroup of S. $\square$

There are two cases:

Case 1: S' is a subgroup of S. Then $(C, +)$ is a subgroup of $(M, +)$, and S' is the quotient group C/B_1. As $(C, +)$ is definable, by Lemma 8.2.4 it must be of the form $C = B(0, \beta)$ (as $B_1 \not\subseteq mM$ for every $m > 1$ with $\gcd(m, p) = 1$). In fact, since $|S'| = n$, it must be that $\beta = \alpha - k$, where k satisfies $n = p^k$. In particular, β is nonstandard. For each i, let H_i be (any choice for) the set of holes of F_i, and let $H = \bigcup_i H_i$. Then we can rewrite $\bigsqcup_{i=1}^n F_i$ as $F = B(0, \beta) \setminus \bigcup H$, which is a single proper cheese, with outer ball $B(0, \beta)$. Let $P = \bigsqcup_{i=1}^n P_i$. Then $F \subseteq P \subseteq B(0, \beta)$, so P is a pseudo proper cheese, and we are done.

Case 2: S' is not a subgroup of S. Then by the claim, there is some $1 \leq i_0 \leq n$ such that for all $a \in B_{i_0}$, $S' - a \neq S'$ (in fact $1 < i_0$). Let $a \in G \cap P_{i_0} \subseteq B_{i_0}$ (which exists because G is dense), and let $A' = A \cap (A - a)$. Then $0 \in A' \neq \emptyset$.

Write $A' = G \cap (\bigsqcup_{i=1}^n \bigsqcup_{j=1}^n P_i \cap (P_j - a))$. Then

$$G \cap \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{n} F_i \cap (F_j - a) \subseteq A' \subseteq G \cap \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{n} B_i \cap (B_j - a).$$

For all $1 \leq i, j \leq n$, $rad(B_i) = rad(B_j) = \alpha$ and therefore, as in Lemma 9.2.10, $B_i \cap (B_j - a) \neq \emptyset \iff B_i = B_j - a \iff F_i \cap (F_j - a) \neq \emptyset$, and in this case, $F_i \cap (F_j - a)$ is a proper cheese with outer ball B_i. We also have that $F_i \cap (F_j - a) \subseteq P_i \cap (P_j - a) \subseteq B_i \cap (B_j - a)$, so $P_i \cap (P_j - a) \neq \emptyset \iff B_i \cap (B_j - a) \neq \emptyset$, and in this case, $P_i \cap (P_j - a)$ is a pseudo proper cheese with outer ball B_i. Therefore, $G \cap (\bigsqcup_{i=1}^n \bigsqcup_{j=1}^n B_i \cap (B_j - a)) = G \cap (\bigsqcup_{i=1}^{n'} B'_i)$, $G \cap (\bigsqcup_{i=1}^n \bigsqcup_{j=1}^n F_i \cap (F_j - a)) = G \cap (\bigsqcup_{i=1}^{n'} F'_i)$, and $A' = G \cap (\bigsqcup_{i=1}^{n'} P'_i)$, where for each i, $B'_i \in S'$, F'_i is a proper swiss cheese with outer ball B'_i, and P'_i is a pseudo proper cheese such that $F'_i \subseteq P'_i \subseteq B'_i$.

Moreover, for every i there is at most one j such that $B_i \cap (B_j - a) \neq \emptyset$, therefore $n' \leq n$. But by the choice of a, $S' - a \neq S'$, so there is an $1 \leq i \leq n$ such that $B_i \neq B_j - a$ for all $1 \leq j \leq n$. Therefore $n' < n$, and by the induction hypothesis we are done. $\square$

9.2.2 Proof of the theorem

The proof of Theorem 9.2.12 is similar to the proof of Theorem 9.1.10 but more involved and relies on Subsection 9.2.1.

Theorem 9.2.12. *Let $(N, +, 0, 1, |_p)$ be an elementary extension of $(\mathbb{Z}, +, 0, 1, |_p)$. Then $(N, +, 0, 1, |_p)$ is $\emptyset$-minimal among the* unstable *$\emptyset$-proper $\emptyset$-expansions of $(N, +, 0, 1)$.*

Proof. Let $\mathcal{N}$ be any unstable structure with universe N, which is a $\emptyset$-proper $\emptyset$-expansion of $(N, +, 0, 1)$ and a $\emptyset$-reduct of $(N, +, 0, 1, |_p)$. We show that $\mathcal{N}$ is $\emptyset$-interdefinable with $(N, +, 0, 1, |_p)$. It is enough to show that $x|_p y$ is definable over $\emptyset$ in $\mathcal{N}$. Let L be the language of $\mathcal{N}$ and $L^- = \{+, 0, 1\}$. As in the proof of Theorem 9.1.10, we may assume that all languages contain $\{-\}\cup\{D_n : n \geq 1\}$, and (by being a $\emptyset$-reduct and $\emptyset$-expansion) that $L^- \subseteq L \subseteq L_p^E$.

