α-Recursion and Randomness

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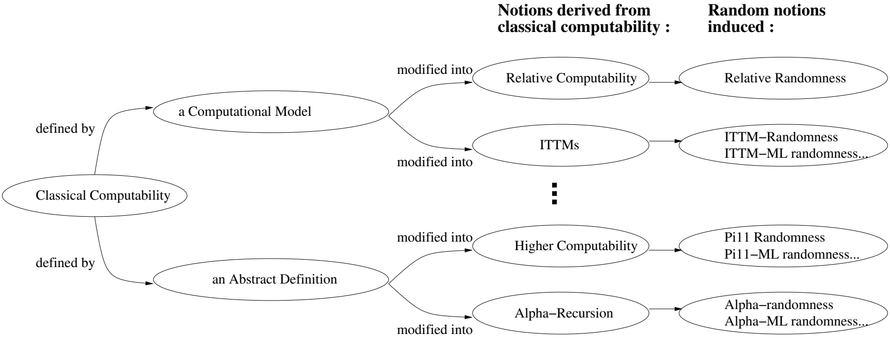
13 avril 2017

Extending computability



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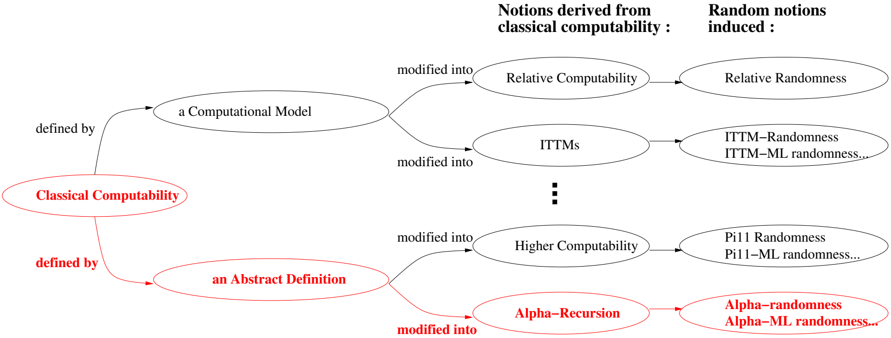
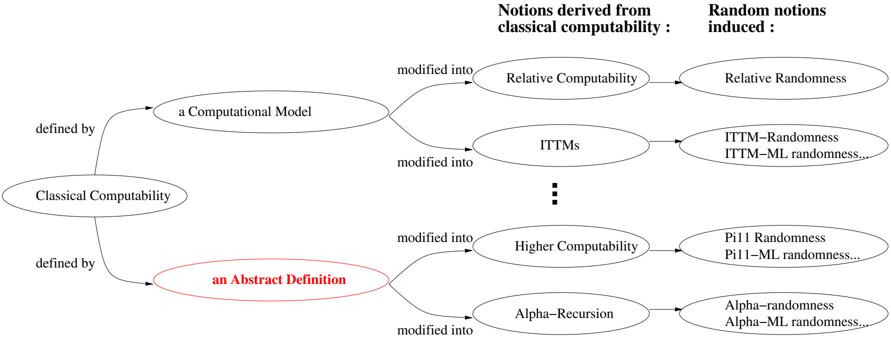


First step


Notions derived from classical computability :
Random notions induced :
Classical Computability
defined by
a Computational Model
defined by
an Abstract Definition
modified into
Relative Computability
ITTMs
Higher Computability
Alpha–Recursion
Relative Randomness
ITTM–Randomness
ITTM–ML randomness...
Pi11 Randomness
Pi11–ML randomness...
Alpha–randomness
Alpha–ML randomness...

Abstract ourselves from computationnal model

Denote HF the set consisting of all hereditarily finite sets. The following theorem caracterise the notion of "being computable" :

Theorem

Let $A \subseteq \mathbb{N}$, then :

1. *A is computable iff A is Δ_1-comprehensible in HF,*
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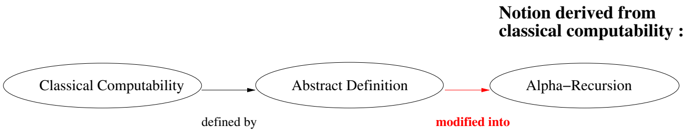

What's next

- We have a definition, parametrized by a set,
- to modify it we need to find the sets for which the definition stays interesting ;
- we will use Godel's constructibles.

