Generic additive subroup of a field of positive characteristic.

Christian d'Elbée

October 16, 2018

1. **Generic Predicate.** [Chatzidakis-Pillay 98] T a model complete $\mathscr{L}$-theory which eliminate $\exists^\infty$. The $\mathscr{L} \cup \{P\}$-theory T with a unary predicate, admits a model-companion TP. If T is simple then so is TP.

Generic constructions

1. **Generic Predicate.** [Chatzidakis-Pillay 98] T a model complete $\mathscr{L}$-theory which eliminate $\exists^\infty$. The $\mathscr{L} \cup \{P\}$-theory T with a unary predicate, admits a model-companion TP. If T is simple then so is TP.
2. **Generic expansion.** T a model complete $\mathscr{L}$-theory, $\mathscr{L} \subset \mathscr{L}'$, does T as an (incomplete) $\mathscr{L}'$-theory admits a model-companion $T_{\mathscr{L}'}$?

Generic constructions

1. **Generic Predicate.** [Chatzidakis-Pillay 98] T a model complete $\mathscr{L}$-theory which eliminate $\exists^\infty$. The $\mathscr{L} \cup \{P\}$-theory T with a unary predicate, admits a model-companion TP. If T is simple then so is TP.
2. **Generic expansion.** T a model complete $\mathscr{L}$-theory, $\mathscr{L} \subset \mathscr{L}'$, does T as an (incomplete) $\mathscr{L}'$-theory admits a model-companion $T_{\mathscr{L}'}$?
    - YES : if (and only if) T eliminate $\exists^\infty$ [Winkler 1975]

Generic constructions

1. **Generic Predicate.** [Chatzidakis-Pillay 98] T a model complete $\mathscr{L}$-theory which eliminate $\exists^\infty$. The $\mathscr{L} \cup \{P\}$-theory T with a unary predicate, admits a model-companion TP. If T is simple then so is TP.
2. **Generic expansion.** T a model complete $\mathscr{L}$-theory, $\mathscr{L} \subset \mathscr{L}'$, does T as an (incomplete) $\mathscr{L}'$-theory admits a model-companion $T_{\mathscr{L}'}$?
   - YES : if (and only if) T eliminate $\exists^\infty$ [Winkler 1975]
   - If T is $NSOP_1$ then so is $T_{\mathscr{L}'}$ [Kruckman-Ramsey 2018]. ([Jeřábek 2018] for $T^{\emptyset}_{\mathscr{L}'}$)

Fix $p > 0$ prime number. $\mathbb{F}_p$ field with p elements, $\overline{\mathbb{F}}_p$ its algebraic closure.

Fix $p > 0$ prime number. $\mathbb{F}_p$ field with p elements, $\overline{\mathbb{F}}_p$ its algebraic closure.
Let $\mathscr{L} = \{+, -, \cdot, ^{-1}, 0, 1\}$ and $\mathscr{L}^G = \mathscr{L} \cup \{G\}$, G unary predicate.

Fix $p > 0$ prime number. $\mathbb{F}_p$ field with p elements, $\overline{\mathbb{F}}_p$ its algebraic closure.
Let $\mathscr{L} = \{+, -, \cdot, ^{-1}, 0, 1\}$ and $\mathscr{L}^G = \mathscr{L} \cup \{G\}$, G unary predicate. Let ACF_p^G be the $\mathscr{L}^G$-theory whose model (F, H) satisfies:

- $F \models ACF_p$;

Fix $p > 0$ prime number. $\mathbb{F}_p$ field with p elements, $\overline{\mathbb{F}}_p$ its algebraic closure.
Let $\mathscr{L} = \{+, -, \cdot, ^{-1}, 0, 1\}$ and $\mathscr{L}^G = \mathscr{L} \cup \{G\}$, G unary predicate. Let ACF_p^G be the $\mathscr{L}^G$-theory whose model (F, H) satisfies:

- $F \models ACF_p$;
- $H = G(F)$ is an additive subgroup of F.

Fix $p > 0$ prime number. $\mathbb{F}_p$ field with p elements, $\overline{\mathbb{F}}_p$ its algebraic closure.
Let $\mathscr{L} = \{+, -, \cdot, ^{-1}, 0, 1\}$ and $\mathscr{L}^G = \mathscr{L} \cup \{G\}$, G unary predicate. Let ACF_p^G be the $\mathscr{L}^G$-theory whose model (F, H) satisfies:

- $F \models ACF_p$;
- $H = G(F)$ is an additive subgroup of F.

