α-Recursion and Randomness

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Randomness in the finite setting

Consider the following game:

Game of Guessing the Random

For every N:

- I choose a sequence in 2^N (deterministically)
- I randomly get another one by throwing N times a coin
- The other player have to bet on which was obtained randomly.

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Which sequence would you bet is obtained randomly ?

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A second player strategy

How to compare the randomness of two sequences sequence ? A random is expected to

- Have no structure
- be not predictable,
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Suppose I moved to 182718525747285286528 Logic Street.

Hi Mom!

Please note my new address is 182718525747285286528 Logic Street.

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How to compare the randomness of two sequences sequence ? A random is expected to

- Have no structure
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- ...

Suppose I moved to 100000000000000000000 Logic Street.

Hi Mom!

Please note my new address is “1” and 20 “0” Logic Street.

Formally

Intuition

The more a string is random the bigger is its shortest description (in some coding).

Definition (Kolmogorov Complexity)

$$C(\sigma) = \min(\{|\tau| : M(\tau) = \sigma\})$$

where

$$M(0^e 1\sigma) = M_e(\sigma)$$

- $182718525747285286528 \rightarrow 0^{e_{id}}1182718525747285286528$.
- $100000000000000000000 \rightarrow 0^e 120$.

Pseudorandomness is not random at all !!!

Solution

Strategy for the second player

Between A and B, choose the sequence with higher Kolmogorov complexity !

(if you can find it...)

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(if you can find it...) Conclusion:

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Infinitary case

Now we consider reals.

Question

How would we define a real in 2^ω obtained by tossing infinitely many coins ?

- Every reals have 0 chance to appear,
- would be awkward if it is definable by a finite sentence.

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There exists several paradigms to define what it is to be random for a sequence of bits :

Paradigm

1. Impredictability,
2. Incompressibility of prefixes,
3. No exceptionnal properties.

We will use the third paradigm.

Formally

Paradigm

A set A is random if it has no sufficiently simple exceptional property.

Definition

Let $\mathcal{C} \subseteq \mathcal{P}(2^\omega)$, and $X \subseteq 2^\omega$. We define $\mathcal{C}$-randomness by:

$$X \text{ is } \mathcal{C}\text{-random if } \forall P \in \mathcal{C}, \text{ if } \lambda(P) = 0 \text{ then } \neg P(X).$$

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$\mathcal{C}$ *countable* ensures that the $\mathcal{C}$-randoms are have measure 1.

Examples :

- the class of Π^0_2 sets, the sets that need two quantifiers (starting with $\forall$) over $\mathbb{N}$ to be defined,
- the class of effectively Borel sets,
- the class of ITTM-semi-recursive.

- Randomness Theory is the study of these different notions, how they relate to each others, what are the properties, computationnal power of random reals, etc.
- Notions are stratified in complexity by logic:
  - Complexity theory studies "low" complexity sets,
    - Polynomial Hierarchy
  - Recursion theory studies "medium" complexity sets,
    - (Hyper)arithmetical Hierarchy
  - Set theory studies "high" complexity sets.
- Let's see how set theory gives us new natural $\mathcal{C}$.

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α-recursion

α-Recursion comes naturally from the theorem :

Theorem

Let $A \subseteq \omega$. Then the following are equivalent:

- *A is recursively enumerable,*
- $\exists \phi$ Σ_1 *such that* $n \in A \Leftrightarrow HF \models \phi(n)$.

Where *HF* is the set of all hereditarily finite sets.

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Where *HF* is the set of all hereditarily finite sets.

Strategy

We'll change *HF* by a level of the Godel Hierarchy!

Godel's constructible universe

Gödel's constructible universe (1938)

Gödel's constructible at rank α, written L_α are defined by induction along ordinals :

1. $L_0 = \emptyset$,
2. $L_{\alpha+1} = Def(L_\alpha)$,
3. $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$.

The constructibles are the elements of $\bigcup_\alpha L_\alpha$.

Definition

$$Def(M) = \left\{ E^M_{\phi,\bar{p}} : \phi \text{ is a formula and } \bar{p} \in M \right\}$$

where

$$E^M_{\phi,\bar{p}} = \{x \in M : \phi(x, \bar{p}) \text{ is true in } M\}$$

Illustration

Omega_squared12.png

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Illustration

Omega_squared11.png

constr8-eps-converted-to.pdf

Illustration

Omega_squared10.png

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Illustration

Omega_squared9.png

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Illustration

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Illustration

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Illustration

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Illustration

Omega_squared3.png

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Illustration

Omega_squared2.png

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Illustration

Omega_squared0.png

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Examples

$L = \bigcup_{\alpha \in Ord} L_\alpha$ is a model of ZFC
These are some particular layers :

1. $L_{n+1} = \mathcal{P}(L_n)$ for n an integer ;
2. $L_\omega = HF$, the hereditarily finite sets ;
3. $L_{\omega_1^{CK}} = HYP$, the sets with hyperarithmetic codes ;
4. $L_\lambda = WRT$, the sets with writable codes.