Introduction to Godel's constructibles

$$\mathbb{N},$$

$$\{n \in \mathbb{N} : n \text{ is even}\},$$

$$\{n \in \mathbb{N} : n \text{ is prime}\},$$

$$\{n \in \mathbb{N} : \text{the } n\text{-th diophantine equation has a solution}\},$$

$$\{n \in \mathbb{N} : \phi(n)\} \text{ where } \phi \text{ is a formula.}$$

Remarks :

1. Are there any other sets than these ?
2. Maybe there are a lot ?

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Remarks :

1. Are there any other sets than these ?
   - Yes, by cardinality... An example ?
2. Maybe there are a lot ?
   - As a study, we can try to have the least possible such sets

A universe of sets with no superfluous : strategy

Idea

1. If we have nothing, we have no superfluous
2. If we have something, M, we need to have the sets shaped like :
$$\{x \in M | \phi(x, p)\}$$
for every formula ϕ and parameters p in M.

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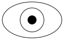

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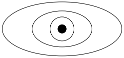

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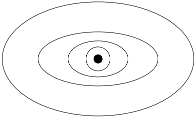

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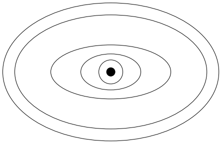

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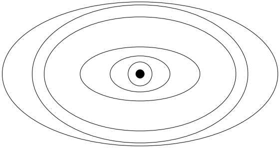

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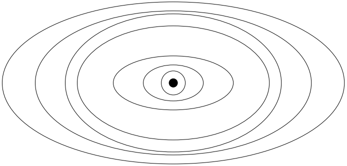

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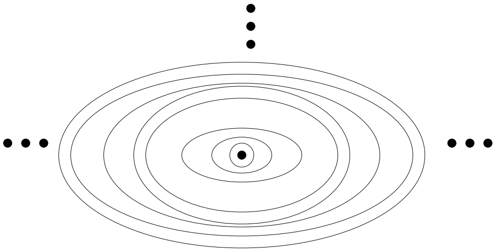
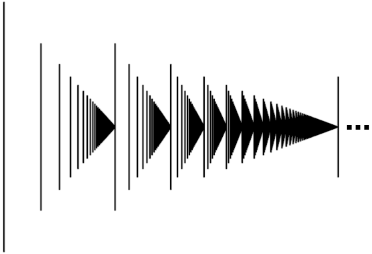


Ordinals

Definition

An ordinal is a set α such that

1. α is transitive : $\forall x \in \alpha, \forall y \in x, y \in \alpha$
2. $(\alpha, \in)$ is a well ordering.



- Some ordinals are successors,
- some ordinals are limits.

A precise definition

Gödel's constructible universe (1938)

Gödel's constructible at rank α, written L_α are defined by induction alons ordinals :

1. $L_0 = \emptyset$,
2. $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$,
3. $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$.

The constructibles are the elements of $\bigcup_\alpha L_\alpha$.

Definition

$$Def(M) = \left\{ E^M_{\phi,\bar{p}} : \phi \text{ is a formula and } \bar{p} \in M \right\}$$

where

$$E^M_{\phi,\bar{p}} = \{x \in M : \phi(x, \bar{p}) \text{ is true in } M\}$$

Illustration

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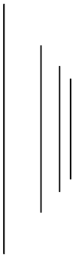

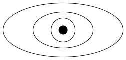

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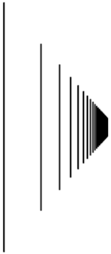

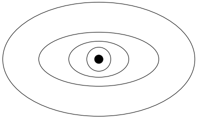

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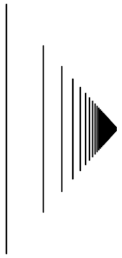

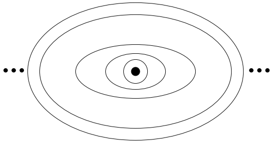

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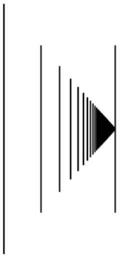

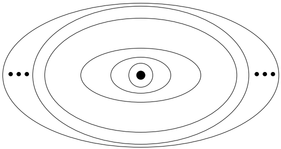

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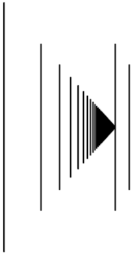

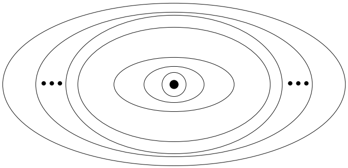

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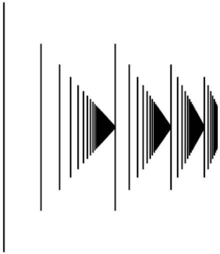