Let $\mathcal{M}$ be a monster model for T_p, so $\mathcal{M}|_L$ is a monster for $Th(\mathcal{N})$. As $(N, +, 0, 1)$ is stable but $\mathcal{N}$ is not, by Lemma 1.4.3 there exist an L-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ and $b \in \mathcal{M}$ such that $\phi(x, b)$ is not L^--definable with parameters in $\mathcal{M}$. By Theorem 8.2.1 (quantifier elimination) and Remark 8.2.3, $\phi(x, b)$ is equivalent to a formula of the form

$$\bigvee_i \left(D_m(x - r_i) \wedge kx \in F_i \wedge \bigwedge_j k'x \neq a_{i,j} \right) \vee \bigvee_{i'} x = c_{i'}$$

where $m, k, k', r_i \in \mathbb{Z}$, $\gcd(m, p) = \gcd(k, p) = 1$, $k' = p^l k$ for some $l \geq 0$, $a_{i,j}, c_{i'} \in \mathcal{M}$ and each F_i is a swiss cheese in $\mathcal{M}$.

The first step of the proof is to show the existence of an L-definable formula which is equivalent to a formula of the form $D_m(x) \wedge x \in B(0, \gamma)$, i.e. $D_m(x) \wedge v(x) \geq \gamma$, for some nonstandard $\gamma \in \Gamma$ and integer m such that $\gcd(m, p) = 1$. Let $\phi'(x, b)$ be the formula

$$\bigvee_i (D_m(x - r_i) \wedge kx \in F_i).$$

The symmetric difference $\phi(x, b) \triangle \phi'(x, b)$ is finite, hence L^--definable, and therefore $\phi'(x, b)$ is also L-definable but not L^--definable. So we may replace $\phi(x, b)$ by $\phi'(x, b)$. For each i, the formula $D_m(x - r_i)$ is equivalent to $D_{km}(kx - kr_i)$, so $\phi(x, b)$ is equivalent to the formula

$$\bigvee_i (D_{km}(kx - kr_i) \wedge kx \in F_i).$$

Let $\phi'(x, b)$ be the formula $D_k(x) \wedge \phi(\frac{x}{k}, b)$. Then $\phi'(x, b)$ is L-definable and equivalent to the formula

$$\bigvee_i (D_{m'}(x - r'_i) \wedge x \in F_i)$$

where $m' = km$ and $r'_i = kr_i$. This substitution is reversible as $\phi(x, b)$ is equivalent to $\phi'(kx, b)$, therefore also $\phi'(x, b)$ is not L^--definable. So again we may replace $\phi(x, b)$ by $\phi'(x, b)$.

We want each F_i to have a nonstandard radiuses. For each i, choose a representation for F_i as a swiss cheese $F_i = B_{i,0} \setminus \bigcup_{j=1}^{n_i} B_{i,j}$, where $B_{i,j} = B(a_{i,j}, \gamma_{i,j})$. Let $J_i = \{1 \leq j \leq n_i : \gamma_{i,j} \notin \mathbb{N}\}$, i.e., the set of indices of the infinite holes, and let

$$B'_{i,0} = \begin{cases} B(0,0) & \gamma_{i,0} \in \mathbb{N} \\ B_{i,0} & \gamma_{i,0} \notin \mathbb{N} \end{cases} \text{ and } B''_{i,0} = \begin{cases} B_{i,0} & \gamma_{i,0} \in \mathbb{N} \\ B(0,0) & \gamma_{i,0} \notin \mathbb{N} \end{cases}$$

(note that $B(0,0) = M$). Let $F'_i = B'_{i,0} \setminus \bigcup_{j \in J_i} B_{i,j}$, and let $F''_i = B''_{i,0} \setminus \bigcup_{j \notin J_i} B_{i,j}$. Then $F_i = F'_i \cap F''_i$, and so $\phi(x, b)$ is equivalent to

$$\bigvee_i (D_{m'}(x - r'_i) \wedge x \in F''_i \wedge x \in F'_i).$$

Each hole of F'_i has nonstandard radius, and its outer ball either has an infinite radius or has radius 0. On the other hand, both the outer ball and all the holes of F''_i have finite radiuses. In general, if $B(a, \gamma)$ has finite radius, then the formula $x \in B(a, \gamma)$ is equivalent to $D_{p^\gamma}(x - a)$. So $x \in F''_i$ is equivalent to a boolean combination of such formulas, and therefore, by Lemma 8.1.3 (1) (choosing the same m'' for all the i's and rearranging the disjunction), $\phi(x, b)$ is equivalent to a formula of the form

$$\bigvee_i (D_{m''}(x - r'_i) \wedge x \in F'_i)$$

where each hole of F'_i has a nonstandard radius, and its outer ball either has an nonstandard radius or has radius 0. Note that now it may be that $p | m''$. By grouping together disjuncts with the same r'_i, we can rewrite this as

