α-Recursion and Randomness

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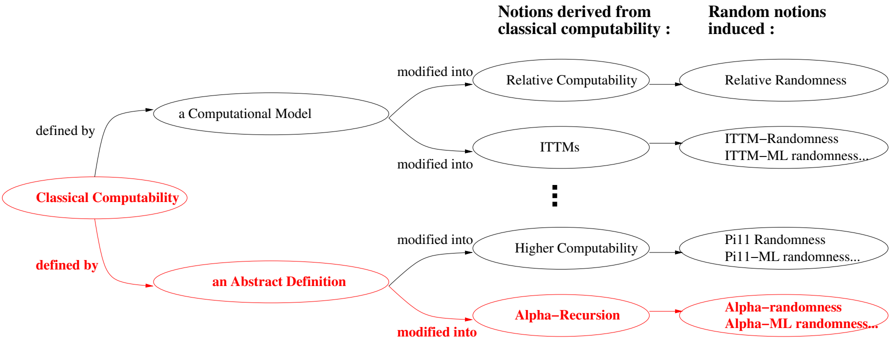

Preliminaries

Three running examples:

- usual Recursion Theory;
- Π^1_1-recursion: Π^1_1 are equivalents of r.e. sets, Δ^1_1 are equivalents of recursive sets;
- Infinite Time Turing Machine. Recall that λ is the supremum of the halting stages of ITTMs.

α-recursion comes naturally from the theorem :

Theorem

Let $A \subseteq \omega$. Then we have :

$$A \text{ is r.e.} \iff \exists \phi \ \Sigma_1 \text{ such that } n \in A \Leftrightarrow L_\omega \models \phi(n)$$

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Definition

Let $A \subseteq \omega$. We say that:

- A is α-r.e. if $n \in A \Leftrightarrow L_\alpha \models \phi(n)$ with ϕ a Σ_1-formula with parameters,
- A is α-recursive. if $n \in A \Leftrightarrow L_\alpha \models \phi(n)$ with ϕ a Δ_1-formula with parameters,
- A is α-finite if $A \in L_\alpha$.

Back to the examples

Theorem (Spector,Gandy)

A set $A \subseteq \mathbb{N}$ is Π^1_1 iff $A = \{n \in \mathbb{N} : L_{\omega_1^{CK}} \models \phi(n)\}$.

So, on $\mathbb{N}$, Π^1_1-recursion is ω_1^{CK}-recursion.

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Theorem (Spector,Gandy)

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So, on $\mathbb{N}$, Π^1_1-recursion is ω_1^{CK}-recursion.

Theorem

A set $A \subseteq \mathbb{N}$ is ITTM-recursive iff A is λ-recursive.

So, on $\mathbb{N}$, ITTM-recursion is λ-recursion.

Admissibility

A condition on α to behave as intended:

Definition

We say that α is admissible if $\forall f$ α-r.e, $\forall a$ α-finite,

$$a \subseteq \mathrm{dom}(f) \Rightarrow f[a] \text{ is } \alpha\text{-finite.}$$

This is $B\Sigma_1$ pendant. It allows swapping quantifiers.
What about our examples ?

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What about our examples ?

Example

- ω is admissible,
- ω_1^{CK} is admissible,
- λ and ζ are admissible, but Σ is not.

Projectibility

Another property on α:

Definition

We say that α is projectible in $\beta < \alpha$ if there exists an α-recursive mapping one-one from α to β.

This is an analogue of $C\Sigma_1$.
What about our examples ?

Projectibility

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We say that α is projectible in $\beta < \alpha$ if there exists an α-recursive mapping one-one from α to β.

