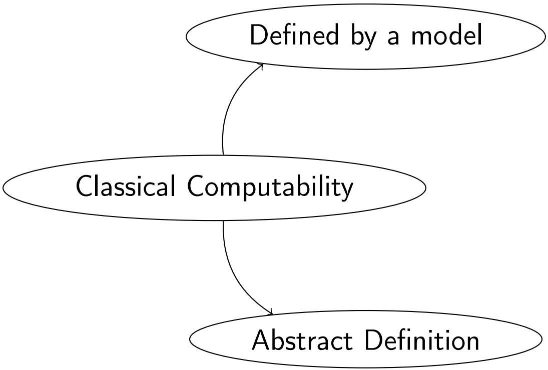
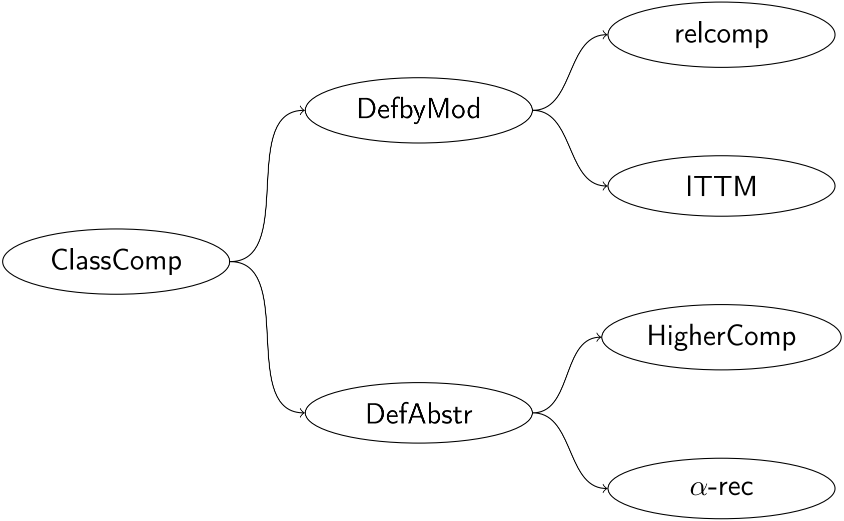
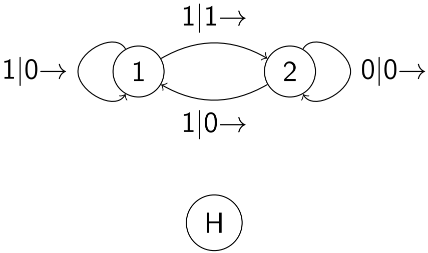
Higher Computability and Randomness