$$\bigvee_i (D_{m''}(x - r'_i) \wedge \bigvee_j x \in F'_{i,j})$$

where for $i_1 \neq i_2$, $r'_{i_1} \not\equiv r'_{i_2} \bmod m''$. As this formula is equivalent to $\phi(x, b)$, which is not L^--definable with parameters in $\mathcal{M}$, there must be an i_0 such that $D_{m''}(x - r'_{i_0}) \wedge \bigvee_j x \in F'_{i_0,j}$ is not L^--definable with parameters in $\mathcal{M}$. This latter formula, which we denote by $\phi_{i_0}(x, b)$, is equivalent to $\phi(x, b) \wedge D_{m''}(x - r'_{i_0})$, and so is L-definable. So we may replace $\phi(x, b)$ by $\phi_{i_0}(x, b)$. For simplicity of notation we rewrite this as

$$D_m(x - r) \wedge \bigvee_i x \in F_i.$$

By Lemma 9.2.4 we may assume that $\{F_i\}_i$ are pairwise disjoint, and still have that for each i, all the holes of F_i have infinite radiuses and its outer ball either has an infinite radius or has radius 0. By Remark 9.2.1 two proper cheeses having the same outer ball must intersect. Applying this to all the F_i's having radius 0 (which are all proper, as all the holes are of infinite radius), we see that there can be at most one i such that F_i has radius 0.

We want all proper cheeses to have infinite radius. If there is i_0 such that the proper cheese F_{i_0} has radius 0, let $\phi'(x,b)$ be the formula $D_m(x-r) \wedge \neg\phi(x,b)$. Then $\phi'(x,b)$ is L-definable and, as $\phi(x,b)$ is equivalent to $D_m(x-r) \wedge \neg\phi'(x,b)$, it is also not L^--definable. The formula $\phi'(x,b)$ is equivalent to

$$D_m(x-r) \wedge \bigwedge_i x \in F_i^c.$$

We may write $F_{i_0} = B(0,0) \setminus \bigcup_{j=1}^n B_j$, where for each j, $rad(B_j)$ is infinite. So $F_{i_0}^c = \bigcup_{j=1}^n B_j$, and $\phi'(x,b)$ is equivalent to

$$D_m(x-r) \wedge \bigvee_{j=1}^n (x \in B_j \wedge \bigwedge_{i \neq i_0} x \in F_i^c).$$

For each $i \neq i_0$, F_i^c is a finite union of swiss cheeses (specifically, a union of a single swiss cheese of radius 0 and a finite number of balls). Therefore, by Remark 9.2.3 *(4)*, for each j, $B_j \cap \bigcap_{i \neq i_0} F_i^c$ is a finite union of swiss cheeses, each of radius at least $rad(B_j)$, so infinite. So $\phi'(x,b)$ is equivalent to a formula of the form

$$D_m(x-r) \wedge \bigvee_i x \in F_i'$$

where each F_i' is a swiss cheese of infinite radius. Again by Lemma 9.2.4, we may assume in addition that $\{F_i'\}_i$ are pairwise disjoint. As $\phi'(x,b)$ is not L^--definable, the disjunction cannot be empty. So we may replace $\phi(x,b)$ by $\phi'(x,b)$ and rename F_i' as F_i.

We may assume that for each i, $D_m(x-r) \wedge x \in F_i$ defines a nonempty set, as otherwise we may just drop the i'th disjunct. Write $m = p^k m'$ with $\gcd(m',p) = 1$. Then $D_m(x-r)$ is equivalent to $D_{m'}(x-r_1) \wedge (v_p(x-r_2) \geq k)$, where $r_1 = r \bmod m'$ and $r_2 = r \bmod p^k$. So $\phi(x,b)$ is equivalent to

$$D_{m'}(x-r_1) \wedge \bigvee_i (v_p(x - r_2 \geq k) \wedge x \in F_i).$$

The formula $v_p(x-r_2) \geq k$ defines the ball $B(r_2,k)$, of finite radius k, and for each i, the outer ball of F_i has an infinite radius. As $D_m(x-r) \wedge x \in F_i$ defines a nonempty set, so too does $v_p(x-r_2) \geq k \wedge x \in F_i$, and hence the outer ball of F_i is contained in $B(r_2,k)$. Therefore $v_p(x-r_2) \geq k \wedge x \in F_i$ is equivalent to just $x \in F_i$, and so $\phi(x,b)$ is equivalent to

$$D_{m'}(x-r_1) \wedge \bigvee_i x \in F_i.$$

By Remark 9.2.3 (3) we may assume that each F_i is a proper cheese. We replace $\phi(x,b)$ by $\phi(x+r_1,b)$, and rename m' as m and each $F_i - r_1$ as F_i. Altogether, $\phi(x,b)$ is equivalent to a formula of the form

$$D_m(x) \wedge \bigvee_i x \in F_i$$

where $\gcd(m,p)=1$, and $\{F_i\}_i$ are disjoint proper cheeses having infinite radiuses. As $\phi(x,b)$ is not L^--definable, the disjunction cannot be empty.