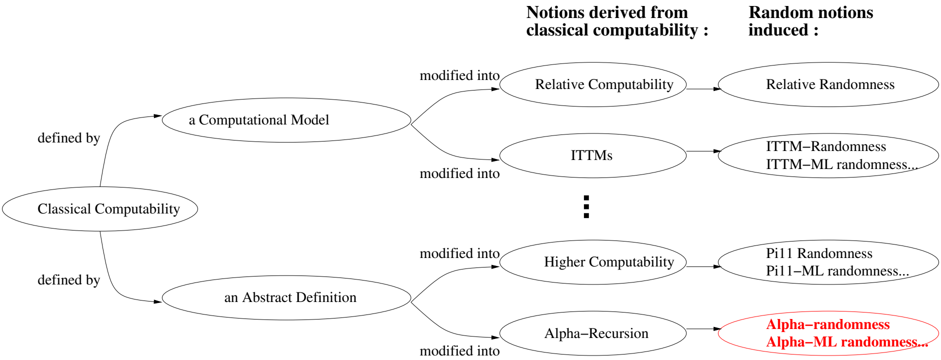
This is an analogue of $C\Sigma_1$.
What about our examples ?

Example

- ω is not projectible,
- ω_1^{CK} is projectible,
- λ is projectible, but ζ is not.

It allows priority arguments!

Randomness Part



Recreation time

Defining randomness

There are three paradigms to define randomness.

- **Incompressibility**: if A is random, then all prefixes are hard to describe ;
- **Impredictability**: given the first n bits of a random set we can't predict the $n+1$th ;
- **No exceptional property**: a random set has no sufficiently simple exceptional property ;

Defining randomness

Definition

A set A is random if it has no sufficiently simple exceptional property.

Definition

Let $\mathcal{C} \subseteq \mathcal{P}(2^\omega)$, and $A \subseteq 2^\omega$. We define $\mathcal{C}$-randomness by:

$$A \text{ is } \mathcal{C}\text{-random if } \forall P \in \mathcal{C},\ \lambda(A) = 0 \Rightarrow \neg P(X).$$

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Examples of classes $\mathcal{C}$:

- If $\mathcal{C}$ is the class of effectively null Π^0_2 set, we call that ML-randomness,
- $\mathcal{C}$ the class of Π^1_1 sets we get Π^1_1-randomness,
- $\mathcal{C}$ the class of ITTM-semi-recursive sets we get ITTM-randomness.

Randomness is Lebesgue pendant of genericity, but is very different.

In the scope of α-recursion :

Definition

A set is α-random if $\neg P(x)$ for all P with $^{\infty}$Borel code in L_α.

What about ML randomness ?

Definition

A is α-ML-random if A is in no effectively null set $\bigcap_n \mathcal{U}_n$ where $\{(n, \sigma) : [\sigma] \subseteq \mathcal{U}_n\}$ is α-recursively enumerable.

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Example

- ω_1^{CK}-randomness is Δ_1^1-randomness, and ω_1^{CK}-ML-randomness is Π_1^1-ML-randomness ;
- λ-randomness and λ-ML-randomness can also be defined in term of Infinite Time Turing Machine

Relation between randomness versions

Theorem

Π^1_1-*ML-randomness is strictly stronger than* Δ^1_1-*randomness.*

Question

Is ITTM-ML-randomness strictly stronger than λ-*randomness ?*

Relation between randomness versions

Theorem

Π^1_1-ML-randomness is strictly stronger than Δ^1_1-randomness.

Question

Is ITTM-ML-randomness strictly stronger than λ-randomness ?

Theorem

Let α be a countable admissible and $L_\alpha \models$"everything is countable". Then the following are equivalent:

1. *α-ML-randomness is strictly stronger than α-randomness,*
2. *α is projectible.*

Proof.

Sketch if we have time... □

Relation between randomness versions

Theorem

A is ITTM-random iff A is Σ-random and $\Sigma^x = \Sigma$.

Question

We have

$$\Sigma\text{-randomness} \supseteq \text{ITTM-randomness} \supseteq \Sigma\text{-ML-randomness}.$$

Which of these inequalities are strict ?

Thank you !

See you on Sentosa Beach ! Meeting with Sabrina at 8:00pm in front of PGPR.