Paul-Elliot Anglès d'Auriac
Benoît Monin

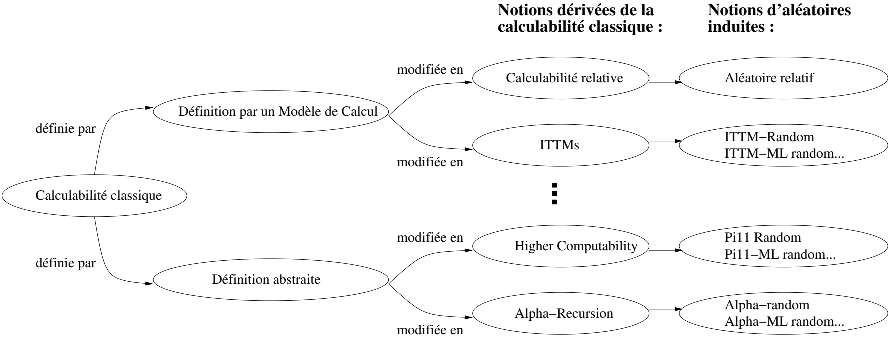
10 mai 2017


Defined by a model
Classical Computability
Abstract Definition

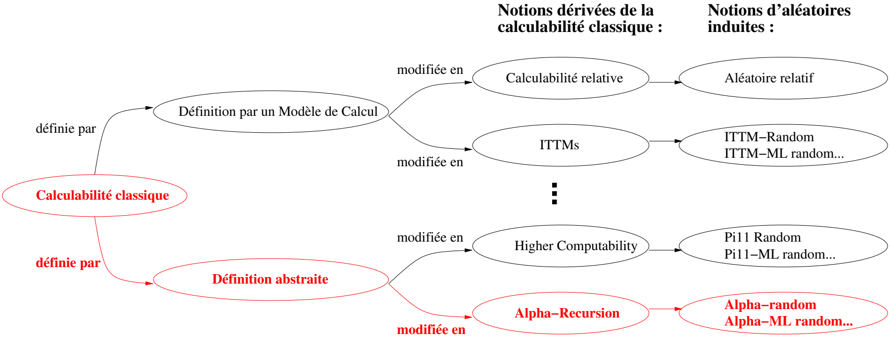

ClassComp
DefbyMod
DefAbstr
relcomp
ITTM
HigherComp
α-rec


1|1→
1|0→
1
2
0|0→
1|0→
H

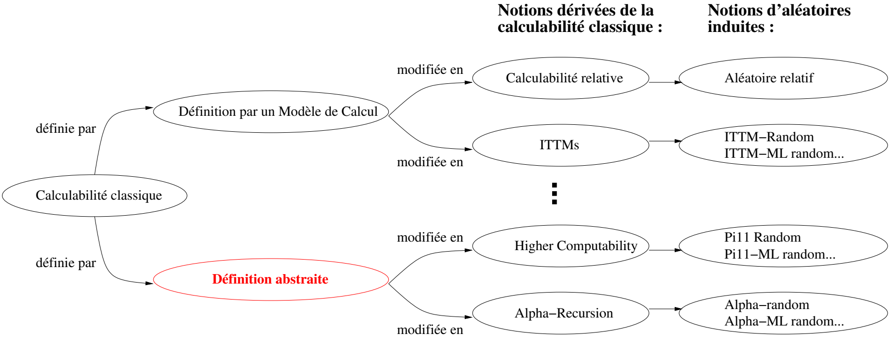
Étendre la calculabilité



Plan de l'exposé



Première étape

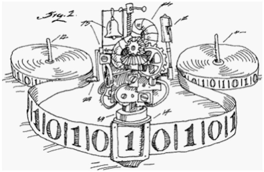


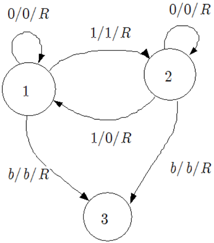
Infinite Time Turing Machine

Définition (Turing Machine)

A Turing Machine is an abstract model of computation, with :

- a finite quantity of states,
- an inifinite tape as memory, with a reading head,
- some transitions between states.





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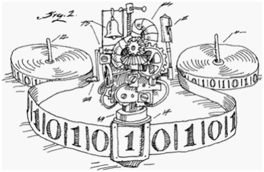
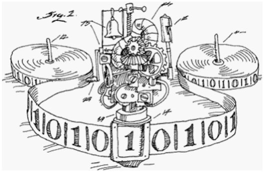


```
10 Input A
15 Input B
20 B = A + 10
30 IF B > 12 GOTO 60
40 C = B / 3
50 IF C < 24 GOTO 10
60 Write C
70 IF Write Failed GOTO 15
80 Input D
```

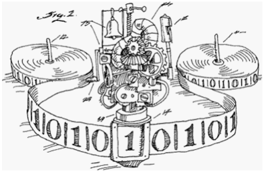
A computation



in state q_0.

A Turing Machine's Clock :

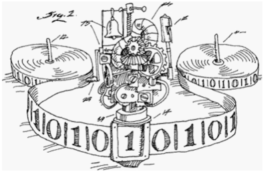
A computation



in state q_1.

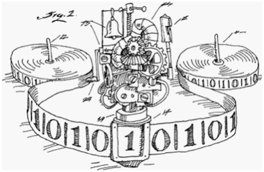
A Turing Machine's Clock :

A computation

 in state q_2.

A Turing Machine's Clock :

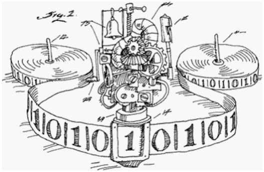
A computation



in state q_3.

A Turing Machine's Clock :

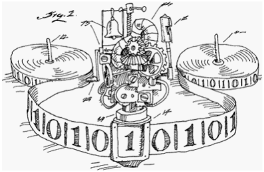
A computation



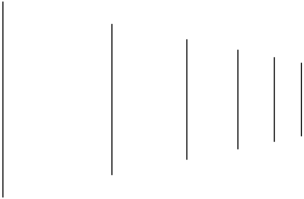
in state q_4.

A Turing Machine's Clock :

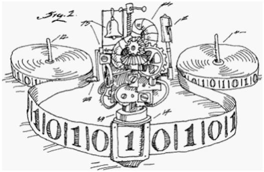
A computation

 in state q_5.

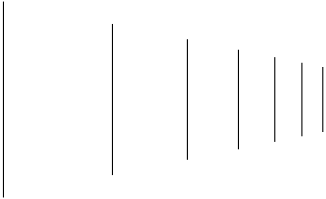
A Turing Machine's Clock :



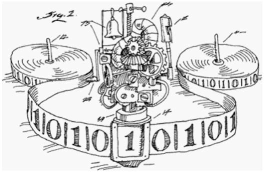
A computation

 in state q_6.