By Remark 9.2.8, $D_m(x)$ defines a dense subgroup of $\mathcal{M}$. By successively applying Lemmas 9.2.10, 9.2.11 and 9.2.9, we get an L-definable formula of the form

$$D_m(x) \wedge x \in B(0,\gamma) \tag{$\star$}$$

with γ nonstandard and $\gcd(m,p)=1$. We will now assume that $\phi(x,b)$ is of this form.

To finish, we need the following:

Claim. Let $\psi(x,z)$ be any L_p-formula with $|x|=1$.

(1) Suppose there exists $a \in \mathcal{M}$ with $v(a)$ nonstandard, for which there exists b such that $\psi(x,b)$ is equivalent to $v(x) \geq v(a)$. Then for any c such that $v(c)$ is nonstandard there is $b' \in \mathcal{M}$ such that $tp(b'/\emptyset) = tp(b/\emptyset)$ (in L_p) and $\psi(x,b')$ is equivalent to $v(x) \geq v(c)$.

(2) Let $\theta(z)$ be an L_p-formula. Then there exists $K \in \mathbb{N}$ such that for any $a \in \mathcal{M}$ with $v(a) \geq K$, if there exists b such that $\theta(b)$ holds and $\psi(x,b)$ is equivalent to $v(x) \geq v(a)$, then for any c such that $v(c) \geq K$ there is $b' \in \mathcal{M}$ such that $\theta(b')$ and $\psi(x,b')$ is equivalent to $v(x) \geq v(c)$. That is, let $\alpha(w)$ be the formula defined by

$$\exists z(\theta(z) \wedge \forall x(\psi(x,z) \leftrightarrow v(x) \geq v(w)))$$

and let $\chi(w)$ be the formula defined by

$$\alpha(w) \to \forall w'(v(w') \geq K \to \alpha(w')).$$

Then $\chi(w)$ is satisfied by any a such that $v(a) \geq K$.

Proof of the claim. Proof of (1). We show that we can find $a' \in \mathcal{M}$ such that $tp(a'/\emptyset) = tp(a/\emptyset)$ and $v(a') = v(c)$. Indeed, let $\Sigma(x)$ be the partial type $tp(a/\emptyset) \cup \{v(x) = v(c)\}$. We show that it is consistent. Let $F \subseteq \Sigma(x)$ be a finite subset. As $v(a)$ is nonstandard, we may assume that F is of the form

$$\{x \neq j \,:\, -n \leq j \leq n\} \cup \{D_{m_k}(x - r_k) \,:\, 1 \leq k \leq s\} \cup \{v(x) = v(c)\}.$$

Let $m = \prod_k m_k$, and write $m = p^l m'$ with $\gcd(m',p)=1$. By **??**, there exists $\tilde{a} \in \mathcal{M}$ satisfying the formula $D_{m'}(x-a) \wedge (v(x) = v(c))$. So $v(\tilde{a}) = v(c)$ is nonstandard. As $v(a)$ is also nonstandard, $\tilde{a}$ also satisfies $D_{p^l}(x-a)$, so it satisfies $D_m(x-a)$, and therefore it satisfies $\{D_{m_k}(x-r_k) \,:\, 1 \leq k \leq s\}$. Also, as $v(\tilde{a})$ is nonstandard, $\tilde{a} \notin \mathbb{Z}$. Together we have that $\tilde{a}$ satisfies F.

So $\Sigma(x)$ is consistent. Let $a' \in \mathcal{M}$ be a realization of $\Sigma(x)$. As $tp(a'/\emptyset) = tp(a/\emptyset)$, there is an automorphism of L_p-structures $\sigma \in Aut(\mathcal{M}/\emptyset)$ such that $\sigma(a) = a'$. Let $b' = \sigma(b)$. So $tp(b'/\emptyset) = tp(b/\emptyset)$ and $\psi(x,b')$ is equivalent to $v(x) \geq v(a')$. As $v(a') = v(c)$, we have what we want.