Theorem

ACF_p^G admits a model companion, we call it ACF_pG.

Fix $p > 0$ prime number. $\mathbb{F}_p$ field with p elements, $\overline{\mathbb{F}}_p$ its algebraic closure.
Let $\mathscr{L} = \{+, -, \cdot, ^{-1}, 0, 1\}$ and $\mathscr{L}^G = \mathscr{L} \cup \{G\}$, G unary predicate. Let ACF_p^G be the $\mathscr{L}^G$-theory whose model (F, H) satisfies:

- $F \models ACF_p$;
- $H = G(F)$ is an additive subgroup of F.

Theorem

ACF_p^G admits a model companion, we call it ACF_pG.

- *Every model of ACF_p^G embeds in a model of ACF_pG;*

Fix $p > 0$ prime number. $\mathbb{F}_p$ field with p elements, $\overline{\mathbb{F}}_p$ its algebraic closure.
Let $\mathscr{L} = \{+, -, \cdot, ^{-1}, 0, 1\}$ and $\mathscr{L}^G = \mathscr{L} \cup \{G\}$, G unary predicate. Let ACF_p^G be the $\mathscr{L}^G$-theory whose model (F, H) satisfies:

- $F \models ACF_p$;
- $H = G(F)$ is an additive subgroup of F.

Theorem

ACF_p^G admits a model companion, we call it ACF_pG.

- *Every model of ACF_p^G embeds in a model of ACF_pG;*
- *Every model of ACF_pG is existentially closed in every model of ACF_p^G extending it.*

Fix $p > 0$ prime number. $\mathbb{F}_p$ field with p elements, $\overline{\mathbb{F}}_p$ its algebraic closure.
Let $\mathscr{L} = \{+, -, \cdot, ^{-1}, 0, 1\}$ and $\mathscr{L}^G = \mathscr{L} \cup \{G\}$, G unary predicate. Let ACF_p^G be the $\mathscr{L}^G$-theory whose model (F, H) satisfies:

- $F \models ACF_p$;
- $H = G(F)$ is an additive subgroup of F.

Theorem

ACF_p^G admits a model companion, we call it ACF_pG.

- *Every model of ACF_p^G embeds in a model of ACF_pG;*
- *Every model of ACF_pG is existentially closed in every model of ACF_p^G extending it.*

ACF_pG is an axiomatisation of the classe of all existentially closed models of ACF_p^G.

Fact (Chatzidakis-Pillay)

If T eliminate $\exists^\infty$ then for any formula $\phi(x, y)$ there is a formula $\theta_\phi(y)$ such that for any tuple b from $\mathscr{M} \models T$ we have

$$\mathscr{M} \models \theta_\phi(b) \iff \text{there exists a tuple } a \text{ from } \mathscr{N} \succ \mathscr{M} \text{ such that } a \cap \mathscr{M} = \emptyset \text{ and } \mathscr{N} \models \phi(a, b)$$

Fact (Chatzidakis-Pillay)

If T eliminate $\exists^\infty$ then for any formula $\phi(x,y)$ there is a formula $\theta_\phi(y)$ such that for any tuple b from $\mathscr{M} \models T$ we have

$$\mathscr{M} \models \theta_\phi(b) \iff \text{there exists a tuple } a \text{ from } \mathscr{N} \succ \mathscr{M} \text{ such that } a \cap \mathscr{M} = \emptyset \text{ and } \mathscr{N} \models \phi(a,b)$$

In our case: For every $\mathscr{L}$-formula $\phi(x,y)$ there exists $\theta_\phi(y)$ such that for all $K \models ACF_p$ and all tuple b from K

$$K \models \theta_\phi(b) \iff \text{there exists a tuple } a \text{ in some field extension of } K \text{ such that } a \text{ is } \mathbb{F}_p\text{-linearly independent over } K \text{ and } \models \phi(a,b)$$

Let $\langle a \rangle$ denote the $\mathbb{F}_p$-vector space spanned by a.