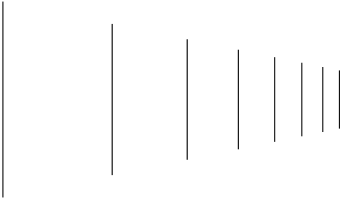
A Turing Machine's Clock :



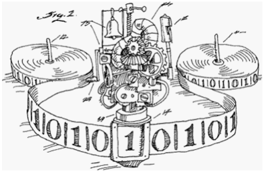
A computation

 in state q_7.

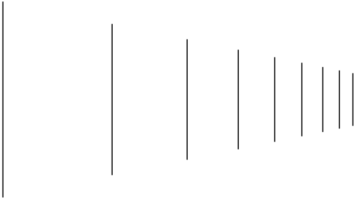
A Turing Machine's Clock :



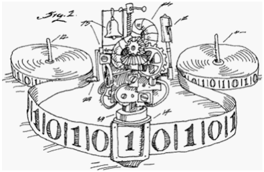
A computation

 in state q_8.

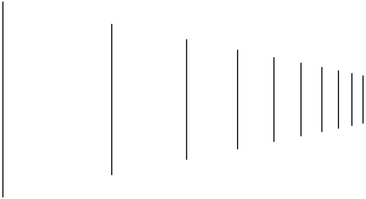
A Turing Machine's Clock :



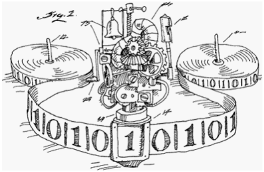
A computation

 in state q_9.

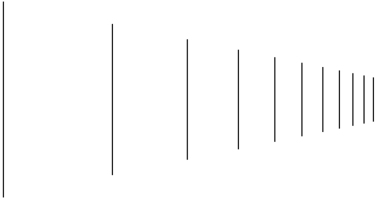
A Turing Machine's Clock :



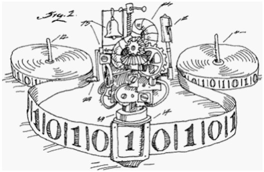
A computation

 in state q_{10}.

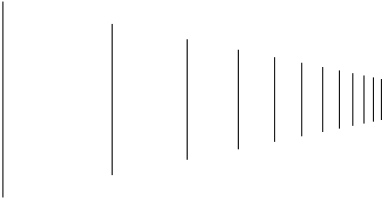
A Turing Machine's Clock :



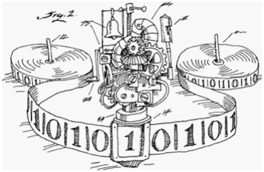
A computation

 in state q_{11}.

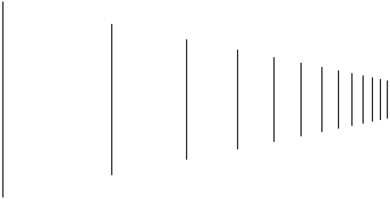
A Turing Machine's Clock :



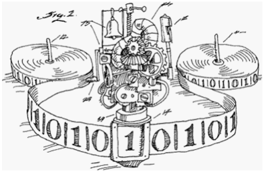
A computation

 in state q_{12}.

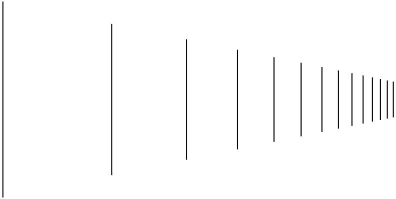
A Turing Machine's Clock :



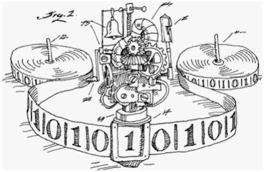
A computation

 in state q_{13}.

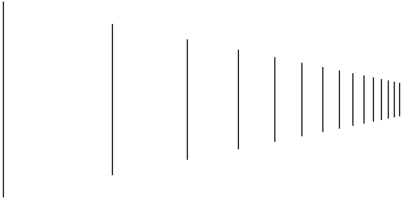
A Turing Machine's Clock :



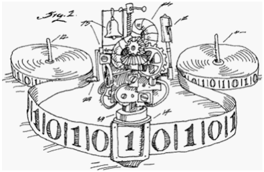
A computation

 in state q_{14}.

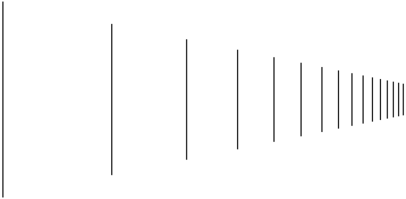
A Turing Machine's Clock :



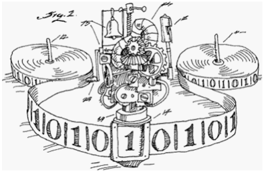
A computation

 in state q_{15}.

