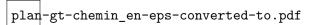
## $\alpha\text{-Recursion}$ and Randomness

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Consider the following game:

Game of Guessing the Random

For every N:

- I choose a sequence in  $2^N$  (deterministically)
- I randomly get another one by throwing N times a coin
- The other player have to bet on which was obtained randomly.

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- Have no structure
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- be hard to remember
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Suppose I moved to 182718525747285286528 Logic Street.

### Hi Mom!

Please note my new address is 182718525747285286528 Logic Street.

How to compare the randomness of two sequences sequence ? A random is expected to

- Have no structure
- be not predictable,
- be hard to remember = being incompressible
- ...

Suppose I moved to 100000000000000000 Logic Street.

### Hi Mom!

Please note my new address is "1" and 20 "0" Logic Street.

#### Intuition

The more a string is random the bigger is its shortest description (in some coding).

Definition (Kolmogorov Complexity)

$$C(\sigma) = \min(\{|\tau| : M(\tau) = \sigma\})$$

where

$$M(0^e 1\sigma) = M_e(\sigma)$$

- $182718525747285286528 \rightarrow 0^{e_{id}} 1182718525747285286528$ .
- $10000000000000000 \rightarrow 0^{e}120.$

Pseudorandomness is not random at all !!!

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## Strategy for the second player

Between A and B, choose the sequence with higher Kolmogorov complexity !

(if you can find it...)

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### Strategy for the second player

Between A and B, choose the sequence with higher Kolmogorov complexity !

(if you can find it...) Conclusion:

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# Infinitary case

#### Now we consider reals.

### Question

How would we define a real in  $2^\omega$  obtained by tossing infinitely many coins ?

- Every reals have 0 chance to appear,
- would be awkward if it is definable by a finite sentence.

# Infinitary case

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There exists several paradigms to define what it is to be random for a sequence of bits :

## Paradigm

- Impredictability,
- Incompressibility of prefixes,
- No exceptionnal properties.

We will use the third paradigm.

## Paradigm

A set A is random if it has no sufficiently simple exceptional property.

#### Definition

Let  $C \subseteq \mathcal{P}(2^{\omega})$ , and  $X \subseteq 2^{\omega}$ . We define *C*-randomness by:

X is C-random if  $\forall P \in C$ , if  $\lambda(P) = 0$  then  $\neg P(X)$ .

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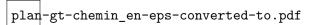
X is C-random if  $\forall P \in C$ , if  $\lambda(P) = 0$  then  $\neg P(X)$ .

 ${\cal C}$  countable ensures that the  ${\cal C}\mbox{-}{\rm randoms}$  are have measure 1. Examples :

- the class of Π<sup>0</sup><sub>2</sub> sets, the sets that need two quantifiers (starting with ∀) over N to be defined,
- the class of effectively Borel sets,
- the class of ITTM-semi-recursive.

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- Randomness Theory is the study of these different notions, how they relate to each others, what are the properties, computationnal power of random reals, etc.
- Notions are stratified in complexity by logic:
  - Complexity theory studies "low" complexity sets,
    - Polynomial Hierarchy
  - Recursion theory studies "medium" complexity sets,
    - (Hyper)arithmetical Hierarchy
  - Set theory studies "high" complexity sets.
- Let's see how set theory gives us new natural  $\mathcal{C}$ .



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 $\alpha\text{-}\mathsf{Recursion}$  comes naturally from the theorem :

#### Theorem

Let  $A \subseteq \omega$ . Then the following are equivalent:

- A is recursively enumerable,
- $\exists \phi \Sigma_1$  such that  $n \in A \Leftrightarrow HF \models \phi(n)$ .

Where HF is the set of all hereditarily finite sets.

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#### Strategy

We'll change HF by a level of the Godel Hierarchy!

## Gödel's constructible universe (1938)

Gödel's constructible at rank  $\alpha,$  written  $L_\alpha$  are defined by induction along ordinals :

- $L_0 = \emptyset$ , •  $L_{\alpha+1} = Def(L_{\alpha})$ ,

The constructibles are the elements of  $\bigcup_{\alpha} L_{\alpha}$ .

### Definition

$$Def(M) = \left\{ E^{M}_{\phi, \bar{p}} : \phi \text{ is a formula and } \bar{p} \in M 
ight\}$$

where

$$E^M_{\phi, \bar{p}} = \{x \in M : \phi(x, \bar{p}) \text{ is true in } M\}$$

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 $L = \bigcup_{\alpha \in Ord} L_{\alpha}$  is a model of ZFC These are some particular layers :

• 
$$L_{n+1} = \mathcal{P}(L_n)$$
 for  $n$  an integer;

2  $L_{\omega} = HF$ , the hereditarily finite sets ;

**③**  $L_{\omega,CK} = HYP$ , the sets with hyperarithmetic codes ;

• 
$$L_{\lambda} = WRT$$
, the sets with writable codes.

We find again HF !

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 $HF = L_{\omega}$ . Recall our theorem

Theorem (Characterization of recursive enumerability)

Let  $A \subseteq \omega$ . Then we have :

A is r.e.  $\iff \exists \phi \ \Sigma_1 \text{ such that } n \in A \Leftrightarrow \underline{L}_{\omega} \models \phi(n)$ 

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The basic definition of  $\alpha$ -recursion :

#### Definition

Let  $\alpha$  be an ordinal and  $A \subseteq L_{\alpha}$ . We say that :

- **0** A is  $\alpha$ -finite if  $A \in L_{\alpha}$ ;
- **2** A is  $\alpha$ -recursive if A is  $\Delta_1$ -comprehensible in  $L_{\alpha}$ ;
- **3** A is  $\alpha$ -recursively enumerable if A is  $\Sigma_1$ -comprehensible in  $L_{\alpha}$ .