Let $\langle a \rangle$ denote the $\mathbb{F}_p$-vector space spanned by a. The theory ACF_pG is obtained by adding to ACF_p the following axiom-schema; for all $\mathscr{L}$-formula $\phi(x, y)$, $x' \subseteq x$, $y' \subseteq y$:

Let $\langle a \rangle$ denote the $\mathbb{F}_p$-vector space spanned by a. The theory ACF_pG is obtained by adding to ACF_p the following axiom-schema; for all $\mathscr{L}$-formula $\phi(x, y)$, $x' \subseteq x$, $y' \subseteq y$:

$$\forall y \left(\theta_\phi(y) \wedge \langle y' \rangle \cap G = \{0\}\right.$$

Let $\langle a\rangle$ denote the $\mathbb{F}_p$-vector space spanned by a. The theory ACF_pG is obtained by adding to ACF_p the following axiom-schema; for all $\mathscr{L}$-formula $\phi(x,y)$, $x' \subseteq x$, $y' \subseteq y$:

$$\forall y \left(\theta_\phi(y) \wedge \langle y'\rangle \cap G = \{0\} \ \to \exists x(\phi(x,y) \wedge \langle xy'\rangle \cap G = \langle x'\rangle)\right)$$

Let $\langle a\rangle$ denote the $\mathbb{F}_p$-vector space spanned by a. The theory ACF_pG is obtained by adding to ACF_p the following axiom-schema; for all $\mathscr{L}$-formula $\phi(x,y)$, $x'\subseteq x$, $y'\subseteq y$:

$$\forall y\left(\theta_\phi(y)\wedge\langle y'\rangle\cap G=\{0\}\ \rightarrow\exists x(\phi(x,y)\wedge\langle xy'\rangle\cap G=\langle x'\rangle)\right)$$

Completions. Completions of ACF_pG are given by the $\mathscr{L}^G$-isomorphism type of $(\overline{\mathbb{F}}_p, G(\overline{\mathbb{F}}_p))$, i.e. for two models (K_1,G_1) and (K_2,G_2) of ACF_pG,

$$(K_1,G_1)\equiv(K_2,G_2)\iff(\overline{\mathbb{F}}_p,G_1(\overline{\mathbb{F}}_p))\text{ and }(\overline{\mathbb{F}}_p,G_2(\overline{\mathbb{F}}_p))\text{ are}$$
$$\mathscr{L}^G-\text{isomorphic.}$$

Let $\langle a\rangle$ denote the $\mathbb{F}_p$-vector space spanned by a. The theory ACF_pG is obtained by adding to ACF_p the following axiom-schema; for all $\mathscr{L}$-formula $\phi(x,y)$, $x'\subseteq x$, $y'\subseteq y$:

$$\forall y\,\left(\theta_\phi(y)\wedge\langle y'\rangle\cap G=\{0\}\;\rightarrow\exists x(\phi(x,y)\wedge\langle xy'\rangle\cap G=\langle x'\rangle)\right)$$

Completions. Completions of ACF_pG are given by the $\mathscr{L}^G$-isomorphism type of $(\overline{\mathbb{F}}_p, G(\overline{\mathbb{F}}_p))$, i.e. for two models (K_1,G_1) and (K_2,G_2) of ACF_pG,

$$(K_1,G_1)\equiv(K_2,G_2)\iff(\overline{\mathbb{F}}_p, G_1(\overline{\mathbb{F}}_p))\text{ and }(\overline{\mathbb{F}}_p, G_2(\overline{\mathbb{F}}_p))\text{ are }\mathscr{L}^G-\text{isomorphic.}$$

Algebraic closure. The algebraic closure in ACF_pG is given by the field theoretic algebraic closure.

Proposition

For every $n \in \mathbb{N}$ and G_0 subgroup of $\mathbb{F}_{p^n}$ there exists $G_0 \subset G \subset \overline{\mathbb{F}}_p$ such that $(\overline{\mathbb{F}}_p, G) \models ACF_pG$.

Proposition

For every $n \in \mathbb{N}$ and G_0 subgroup of $\mathbb{F}_{p^n}$ there exists $G_0 \subset G \subset \overline{\mathbb{F}}_p$ such that $(\overline{\mathbb{F}}_p, G) \models ACF_pG$.