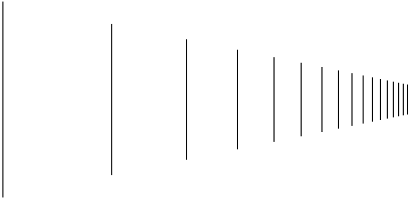
A Turing Machine's Clock :



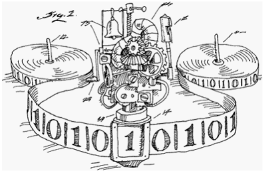
A computation

 in state q_{16}.

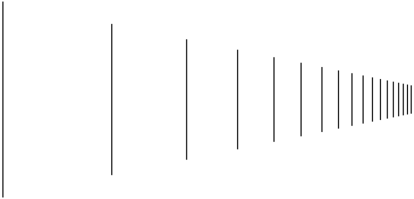
A Turing Machine's Clock :



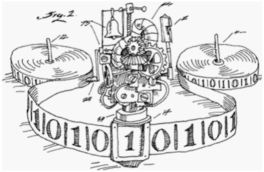
A computation

 in state q_{17}.

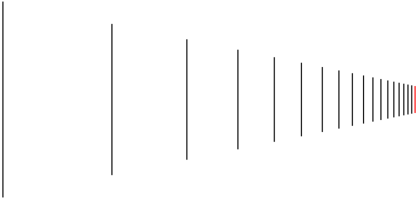
A Turing Machine's Clock :



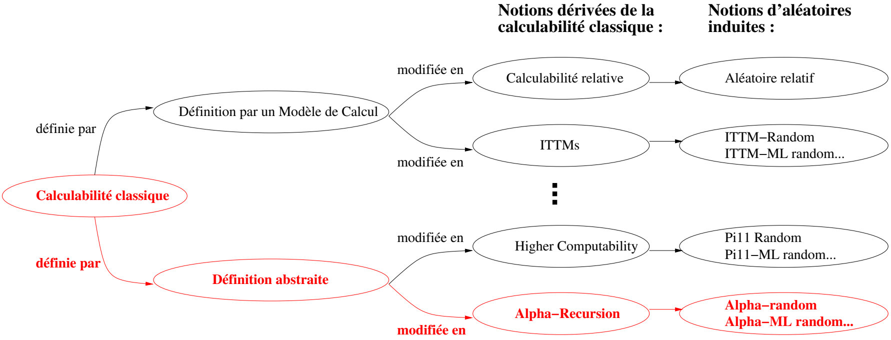
A computation


in state H.

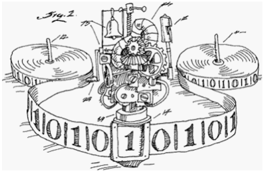
A Turing Machine's Clock :



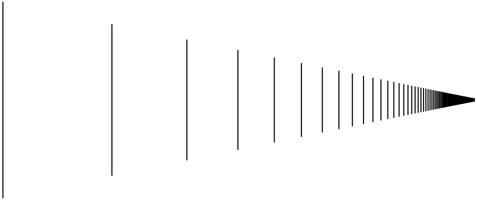
Plan de l'exposé



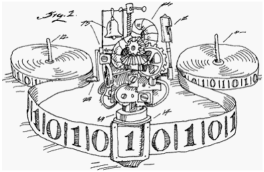
Hardware Clock



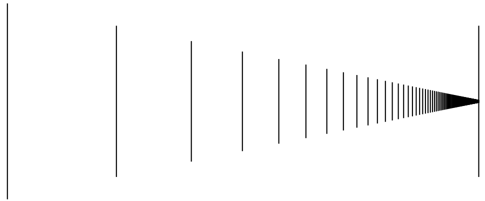
The Hardware Clock of all Turing Machines (compactified) :



New Hardware Clock



What if we consider instead :



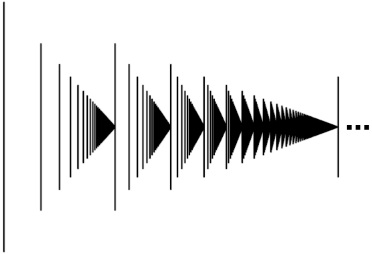
or even more ticks ?

Ordinals

Définition

An ordinal is a set α such that

1. α is transitive : $\forall x \in \alpha, \forall y \in x, y \in \alpha$
2. $(\alpha, \in)$ is a well ordering.



- Some ordinals are successors,
- some ordinals are limits.

Behaviour

Question

What is the behaviour at a limit stage ?

Answer

- The current state become the "limit" state,
- each cell become the liminf of its previous values.