We find again HF !

$HF = L_\omega$. Recall our theorem

Theorem (Characterization of recursive enumerability)

Let $A \subseteq \omega$. Then we have :

$$A \text{ is r.e.} \iff \exists \phi\ \Sigma_1 \text{ such that } n \in A \Leftrightarrow L_\omega \models \phi(n)$$

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The basic definition of α-recursion :

Definition

Let α be an ordinal and $A \subseteq L_\alpha$. We say that :

1. A is α-finite if $A \in L_\alpha$;
2. A is α-recursive if A is Δ_1-comprehensible in L_α ;
3. A is α-recursively enumerable if A is Σ_1-comprehensible in L_α.

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- Intuition: In α-recursion, we see a computation as a search into all the α-finite sets.
- Some α will reveal more interesting than others,
- A is a set of α-finite elements, not only integers.

Premature birth of a conclusion

- We can define new classes $\mathcal{C}$ of "sufficiently simple" properties by mimiking the classical case,
- for example,
  - from Π^0_2 to α-recursive $\mathbf{\Pi^0_2}$,
  - from ML tests to α-ML tests...
- What happens to the relation between higher counterparts of classical notions ?

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Thank you for your attention!

Intuition

We see a computation as a search into all the α-finite sets.

It is not yet finished ! Because :

Remark

Some α will reveal more interesting than others...

Which α ? Or, a better question would be which L_α ?

Admissibility II

Definition

An α is said admissible of the image of any α-finite set over an α-recursive function is α-finite.

- ω, ω_1^{CK}, ${\omega_1^{CK}}^A$, λ, ω_1 are admissibles. $\omega \cdot 2$ is not.

Intuition

An ordinal α is admissible if the α-recursion is not too far from computability.

What did we defined ?

Intuition

We see a computation as a search into all the α-finite sets.

- ω-recursion, is classical computability ;
- ω_1^{CK}-recursion, is higher computability ;
- λ-recursion, is ITTM computability.

We have a general and satisfying definition of computability.

α-randomness

Following this principle, we define the tests in L_α.

Definition

X is random over L_α (or α-random) if X do not belong to any null borel set with code in L_α.

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Figure: A borel code

- ω_1^{CK}-randomness is Δ_1^1-randomness,
- λ-randomness is ITTM-randomness.

α-ML-randomness

We continue the process to generalise Martin-Löf's idea :

Definition

- A ML test is a set $A \subseteq 2^\omega$, with $A = \bigcap \mathcal{U}_n$ a Π^0_2 set with $\lambda(\mathcal{U}_n) \leq 2^{-n}$,
- an α-ML test is a Martin-Löf test $\mathcal{U} \subseteq \omega \times 2^{<\omega}$ which is α-recursively enumerable,
- X is α-ML random if it is in no α-ML tests.

- ω-ML randomness is ML random,
- ω_1^{CK}-ML randomness is Π^1_1-ML randomness,
- λ-ML randomness is ITTM_{ML} randomness

A question

Question

For every α, do the notions of "α-random" and "α-ML random" coincide ?

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For every α, do the notions of "α-random" and "α-ML random" coincide ?

Theorem

Δ^1_1-randomness and Π^1_1-ML randomness are different notion.

This answers the quesion in a particular case. We would like a condition on α for it to be true.

Projectibility

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Definition

α is projectible into β if there exists an α-recursive function, one-one from α to β.

- ω_1^{CK}, λ are projectible into ω ;
- it means the whole is being projectible into the α-finite ;
- not every ordinals are projective into a smaller ordinal than themselves.

A fine structure property of the universe of set!

An equivalence

Theorem

The following are equivalent :

1. α *is projectible into* ω*, and*
2. α*-randomness and* α*-ML randomness are different notions.*

Corollary

ITTM-randomness and $ITTM_{ML}$ *randomness are two different notions.*

Conclusion

- α-recursion extends computability, and includes other extensions ;
- it allows us to define new notions of randomness ;
- we have an equivalence between a property of set theory and a property of algorithmic randomness.

Thanks for your attention !