Consider a non principal ultrafilter $\mathscr{U}$ on the set of prime numbers, and a model $(\overline{\mathbb{F}}_q, G_q)$ of ACF_qG, for each prime q. What is

$$\prod_{q \in \mathscr{U}} (\overline{\mathbb{F}}_q, G_q) \ ?$$

Proposition

For every $n \in \mathbb{N}$ and G_0 subgroup of $\mathbb{F}_{p^n}$ there exists $G_0 \subset G \subset \overline{\mathbb{F}}_p$ such that $(\overline{\mathbb{F}}_p, G) \models ACF_pG$.

Consider a non principal ultrafilter $\mathscr{U}$ on the set of prime numbers, and a model $(\overline{\mathbb{F}}_q, G_q)$ of ACF_qG, for each prime q. What is

$$\prod_{q \in \mathscr{U}} (\overline{\mathbb{F}}_q, G_q) \text{ ?}$$

Remark (Characteristic 0?)

If (K, G) is an existentially closed models of the the class of (K, G) with $char(K) = 0$, then $Stab(G) = \mathbb{Z}$. ***Not*** *axiomatisable.*

Definition

Let T be an arbitrary theory and $\phi(x, y)$ a formula in the language of T.

1. *We say that $\phi(x, y)$ has the 1-strong order property (SOP_1) if there exists a tree of tuple $(b_\eta)_{\eta \in 2^{<\omega}}$ such that*
   - *for all $\eta \in 2^\omega$ $\{\phi(x, b_{\eta \restriction \alpha} \mid \alpha < \omega\}$ is consistent*
   - *for all $\eta \in 2^{<\omega}$ if $\eta^\frown 0 \lhd \nu$ then $\{\phi(x, b_\nu), \phi(x, b_{\eta^\frown 1}\}$ is inconsistent.*

Definition

Let T be an arbitrary theory and $\phi(x, y)$ a formula in the language of T.

Definition

Let T be an arbitrary theory and $\phi(x,y)$ a formula in the language of T.

1. *We say that $\phi(x,b)$ Kim-divides over A if for some A-invariant global extension $p(x)$ of $tp(b/A)$ and $(b_i)_{i<\omega}$ such that $b_i \models p \upharpoonright Ab_{<i}$ then $\bigwedge_{i<\omega} \phi(x,b_i)$ is inconsistent.*

Definition

Let T be an arbitrary theory and $\phi(x, y)$ a formula in the language of T.

1. *We say that $\phi(x, b)$ Kim-divides over A if for some A-invariant global extension $p(x)$ of $tp(b/A)$ and $(b_i)_{i<\omega}$ such that $b_i \models p \restriction Ab_{<i}$ then $\bigwedge_{i<\omega} \phi(x, b_i)$ is inconsistent.*
2. *We say that $\phi(x, b)$ Kim-forks over A if $\phi(x, b)$ implies a disjunction of formula each of which Kim-divides over A.*

Definition

Let T be an arbitrary theory and $\phi(x, y)$ a formula in the language of T.

1. *We say that $\phi(x, b)$ Kim-divides over A if for some A-invariant global extension $p(x)$ of $tp(b/A)$ and $(b_i)_{i<\omega}$ such that $b_i \models p \restriction Ab_{<i}$ then $\bigwedge_{i<\omega} \phi(x, b_i)$ is inconsistent.*
2. *We say that $\phi(x, b)$ Kim-forks over A if $\phi(x, b)$ implies a disjunction of formula each of which Kim-divides over A.*
3. *We define the Kim-forking independence relation*

$$A \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{C} B \iff tp(A/BC) \text{ does not Kim-forks over } C.$$

Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant ternary relation in T.

Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant ternary relation in T.

- **Symmetry**.
  If $A \mathop{\smile\!\!\!\!\!\!|}_{\mathscr{M}} B$ then $B \mathop{\smile\!\!\!\!\!\!|}_{\mathscr{M}} A$.

Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant ternary relation in T.

- **Symmetry**.
- **Monotonicity**.
  If $A \mathop{\smile\!\!\!\!\!\!|}_{\mathscr{M}} BD$ then $A \mathop{\smile\!\!\!\!\!\!|}_{\mathscr{M}} B$.

Let $\downarrow$ be an invariant ternary relation in T.

- **Symmetry**.
- **Monotonicity**.
- **Existence**.
  For any $\mathscr{M}$ and a we have $a \downarrow_{\mathscr{M}} \mathscr{M}$

Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant ternary relation in T.