Cellule 0 was : 1,0,0,1,1,1,1,1,1,1,1,... $\rightarrow$ now it is 1.
Cellule 1 was : 1,1,1,0,1,0,1,1,0,0,1,... $\rightarrow$ now it is 0.

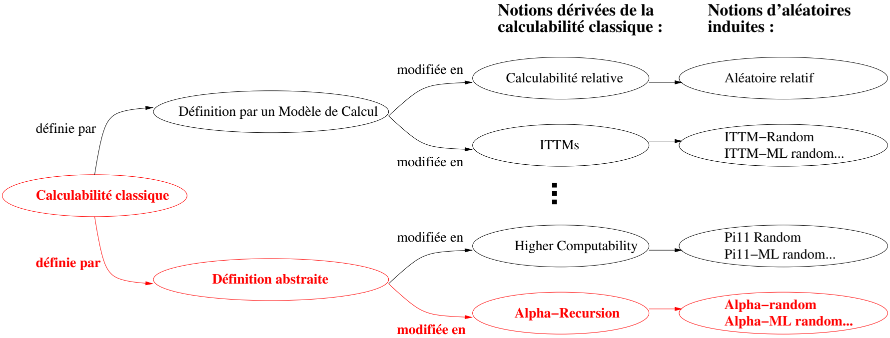
Links between Définability and Computability

- Is the behaviour original as far as we go through the ordinals ?
- What can these machines compute ?
- What's all that for ?

Links between Définability and Computability

- Is the behaviour original as far as we go through the ordinals ?
  - No, no machine would stop if it reach a certain ordinal, λ, no machine would stabilize if it reach another ordinal, ζ, and all machine will loop if it reaches a third ordinal ordinal, Σ.
- What can these machines compute ?
  - For example, the Halting Problem can be written on a tape ;
  - however, only countably many strings are writable ;
  - we will see soon a caracterization of what can be computed.
- What's all that for ?
  - The ordinals λ, ζ and Σ have interesting properties.
  - A new notion helps us understand the previous one.

Plan de l'exposé



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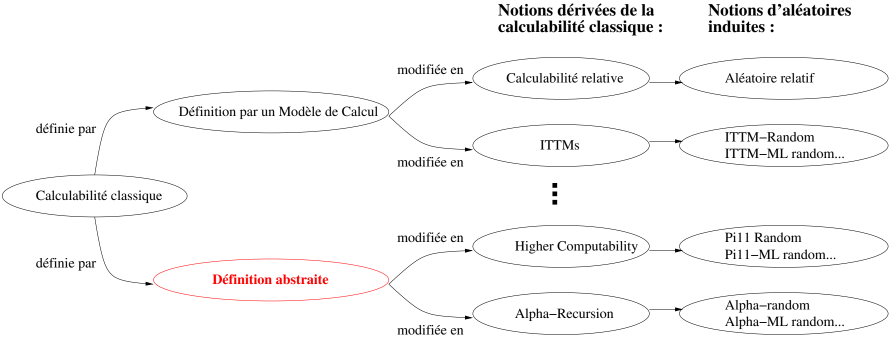

Abstract ourselves from computationnal model

Denote HF the set consisting of all hereditarily finite sets. The following theorem caracterise the notion of "being computable" :

Théorème

Let $A \subseteq \mathbb{N}$, then :

1. *A is computable iff A is Δ_1-comprehensible in HF,*
2. *A is recursively enumerable iff A is Σ_1-comprehensible in HF,*

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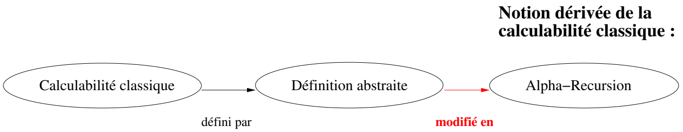

What's next

- We have a definition, parametrized by a set,
- to modify it we need to find the sets for which the definition stays interesting ;
- we will use Godel's constructibles.

Introduction to Godel's constructibles

$$\mathbb{N},$$

$$\{n \in \mathbb{N} : n \text{ is even}\},$$

$$\{n \in \mathbb{N} : n \text{ is prime}\},$$

$$\{n \in \mathbb{N} : \text{the } n\text{-th diophantine equation has a solution}\},$$

$$\{n \in \mathbb{N} : \phi(n)\} \text{ where } \phi \text{ is a formula.}$$

Remarks :

1. Are there any other sets than these ?
2. Maybe there are a lot ?

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Remarks :

1. Are there any other sets than these ?
   - Yes, by cardinality... An example ?
2. Maybe there are a lot ?
   - As a study, we can try to have the least possible such sets

A universe of sets with no superfluous : strategy

Idea

1. If we have nothing, we have no superfluous
2. If we have something, M, we need to have the sets shaped like :

$$\{x \in M | \phi(x, p)\}$$

   for every formula ϕ and parameters p in M.