Proof of (2). Let $\xi(w,w')$ be the formula defined by $\alpha(w) \to \alpha(w')$. By (1), $\xi(a,c)$ holds for any a,c such that $v(a)$ and $v(c)$ are nonstandard, so the result follows by compactness. $\square$

Now, let $\theta(z)$ be the formula expressing that $(\phi(x,z),+)$ is a subgroup. By Lemma 8.2.4 there are $n_1,\dots,n_k$, having $\gcd(n_i,p)=1$ for each i, such that for all $c\in\mathcal{M}$ for which $\theta(c)$ holds, $\phi(x,c)$ is equivalent to a formula of the form $D_{n_i}(x)\wedge v(x)\geq v(d)$ for some i and some $d\in\mathcal{M}$. As $(N,+,0,|_p)$ is an elementary substructure, if $c\in N$ then there exists such $d\in N$. Let $n=\prod_i n_i$, and let $\psi(x,z)$ be the formula $\phi(nx,z)$. Then for all $c\in\mathcal{M}$ for which $\theta(c)$ holds, $\psi(x,c)$ is equivalent to $v(x)\geq v(d)$, for the same d corresponding to $\phi(x,c)$ (as $v(n)=0$).

Let $K\in\mathbb{N}$ be as given by the claim for $\psi(x,z)$ and $\theta(z)$, and let $\alpha(w)$ and $\chi(w)$ be as in the claim. We have that $\psi(x,b)$ is equivalent to $v(x)\geq\gamma$. In particular, the formula $\rho(z)$ defined by

$$\theta(z)\wedge\exists w(v(w)\geq K\wedge\forall x(\psi(x,z)\leftrightarrow v(x)\geq v(w)))$$

is satisfied by b. Since $\rho(z)$ contains no parameters, there exists $c\in N$ such that $(N,+,0,|_p)\vDash\rho(c)$. So $\theta(c)$ holds and there exists $d\in N$ such that $v(d)\geq K$ and $\psi(x,c)$ is equivalent to $v(x)\geq v(d)$. So $(N,+,0,|_p)\vDash\alpha(d)$. As $v(d)\geq K$, by the claim we have $\mathcal{M}\vDash\chi(d)$. Since $\chi(w)$ contains no parameters, also $(N,+,0,|_p)\vDash\chi(d)$. Hence, as v_p is surjective, for every $\gamma\in\Gamma(N)$ such that $\gamma\geq K$ there exists $c_\gamma\in N$ such that $\theta(c_\gamma)$ holds and $\psi(x,c_\gamma)$ is equivalent to $v(x)\geq\gamma$.

Let $\delta(x,y)$ be the formula

$$\bigwedge_{k=1}^{K-1}(D_{p^k}(x)\to D_{p^k}(y))\wedge\forall z(\theta(z)\to(\psi(x,z)\to\psi(y,z))).$$

Then $\delta(x,y)$ is L-definable over $\emptyset$, and we claim that it defines $v(x)\leq v(y)$ in $\mathcal{N}$: Let $a_1,a_2\in N$, and suppose $v(a_1)\leq v(a_2)$. Then of course $\bigwedge_{k=1}^{K-1}(D_{p^k}(a_1)\to D_{p^k}(a_2))$. Let $c\in N$ such that $\theta(c)$. Then there exists $d\in N$ such that $\psi(x,c)$ is equivalent to $v(x)\geq v(d)$, and therefore also $\psi(a_1,c)\to\psi(a_2,c)$. So we have $\delta(a_1,a_2)$. On the other hand, suppose $\delta(a_1,a_2)$. If $v(a_1)\leq K-1$, then by $\bigwedge_{k=1}^{K-1}(D_{p^k}(a_1)\to D_{p^k}(a_2))$ we get $v(a_1)\leq v(a_2)$. Otherwise, we have that $\gamma:=v(a_1)\geq K$ and hence $\psi(a_1,c_\gamma)$. From $\forall z(\theta(z)\to(\psi(a_1,z)\to\psi(a_2,z)))$, as $\theta(c_\gamma)$ holds, we get in particular $\psi(a_1,c_\gamma)\to\psi(a_2,c_\gamma)$, and therefore we get $\psi(a_2,c_\gamma)$, which means $v(a_2)\geq\gamma=v(a_1)$. Therefore, $v(x)\leq v(y)$ is definable over $\emptyset$ in $\mathcal{N}$. $\square$

Combined with Fact 9.1.5 and Theorem 8.3.2, we obtain:

Theorem 9.2.13. *Let $(N,+,0,1,|_p)$ be an elementary extension of $(\mathbb{Z},+,0,1,|_p)$. Then $(N,+,0,1,|_p)$ is $\emptyset$-minimal among the $\emptyset$-proper $\emptyset$-expansions of $(N,+,0,1)$.*

Proof. Identical to the proof of Corollary 9.1.9 from Theorem 9.1.10. $\square$

In particular:

Corollary 9.2.14. *$(\mathbb{Z},+,0,1,|_p)$ is minimal among the proper expansions of $(\mathbb{Z},+,0,1)$.*

Proof. Identical to the proof of Fact 9.1.7 from Corollary 9.1.9. $\square$

9.3 Intermediate structures in elementary extensions: some counter-examples

In this section, we show that Fact 9.1.5, Fact 9.1.7 and Corollary 9.2.14 are no longer true if we replace $\mathbb{Z}$ by an elementarily equivalent structure. In the case of Corollary 9.2.14, there are both stable and unstable counterexamples. For Fact 9.1.7 there are unstable counterexamples, but we do not know whether there are stable ones.