- **Symmetry**.
- **Monotonicity**.
- **Existence**.
- **Strong Finite Character**. For any model $\mathscr{M}$, if $a \not\mathop{\smile\!\!\!\!\!\!|}_{\mathscr{M}} b$, then there is a formula $\phi(x, b, m) \in tp(a/b\mathscr{M})$ such that for all a', if $a' \models \phi(x, b, m)$ then $a' \not\mathop{\smile\!\!\!\!\!\!|}_{\mathscr{M}} b$.

Let $\mathop{\smile\!\!\!\!\!\mid}$ be an invariant ternary relation in T.

- **Symmetry**.
- **Monotonicity**.
- **Existence**.
- **Strong Finite Character**.
- 3-**amalgamation**.
  For all model $\mathscr{M}$ if there exists tuples c_1, c_2 and sets A, B such that
  - $c_1 \equiv_{\mathscr{M}} c_2$
  - $A \mathop{\smile\!\!\!\!\!\mid}_E B$
  - $c_1 \mathop{\smile\!\!\!\!\!\mid}_{\mathscr{M}} A$ and $c_2 \mathop{\smile\!\!\!\!\!\mid}_{\mathscr{M}} B$

  then there exists $c \mathop{\smile\!\!\!\!\!\mid}_{\mathscr{M}} A, B$ such that $c \equiv_A c_1$ and $c \equiv_B c_2$.

Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant ternary relation in T.

- **Symmetry**.
- **Monotonicity**.
- **Existence**.
- **Strong Finite Character**.
- 3-**amalgamation**.
- **Witnessing**.
  Let $a, b, (b_i)_{i<\omega}$ and a model $\mathscr{M}$ and assume the following:
  - $a \not\mathop{\smile\!\!\!\!\!\!|}_{\mathscr{M}} b$, witnessed by $\phi(x, b, m)$
  - for p some global $\mathscr{M}$-invariant extension of $tp(b/\mathscr{M})$, $b_i \models p \restriction Cb_{<i}$.

  Then $\bigwedge_{i<\omega} \phi(x, b_i, m)$ is inconsistent.

Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant ternary relation in T.

- **Symmetry**.
- **Monotonicity**.
- **Existence**.
- **Strong Finite Character**.
- 3-**amalgamation**.
- **Witnessing**.
- **Base Monotonicity**.
  For $\mathscr{M} \subset \mathscr{N}$ if $A \mathop{\smile\!\!\!\!\!\!|}_{\mathscr{M}} B\mathscr{N}$ then $A \mathop{\smile\!\!\!\!\!\!|}_{\mathscr{N}} B$.

Let $\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.2ex\hbox{$\mid$}}$ be an invariant ternary relation in T.

- **Symmetry**.
- **Monotonicity**.
- **Existence**.
- **Strong Finite Character**.
- **3-amalgamation**.
- **Witnessing**.
- **Base Monotonicity**.

Classification for ACF_pG
Chernikov-Ramsey (2015) ; Kaplan-Ramsey (2017)

- Symmetry.
- Monotonicity.
- Existence.
- Strong Finite Character.
- 3-amalgamation.

$\Big\}$ $NSOP_1$

- Witnessing.

$\Big\}$ Kim-Forking

- Base Monotonicity.

$\Big\}$ $\boxed{\text{Simple}}$

We define for A, B, C algebraically closed sets in a model of ACF_pG

We define for A, B, C algebraically closed sets in a model of ACF_pG

$$A \mathop{\smile\!\!\!\!\!\!\,|}^{w}_{C} B$$

We define for A, B, C algebraically closed sets in a model of ACF_pG

$$A \mathop{\smile\!\!\!\!\!\!\,|}_C^{w} B \iff A \mathop{\smile\!\!\!\!\!\!\,|}_C^{ACF} B \text{ and}$$

We define for A, B, C algebraically closed sets in a model of ACF_pG

$$A \mathop{\smile\!\!\!\!\!\!\,|}_{C}^{w} B \iff A \mathop{\smile\!\!\!\!\!\!\,|}_{C}^{ACF} B \text{ and } G(\overline{AC} + \overline{BC}) = G(\overline{AC}) + G(\overline{BC})$$