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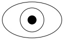

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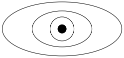

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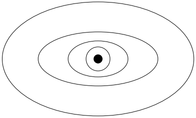

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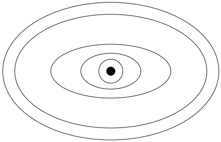

A universe of sets with no superfluous : strategy

Idea

1. If we have nothing, we have no superfluous
2. If we have something, M, we need to have the sets shaped like :

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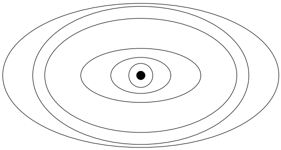

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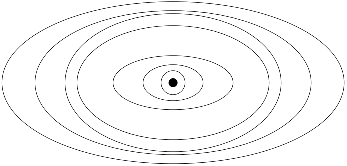

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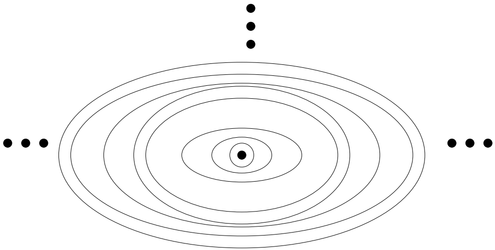
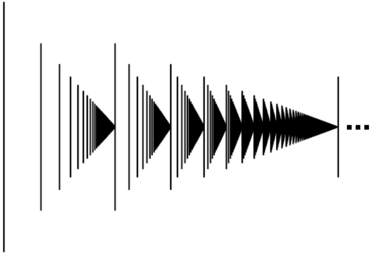


Ordinals

Définition

An ordinal is a set α such that

1. α is transitive : $\forall x \in \alpha, \forall y \in x, y \in \alpha$
2. $(\alpha, \in)$ is a well ordering.



- Some ordinals are successors,
- some ordinals are limits.

A precise definition

Gödel's constructible universe (1938)

Gödel's constructible at rank α, written L_α are defined by induction alons ordinals :

1. $L_0 = \emptyset$,
2. $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$,
3. $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$.

The constructibles are the elements of $\bigcup_\alpha L_\alpha$.

Définition

$$Def(M) = \left\{ E^M_{\phi,\bar{p}} : \phi \text{ is a formula and } \bar{p} \in M \right\}$$

where

$$E^M_{\phi,\bar{p}} = \{x \in M : \phi(x, \bar{p}) \text{ is true in } M\}$$

Illustration

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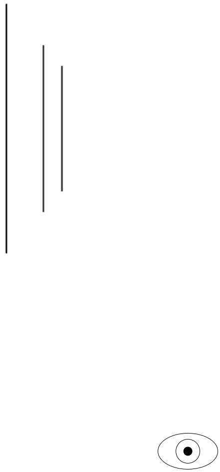

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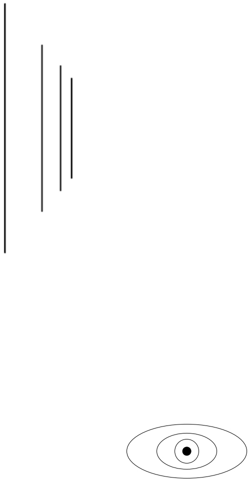

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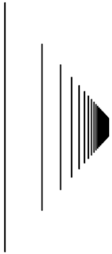

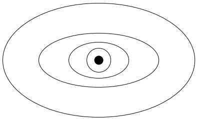

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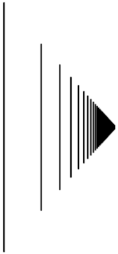

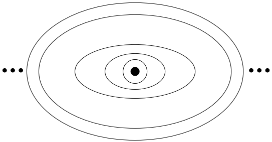

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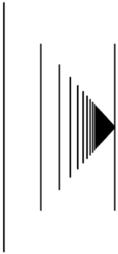

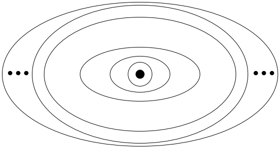

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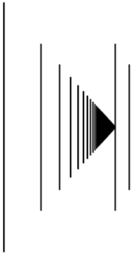

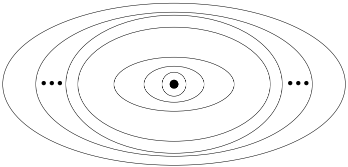

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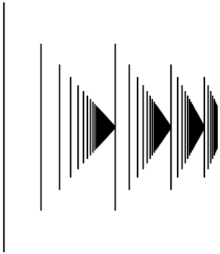