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The basic definition of  $\alpha$ -recursion :

#### Definition

Let  $\alpha$  be an ordinal and  $A \subseteq L_{\alpha}$ . We say that :

- **1** A is  $\alpha$ -finite if  $A \in L_{\alpha}$ ;
- **2** A is  $\alpha$ -recursive if A is  $\Delta_1$ -comprehensible in  $L_{\alpha}$ ;
- **3** A is  $\alpha$ -recursively enumerable if A is  $\Sigma_1$ -comprehensible in  $L_{\alpha}$ .
  - Intuition: In  $\alpha$ -recursion, we see a computation as a search into all the  $\alpha$ -finite sets.
  - Some  $\alpha$  will reveal more interesting than others,
  - A is a set of  $\alpha$ -finite elements, not only integers.

- We can define new classes C of "sufficiently simple" properties by mimiking the classical case,
- for example,
  - from  $\Pi_2^0$  to  $\alpha$ -recursive  $\Pi_2^0$ ,
  - from ML tests to  $\alpha\text{-ML}$  tests...
- What happens to the relation between higher counterparts of classical notions ?

- We can define new classes C of "sufficiently simple" properties by mimiking the classical case,
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  - from  $\Pi_2^0$  to  $\alpha$ -recursive  $\Pi_2^0$ ,
  - $\bullet\,$  from ML tests to  $\alpha\text{-ML}$  tests...
- What happens to the relation between higher counterparts of classical notions ?

Thank you for your attention!

### Intuition

We see a computation as a search into all the  $\alpha$ -finite sets.

It is not yet finished ! Because :

#### Remark

Some  $\alpha$  will reveal more interesting than others...

Which  $\alpha$  ? Or, a better question would be which  $L_{\alpha}$  ?

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# Definition

An  $\alpha$  is said admissible of the image of any  $\alpha\text{-finite}$  set over an  $\alpha\text{-recursive}$  function is  $\alpha\text{-finite}.$ 

• 
$$\omega$$
,  $\omega_1^{CK}$ ,  $\omega_1^{CKA}$ ,  $\lambda$ ,  $\omega_1$  are admissibles.  $\omega \cdot 2$  is not.

#### Intuition

An ordinal  $\alpha$  is admissible if the  $\alpha\text{-recursion}$  is not too far from computability.

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### Intuition

We see a computation as a search into all the  $\alpha$ -finite sets.

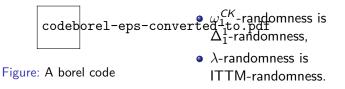
- $\omega$ -recursion, is classical computability ;
- $\omega_1^{\mathit{CK}}\text{-recursion, is higher computability ;}$
- $\lambda$ -recursion, is ITTM computability.

We have a general and satisfying definition of computability.

Following this principle, we define the tests in  $L_{\alpha}$ .

### Definition

X is random over  $L_{\alpha}$  (or  $\alpha$ -random) if X do not belong to any null borel set with code in  $L_{\alpha}$ .



We continue the process to generalise Martin-Löf's idea :

## Definition

- A ML test is a set  $A \subseteq 2^{\omega}$ , with  $A = \bigcap \mathcal{U}_n$  a  $\Pi_2^0$  set with  $\lambda(\mathcal{U}_n) \leq 2^{-n}$ ,
- an  $\alpha$ -ML test is a Martin-Löf test  $\mathcal{U} \subseteq \omega \times 2^{<\omega}$  which is  $\alpha$ -recursively enumerable,
- X is  $\alpha$ -ML random if it is in no  $\alpha$ -ML tests.
- $\omega$ -ML randomness is ML random,
- $\omega_1^{CK}$ -ML randomness is  $\Pi_1^1$ -ML randomness,
- $\lambda$ -ML randomness is ITTM<sub>ML</sub> randomness

# Question

For every  $\alpha,$  do the notions of " $\alpha\text{-random"}$  and " $\alpha\text{-ML}$  random" coincide ?

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## Question

For every  $\alpha,$  do the notions of " $\alpha\text{-random"}$  and " $\alpha\text{-ML}$  random" coincide ?

### Theorem

 $\Delta_1^1$ -randomness and  $\Pi_1^1$ -ML randomness are different notion.

This answers the quesion in a particular case. We would like a condition on  $\alpha$  for it to be true.

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# Definition

 $\alpha$  is projectible into  $\beta$  if there exists an  $\alpha\text{-recursive}$  function, one-one from  $\alpha$  to  $\beta.$ 

- $\omega_1^{\mathit{CK}}$ ,  $\lambda$  are projectible into  $\omega$  ;
- $\bullet$  it means the whole is being projectible into the  $\alpha\mbox{-finite}$  ;
- not every ordinals are projective into a smaller ordinal than themselves.

A fine structure property of the universe of set!

### Theorem

The following are equivalent :

- **1**  $\alpha$  is projectible into  $\omega$ , and
- 2  $\alpha$ -randomness and  $\alpha$ -ML randomness are different notions.

# Corollary

ITTM-randomness and ITTM<sub>ML</sub> randomness are two different notions.

- $\alpha\text{-recursion}$  extends computability, and includes other extensions ;
- it allows us to define new notions of randomness ;
- we have an equivalence between a property of set theory and a property of algorithmic randomness.

Thanks for your attention !

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