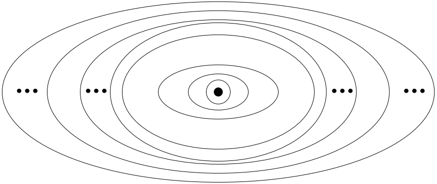

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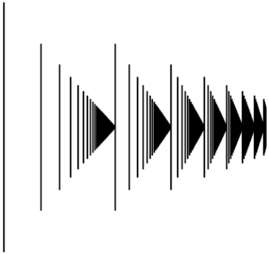

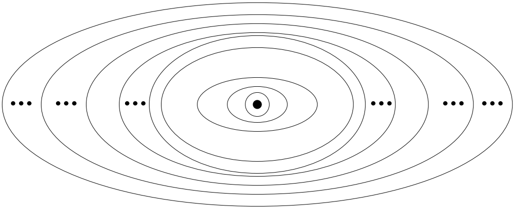

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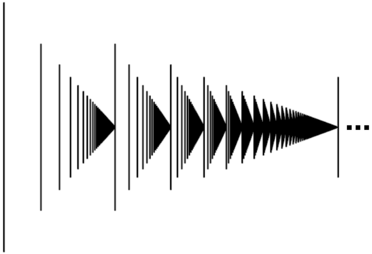

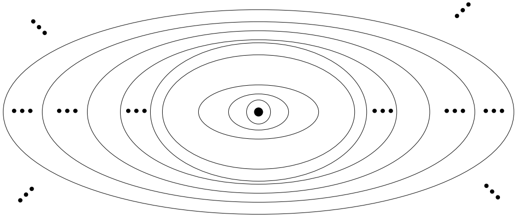

Examples

The constructibles are constructed layer by layer. These are some particular layers :

1. $L_{n+1} = \mathcal{P}(L_n)$ for n an integer ;
2. $L_\omega = \mathrm{HF}$, the hereditarily finite sets ;
3. $L_{\omega_1^{CK}} = \mathrm{HYP}$, the sets with hyperarithmetic codes ;
4. $L_\lambda = \mathrm{WRT}$, the sets with writable codes.

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Computability in a space of sets

The basic definition of α-recursion :

Definition

Let α be an ordinal and $A \subseteq L_\alpha$. We say that :

1. A is α-finite if $A \in L_\alpha$;
2. A is α-recursive if A is Δ_1-comprehensible in L_α ;
3. A is α-recursively enumerable if A is Σ_1-comprehensible in L_α.

- Some α will reveal more interesting than others,
- A is a set of α-finite elements, not only integers.

Intuition

We see a computation as a search into all the α-finite sets.

Admissibility I

It is not yet finished ! Because :

Remark

Some α will reveal more interesting than others...

- Which α ?
- Then, what are the properties of L_α ?

Admissibility I

It is not yet finished ! Because :

Remark

Some α will reveal more interesting than others...

- Which α ?
  - The admissibles ordinals, the ω_1^X for any $X \in 2^\omega$.
- Then, what are the properties of L_α ?
  - L_α is then admissible, it verifies the Kripke Platek axioms : L_α is a model of Δ_1-comprehension et Σ_1-collection.

Admissibility II

Definition

- A set is said admissible if it verifies the Kripke-Platek axioms, of which the most notable are Δ_1-comprehension and Σ_1-collection.
- An ordinal α is said to be admissible if L_α is admissible.

- L_ω, $L_{\omega_1^{CK}}$, L_λ are admissibles.
- If α is admissible, the mapping of an α-finite by a function of α-recursive graph is α-finite.

Intuition

An ordinal α is admissible if the α-recursion is not too far from computability.

What did we defined ?

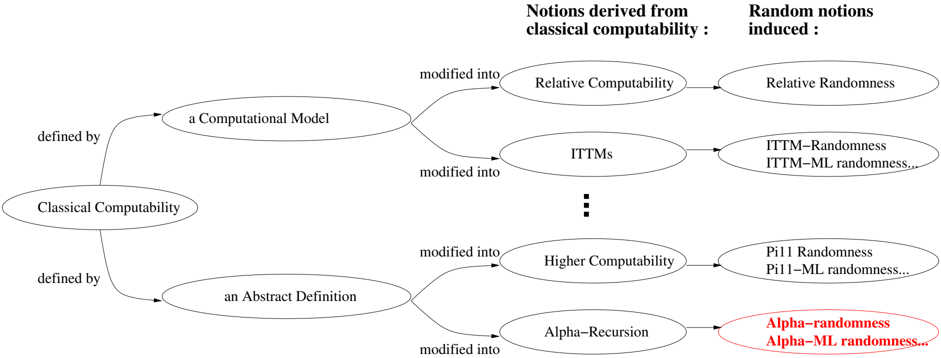
Intuition

We see a computation as a search into all the α-finite sets.