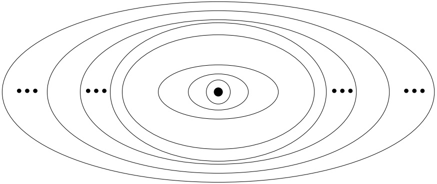

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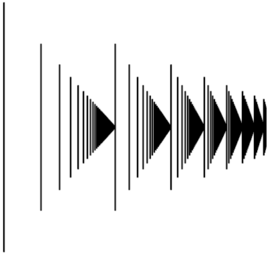

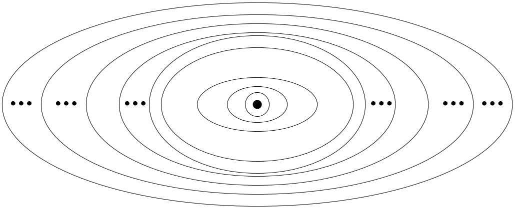

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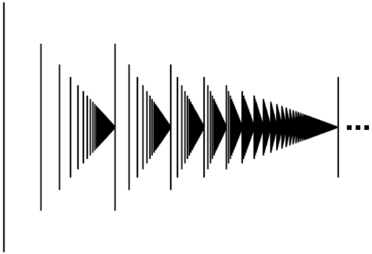

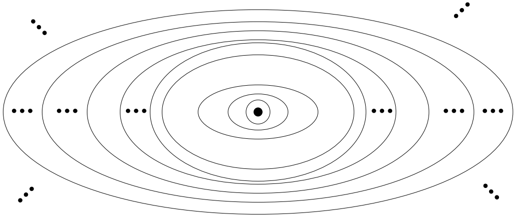

Examples

The constructibles are constructed layer by layer. These are some particular layers :

1. $L_{n+1} = \mathcal{P}(L_n)$ for n an integer ;
2. $L_\omega = \mathrm{HF}$, the hereditarily finite sets ;
3. $L_{\omega_1^{CK}} = \mathrm{HYP}$, the sets with hyperarithmetic codes ;
4. $L_\lambda = \mathrm{WRT}$, the sets with writable codes.

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Théorème

Let $A \subseteq \mathbb{N}$, then :

1. *A is computable iff A is Δ_1-comprehensible in L_ω,*
2. *A is recursively enumerable iff A is Σ_1-comprehensible in L_ω,*

Computability in a space of sets

The basic definition of α-recursion :

Definition

Let α be an ordinal and $A \subseteq L_\alpha$. We say that :

1. A is α-finite if $A \in L_\alpha$;
2. A is α-recursive if A is Δ_1-comprehensible in L_α ;
3. A is α-recursively enumerable if A is Σ_1-comprehensible in L_α.

- Some α will reveal more intersting than others,
- A is a set of α-finite elements, not only integers.

Intuition

We see a computation as a search into all the α-finite sets.

Admissibility I

It is not yet finished ! Because :

Remark

Some α will reveal more intersting than others...

- Which α ?
- Then, what are the properties of L_α ?

Admissibility I

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Some α will reveal more intersting than others...

- Which α ?
  - The admissibles ordinals, the ω_1^X for any $X \in 2^\omega$.
- Then, what are the properties of L_α ?
  - L_α is then admissible, it verifies the Kripke Platek axioms : L_α is a model of Δ_1-comprehension et Σ_1-collection.

Admissibility II

Définition

- A set is said admissible if it verifies the Kripke-Platek axioms, of which the most notable are Δ_1-comprehension ans Σ_1-collection.
- An ordinal α is said to be admissible if L_α is admissible.

- L_ω, $L_{\omega_1^{CK}}$, L_λ are admissibles.
- If α is admissible, the mapping of an α-finite by a function of α-recursive graph is α-finite.

Intuition

An ordinal α is admissible if the α-recursion is not too far from computability.

What did we defined ?

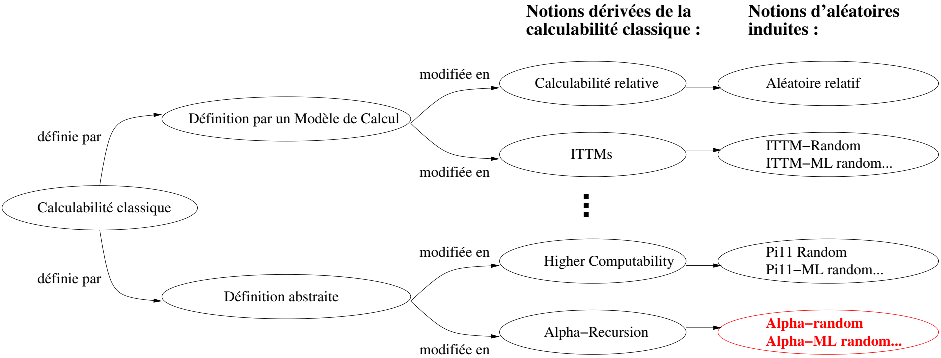
Intuition

We see a computation as a search into all the α-finite sets.