For each of the above, we give a family of counterexamples.

Remark 9.3.1. Let $L \subseteq L^+$ be two first-order languages, let $\phi(x, y)$ be an L^+-formula, and let P be a new relation symbol. Let $\mathcal{N}$ be an L^+-structure, let $a, b \in \mathcal{N}$ be such that $tp(a/\emptyset) = tp(b/\emptyset)$ (in L^+), and let $A = \phi(\mathcal{N}, a)$, $B = \phi(\mathcal{N}, b)$. Let $\mathcal{N}_1$,$\mathcal{N}_2$ be two reducts of $\mathcal{N}$, both in the language $L \cup \{P\}$, such that $\mathcal{N}_1|_L = \mathcal{N}_2|_L = \mathcal{N}|_L$, $P(\mathcal{N}_1) = A$, $P(\mathcal{N}_2) = B$. Then $\mathcal{N}_1 \equiv \mathcal{N}_2$.

Proposition 9.3.2. *Let $(N, +, 0, 1, |_p)$ be a nontrivial elementary extension of $(\mathbb{Z}, +, 0, 1, |_p)$, let γ be a nonstandard element from Γ. Let $B = \{a \in N : v_p(a) \geq \gamma\}$. Then $(N, +, 0, 1, B)$ is a stable proper expansion of $(N, +, 0, 1)$ of dp-rank 1. In particular, it is a proper reduct of $(N, +, 0, 1, |_p)$.*

Proof. It is clear that $(N, +, 0, 1, B)$ is a proper expansion of $(N, +, 0, 1)$, and, as a reduct of $(N, +, 0, 1, |_p)$, by Theorem 8.3.2 it is of dp-rank 1. It remains to show stability. This follows from a theorem of Wagner, see Remark 9.3.3, but we also give a direct proof. First, we show that $Th(N, +, 0, 1, B)$ does not depend on N or b, as long as $v_p(b)$ is infinite, so it is enough to prove stability for just one particular choice of $(N, +, 0, 1, |_p)$ and b. Let $(N_2, +, 0, 1, |_p) \equiv (N, +, 0, 1, |_p)$, let $c \in N_2$ be such that $\delta := v_p(c)$ is nonstandard, and let $C = \{a \in N_2 : c|_p a\} = \{a \in N_2 : v_p(a) \geq \delta\}$. Let $(M, +, 0, 1, |_p)$ be a monster model, and let $B' = \{a \in M : b|_p a\}$, $C' = \{a \in M : c|_p a\}$. So $B = B' \cap N$, $C = C' \cap N_2$, and $(N, +, 0, 1, B) \prec (M, +, 0, 1, B')$, $(N_2, +, 0, 1, C) \prec (M, +, 0, 1, C')$. By Claim 9.2.2 (1), there exists $d \in M$ such that $tp(d/\emptyset) = tp(b/\emptyset)$ (in $\{+, 0, 1, |_p\}$) and $v_p(d) = v_p(c)$. Let $D' = \{a \in M : d|_p a\}$. Then $D' = C'$, and by Remark 9.3.1, $(M, +, 0, 1, D') \equiv (M, +, 0, 1, B')$. So $(N_2, +, 0, 1, C) \equiv (M, +, 0, 1, C') = (M, +, 0, 1, D') \equiv (M, +, 0, 1, B') \equiv (N, +, 0, 1, B)$.

Now, consider the valued ring $(\mathbb{Z}, +, \cdot, 0, 1, |_p)$, and let $\mathcal{M}_1 = (M, +, \cdot, 0, 1, |_p)$ be a monster model for its theory. Consider the partial type $\Sigma(x) = \{p^n|_p x : n \in \mathbb{N}\} \cup \{\forall y(x|_p y \leftrightarrow \exists z(y = x \cdot z))\}$. Then for each $n_0 \in \mathbb{N}$, p^{n_0} satisfies $\{p^n|_p x : n \leq n_0\} \cup \{\forall y(x|_p y \leftrightarrow \exists z(y = x \cdot z))\}$, so Σ is consistent. Let $b \vDash \Sigma$. Let $\mathcal{M}_2 = (M, +, 0, 1, \{\tilde{r}\}_{r \in M})$, where for each $r \in M$, $\tilde{r} : M \to M$ is the function $\tilde{r}(a) := r \cdot a$. So $\mathcal{M}_2$ is an $\mathcal{M}_1$-module in the language of $\mathcal{M}_1$-modules (expanded by the constant 1), and therefore it is stable (see e.g. [Poi00, Theorem 13.14]). Let $B = \{a \in M : b|_p a\}$, and let $\mathcal{M}_3 = (M, +, 0, 1, B)$. As $b \vDash \Sigma$, $B = \{a \in M : \exists z(a = b \cdot z)\} = \{a \in M : \exists z(a = \tilde{b}(z))\}$, so B is definable in $\mathcal{M}_2$. Hence $\mathcal{M}_3$ is a reduct of $\mathcal{M}_2$, and therefore it is stable. $\square$