- ω-recursion, is classical computability ;
- ω_1^{CK}-recursion, is higher computability ;
- λ-recursion, is ITTM computability.

We have a general and satisfying definition of computability.

Randomness Part



Defining randomness...

A randomly chosen sequence of bits

0 1 1 1 0 0 1 1 1 1 1 0 1 0 0 0 1 0 1 0 1 . . .

There exists several paradigms to define what it is to be random for a sequence of bits :

1. Impredictability,
2. Incompressibility of prefixes,
3. No exceptionnal properties.

We will use the third paradigm.

Algorithmic randomness ?

Question

For $X \in 2^\omega$, what does it means for X to be a random set ?

Algorithmic randomness ?

Question

For $X \in 2^\omega$, what does it means for X to be a random set ?

1. Has no more even numbers than odd ones,
2. is not computable,
3. Is not like $b_0 0 b_1 0 b_2 0 \ldots$

We define randomness by the negative : we remove those which do not seem random.

Formally

Paradigm

X is random if X has no exceptionnal property

Becomes

Definition

X is $\mathscr{C}$-random if $\forall P \in \mathscr{C}$ such that $\lambda(P) = 0$, $X \notin P$

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Examples of $\mathscr{C}$:

1. the null Π^0_2,
2. the null Δ^1_1,
3. the Martin-Löf tests...

$\mathscr{C}$ countable ensures us that the $\mathscr{C}$-randoms are conull.

Martin-Löf Random

- Martin-Löf randomness has been the most studied.
- It has a definition for every of the three paradigm : impredictability, incompressibility of prefixes, and no exceptionnal properties.

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Definition (Martin-Löf's tests)

A Martin-Löf test is an intersection $\bigcap_n \mathcal{U}_n$, where $(\mathcal{U}_n)$ is recursively enumerable, and $\lambda(\mathcal{U}_n) \leq 2^{-n}$.

Also called Π^0_2 effectively null.

Definition (Martin-Löf Random)

X is Martin-Löf Random if X do not belong to any Martin-Löf test.

α-randomness

Following this principle, we define the tests in L_α.

Definition

X is random over L_α (or α-random) if X do not belong to any null borel set with code in L_α.

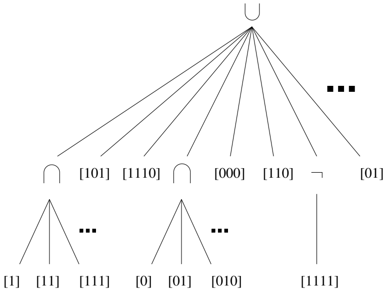

Figure – A borel code

- ω_1^{CK}-randomness is Δ_1^1-randomness,
- λ-randomness is ITTM-randomness.

α-ML-randomness

We continue the process to generalise Martin-Löf's idea :

Definition

- An α-ML test is a Martin-Löf test $\mathcal{U} \subseteq \omega \times 2^{<\omega}$ which is α-recursively enumerable.
- X is α-ML random if it is in no α-ML tests.

- ω-ML randomness is ML random,
- ω_1^{CK}-ML randomness is Π_1^1-ML randomness,
- λ-ML randomness is $\mathrm{ITTM}_{\mathrm{ML}}$ randomness

A question

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For every α, do the notions of "α-random" and "α-ML random" coincide ?

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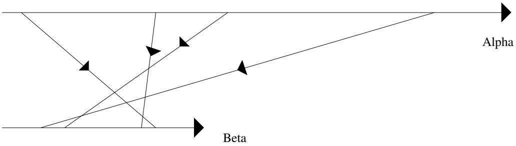
Theorem

Δ^1_1-randomness and Π^1_1-ML randomness are different notion.

This answers the quesion in a particular case. We would like a condition on α for it to be true.

Projectibility



Definition

α is projectible into β if there exists anα-recursive function, one-one from α to β.

- ω_1^{CK}, λ are projectible into ω ;
- not every ordinals are projective into a smaller ordinal thant themselves.

An equivalence

Theorem

The following are equivalent :

1. α *is projectible into* ω*, and*
2. α*-randomness and* α*-ML randomness are different notions.*

Being projectible into ω allows us to reduce "space" and "time" into a single dimension.

Corollary

ITTM-randomness et ITTM-ML randomness are two different notions.

Conclusion

- L'α-recursion extends computability, and includes other extensions ;
- it allows us to define new notions of randomness ;
- we have an equivalence between a property of set theory and a property of algorithmic randomness.

Thanks for your attention !