- ω-recursion, is classical computability ;
- ω_1^{CK}-recursion, is higher computability ;
- λ-recursion, is ITTM computability.

We have a general and satisfying definition of computability.

Randomness Part



Defining randomness...

A randomly chosen sequence of bits

0 1 1 1 0 0 1 1 1 1 1 0 1 0 0 0 1 0 1 0 1 . . .

There exists several paradigms to define what it is to be random for a sequence of bits :

1. Impredictability,
2. Incompressibility of prefixes,
3. No exceptionnal properties.

We will use the third paradigm.

Algorithmic randomness ?

Question

For $X \in 2^\omega$, what does it means for X to be a random set ?

Algorithmic randomness ?

Question

For $X \in 2^\omega$, what does it means for X to be a random set ?

1. Has no more even numbers than odd ones,
2. is not computable,
3. Is not like $b_0 0 b_1 0 b_2 0 \ldots$

We define randomness by the negative : we remove those which do not seem random.

Formally

Paradigme

X is random if X has no exceptionnal property

Becomes

Définition

X is $\mathscr{C}$-random if $\forall P \in \mathscr{C}$ such that $\lambda(P) = 0$, $X \notin P$

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Examples of $\mathscr{C}$:

1. the null Π^0_2,
2. the null Δ^1_1,
3. the Martin-Löf tests...

$\mathscr{C}$ countable ensures us that the $\mathscr{C}$-randoms are conull.

Martin-Löf Random

- Martin-Löf randomness has been the most studied.
- It has a definition for every of the three paradigm : impredictability, incompressibility of prefixes, and no exceptionnal properties.

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Définition (Martin-Löf's tests)

A Martin-Löf test is an intersection $\bigcap_n \mathcal{U}_n$, where $(\mathcal{U}_n)$ is recursively enumerable, and $\lambda(\mathcal{U}_n) \leq 2^{-n}$.

Also called Π^0_2 effectively null.

Définition (Martin-Löf Random)

X is Martin-Löf Random if X do not belong to any Martin-Löf test.

α-randomness

Following this principle, we define the tests in L_α.

Définition

X is random over L_α (or α-random) if X do not belong to any null borel set with code in L_α.

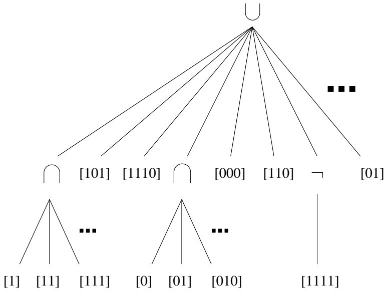

Figure – A borel code

- ω_1^{CK}-randomness is Δ^1_1-randomness,
- λ-randomness is ITTM-randomness.

α-ML-randomness

We continue the process to generalise Martin-Löf's idea :

Définition

- An α-ML test is a Martin-Löf test $\mathcal{U} \subseteq \omega \times 2^{<\omega}$ which is α-recursively enumerable.
- X is α-ML random if it is in no α-ML tests.

- ω-ML randomness is ML random,
- ω_1^{CK}-ML randomness is Π^1_1-ML randomness,
- λ-ML randomness is $\mathrm{ITTM}_{\mathrm{ML}}$ randomness

A question

Question

For every α, do the notions of "α-random" and "α-ML random" coincide ?

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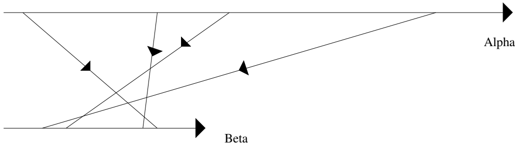
Théorème

Δ^1_1-randomness and Π^1_1-ML randomness are different notion.

This answers the quesion in a particular case. We would like a condition on α for it to be true.

Projectibility



Définition

α is projectible into β if there exists anα-recursive function, one-one from α to β.

- ω_1^{CK}, λ are projectible into ω ;
- not every ordinals are projective into a smaller ordinal thant themselves.

An equivalence

Théorème

The following are equivalent :

1. α *is projectible into* ω, *and*
2. α*-randomness and* α*-ML randomness are different notions.*

Being projectible into ω allows us to reduce "space" and "time" into a single dimension.

Corollaire

ITTM-randomness et ITTM-ML randomness are two different notions.

Conclusion

- L'α-recursion extends computability, and includes other extensions ;
- it allows us to define new notions of randomness ;
- we have an equivalence between a property of set theory and a property of algorithmic randomness.

Thanks for your attention !