Remark 9.3.3. In [Wag97, Example 0.3.1 and Theorem 4.2.8], Wagner defines an abelian structure to be an abelian group together with some predicates for subgroups of powers of this group. Every module is an abelian structure. Wagner proves that, as with modules, in an abelian structure every definable set is equal to a boolean combination of cosets of $\mathrm{acl}(\emptyset)$-definable subgroups. As a consequence, every abelian structure is stable. Under the assumptions of Proposition 9.3.2, B is a subgroup of N, so $(N, +, 0, 1, B)$ is an abelian structure. This immediately proves its stability.

Let $(N, +, 0, 1, |_p)$ be a nontrivial elementary extension of $(\mathbb{Z}, +, 0, 1, |_p)$. For $\gamma \in \Gamma$ we define

$$C_\gamma = \left\{(a, b) \in N^2 \,:\, v_p(a) \le \gamma \wedge v_p(b) \le \gamma \wedge v_p(a) \le v_p(b)\right\}.$$

Proposition 9.3.4. *Let $(N, +, 0, 1, |_p) \succ (\mathbb{Z}, +, 0, 1, |_p)$ be a nontrivial elementary extension, let $\gamma \in \Gamma$ be nonstandard. Then $(N, +, 0, 1, C_\gamma)$ is an unstable proper expansion of $(N, +, 0, 1)$ and a proper reduct of $(N, +, 0, 1, |_p)$.*

Proof. Let R be the relation symbol corresponding to C. It is clear that $(N, +, 0, 1, C_\gamma)$ is an unstable proper expansion of $(N, +, 0, 1)$. We show that $|_p$ is not definable with parameters in $(N, +, 0, 1, C)$. First, exactly as in Proposition 9.3.2, $Th(N, +, 0, 1, |_p, C)$ does not depend on N or c, as long as γ is nonstandard. That is, if $(N_2, +, 0, 1, |_p) \equiv (N, +, 0, 1, |_p)$, $d \in N_2$ is such that $\delta := v_p(d)$ is nonstandard, then $(N, +, 0, 1, |_p, C_\delta) \equiv (N, +, 0, 1, |_p, C_\gamma)$. So it is enough to prove this for just one particular choice of $(N, +, 0, 1, |_p)$ and γ..

For each $m \in \mathbb{N}$, let

$$\begin{aligned} C_m &= \{(a, b) \in \mathbb{Z}^2 \,:\, \neg D_{p^{m+1}}(a) \wedge \neg D_{p^{m+1}}(b) \wedge \bigwedge_{i=1}^{m} (D_{p^i}(a) \to D_{p^i}(b))\} \\ &= \{(a, b) \in \mathbb{Z}^2 \,:\, a|_p p^m \wedge b|_p p^m \wedge a|_p b\} \end{aligned}$$

and let $\mathcal{Z}_m = (\mathbb{Z}, +, 0, 1, |_p, C_m)$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and let $\mathcal{N} = \prod_{\mathcal{U}} \mathcal{Z}_m = (N, +, 0, 1, |_p, C)$ be the ultraproduct of $\{\mathcal{Z}_m\}_m$ with respect to $\mathcal{U}$. Let $\psi(z)$ be the formula $\forall x, y(R(x, y) \leftrightarrow x|_p z \wedge y|_p z \wedge x|_p y)$. For each $k \in \mathbb{N}$, for every $m \ge k$, $\mathcal{Z}_m \models (\exists z \psi(z)) \wedge \forall z(\psi(z) \to p^k|_p z)$, and therefore also $\mathcal{N} \models (\exists z \psi(x)) \wedge \forall z(\psi(z) \to p^k|_p z)$. Hence there exists $c \in N$ such that $\gamma := v_p(c)$ is infinite and $C = C_\gamma$.

Suppose for a contradiction that $|_p$ is definable in $(N, +, 0, 1, C)$. Then there is a formula $\phi(x, y, z)$ in the language of $(N, +, 0, 1, C)$ with $|x| = |y| = 1$, and there is $d \in N$, such that $\mathcal{N} \models \forall x, y(x|_p y \leftrightarrow \phi(x, y, d))$. Let $(d_m)_{m\in\mathbb{N}}$ be a representative for d mod $\mathcal{U}$. Then $\{m \in \mathbb{N} : \mathcal{Z}_m \models \forall x, y(x|_p y \leftrightarrow \phi(x, y, d_m))\} \in \mathcal{U}$. In particular, this set is not empty, so there exists $m \in \mathbb{N}$ such that $\mathcal{Z}_m \models \forall x, y(x|_p y \leftrightarrow \phi(x, y, d_m))$. Hence $|_p$ is definable in $(\mathbb{Z}, +, 0, 1, C_m)$. But C_m is definable in $(\mathbb{Z}, +, 0, 1)$, a contradiction. $\square$

Proposition 9.3.5. *Let $(N, +, 0, 1, <) \succ (\mathbb{Z}, +, 0, 1, <)$ be a non-trivial elementary extension, let $b \in N$ be a positive infinite element, and let $B = [0, b]$. Then $(N, +, 0, 1, B)$ is an unstable proper expansion of $(N, +, 0, 1)$ and a proper reduct of $(N, +, 0, 1, <)$.*

Proof. Let P be the relation symbol corresponding to B. It is clear that $(N,+,0,1,B)$ is a proper expansion of $(N,+,0,1)$. The formula $P(y-x)$ defines the ordering on B, so this structure is unstable. It remains to show that $<$ is not definable with parameters in $(N,+,0,1,B)$. First, we show that it is enough to prove this for a single choice of N and b (though in this case, the theory does depend on $tp(b/\emptyset)$). Let $(N_2,+,0,1,<) \equiv (N,+,0,1,<)$, let $c \in N_2$ be a positive infinite element, and let $C = \{a \in N : 0 \leq a \leq c\} = [0,c]$. Suppose that $<$ is not definable with parameters in $(N_2,+,0,1,C)$. Let $(M,+,0,1,<)$ be a monster model, and let $B' = \{a \in M : 0 \leq a \leq b\}$, $C' = \{a \in M : 0 \leq a \leq c\}$. So $B = B' \cap N$, $C = C' \cap N_2$. By Lemma 9.1.2 (with $A = \{c\}$), $(N_2,+,0,1,C) \prec (M,+,0,1,C')$ and $<$ is not definable with parameters in $(M,+,0,1,C')$. Similarly, $(N,+,0,1,B) \prec (M,+,0,1,B')$, and $<$ is definable with parameters in $(N,+,0,1,B)$ if and only if it is definable with parameters in $(M,+,0,1,B')$. As c is a positive infinite element, $tp(c/\emptyset)$ in $\{+,0,1,<\}$ is unbounded from above in $(M,+,0,1,<)$. Let $d \in M$ such that $d > b$ and $tp(d/\emptyset) = tp(c/\emptyset)$. Let $D' = \{a \in M : 0 \leq a \leq d\}$. By Remark 9.3.1 (with $L = L^+ = \{+,0,1,<\}$), $(M,+,0,1,<,C') \equiv (M,+,0,1,<,D')$, so in particular, $<$ is not definable in $(M,+,0,1,D')$. As $d > b$, $[0,b] = [0,d] \cap [-d+b,b]$, and so the formula $P(x) \wedge P(-x+b)$ defines B' in $(M,+,0,1,D')$. So $(M,+,0,1,B')$ is a reduct of $(M,+,0,1,D')$, and hence $<$ is not definable in $(M,+,0,1,B')$.

Now, for each $m \in \mathbb{N}$, let $B_m = [0,m]$, and let $\mathcal{Z}_m = (\mathbb{Z},+,0,1,<,B_m)$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and let $\mathcal{N} = \prod_{\mathcal{U}} \mathcal{Z}_m = (N,+,0,1,<,B)$ be the ultraproduct of $\{\mathcal{Z}_m\}_m$ with respect to $\mathcal{U}$. For each $k \in \mathbb{N}$, for every $m \geq k$,

$$\mathcal{Z}_m \models \exists! x((\forall y(P(y) \leftrightarrow 0 \leq y \leq x)) \wedge x \geq \underline{k})$$

and therefore also $\mathcal{N} \models \exists! x((\forall y(P(y) \leftrightarrow 0 \leq y \leq x)) \wedge x \geq \underline{k})$. Hence there exists a positive infinite element $b \in N$ such that $B = [0,b]$.

Suppose for a contradiction that $<$ is definable in $(N,+,0,1,B)$. Then there is a formula $\phi(x,y,z)$ in the language of $(N,+,0,1,B)$ with $|x| = |y| = 1$, and there is $c \in N$, such that $\mathcal{N} \models \forall x,y(x < y \leftrightarrow \phi(x,y,c))$. Let $(c_m)_{m \in \mathbb{N}}$ be a representative for c mod $\mathcal{U}$. Then $\{m \in \mathbb{N} : \mathcal{Z}_m \models \forall x,y(x < y \leftrightarrow \phi(x,y,c_m))\} \in \mathcal{U}$. In particular, this set is not empty, so there exists $m \in \mathbb{N}$ such that $\mathcal{Z}_m \models \forall x,y(x < y \leftrightarrow \phi(x,y,c_m))$. Hence $<$ is definable in $(\mathbb{Z},+,0,1,B_m)$, a contradiction. ∎

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