We define for A, B, C algebraically closed sets in a model of ACF_pG

$$A \mathop{\smile\!\!\!\!\!\!|}_{C}^{w} B \iff A \mathop{\smile\!\!\!\!\!\!|}_{C}^{ACF} B \text{ and } G(\overline{AC} + \overline{BC}) = G(\overline{AC}) + G(\overline{BC})$$

$$A \mathop{\smile\!\!\!\!\!\!|}_{C}^{st} B \iff A \mathop{\smile\!\!\!\!\!\!|}_{C}^{ACF} B \text{ and } G(\overline{ABC}) = G(\overline{AC}) + G(\overline{BC})$$

We define for A, B, C algebraically closed sets in a model of ACF_pG

$$A \underset{C}{\mathop{\smile\!\!\!\!\!\!\;|}}^{w} B \iff A \underset{C}{\mathop{\smile\!\!\!\!\!\!\;|}}^{ACF} B \text{ and } G(\overline{AC} + \overline{BC}) = G(\overline{AC}) + G(\overline{BC})$$

$$A \underset{C}{\mathop{\smile\!\!\!\!\!\!\;|}}^{st} B \iff A \underset{C}{\mathop{\smile\!\!\!\!\!\!\;|}}^{ACF} B \text{ and } G(\overline{ABC}) = G(\overline{AC}) + G(\overline{BC})$$

Theorem

- $\mathop{\smile\!\!\!\!\!\!\;|}^{w}$ *satisfies* ***Symmetry***, ***Monotonicity***, ***Existence***, ***Strong Finite Character***, *3-**amalgamation** so ACF_pG is NSOP1. It also satisfies* ***Witnessing***, *so* $\mathop{\smile\!\!\!\!\!\!\;|}^{w}$ *agrees with Kim-forking over models.* $\mathop{\smile\!\!\!\!\!\!\;|}^{w}$ *doesn't satisfy* ***Base Monotonicity***, *so ACF_pG is not simple, has TP_2.*

We define for A, B, C algebraically closed sets in a model of ACF_pG

$$A \mathop{\smile\!\!\!\!\!\!|}^{w}_{C} B \iff A \mathop{\smile\!\!\!\!\!\!|}^{ACF}_{C} B \text{ and } G(\overline{AC} + \overline{BC}) = G(\overline{AC}) + G(\overline{BC})$$

$$A \mathop{\smile\!\!\!\!\!\!|}^{st}_{C} B \iff A \mathop{\smile\!\!\!\!\!\!|}^{ACF}_{C} B \text{ and } G(\overline{ABC}) = G(\overline{AC}) + G(\overline{BC})$$

Theorem

- $\mathop{\smile\!\!\!\!\!\!|}^{w}$ *satisfies* ***Symmetry***, ***Monotonicity***, ***Existence***, ***Strong Finite Character***, *3-**amalgamation** so ACF_pG is NSOP1. It also satisfies **Witnessing**, so $\mathop{\smile\!\!\!\!\!\!|}^{w}$ agrees with Kim-forking over models. $\mathop{\smile\!\!\!\!\!\!|}^{w}$ doesn't satisfy **Base Monotonicity**, so ACF_pG is not simple, has TP_2.*
- $\mathop{\smile\!\!\!\!\!\!|}^{st}$ *satisfies every property except **Strong Finite Character** and **Witnessing**.*

Remark (More properties for $\mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^{w}$ and $\mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^{st}$)

Actually, all properties listed in the last slide are satisfied over ***algebraically closed sets****.*

Remark (More properties for $\mathop{\smile\!\!\!\!\!\!\mid}^w$ and $\mathop{\smile\!\!\!\!\!\!\mid}^{st}$)

Actually, all properties listed in the last slide are satisfied over ***algebraically closed sets****. Furthermore both $\mathop{\smile\!\!\!\!\!\!\mid}^w$ and $\mathop{\smile\!\!\!\!\!\!\mid}^{st}$ satisfies* ***Finite Character****,* ***Extension*** *and* ***Transitivity****. $\mathop{\smile\!\!\!\!\!\!\mid}^w$ satisfies* ***Local Character*** *and $\mathop{\smile\!\!\!\!\!\!\mid}^{st}$ doesn't. $\mathop{\smile\!\!\!\!\!\!\mid}^{st}$ is stationnary over algebraically closed sets.*

Remark (More properties for $\mathop{\smile\!\!\!\!\!|}^{w}$ and $\mathop{\smile\!\!\!\!\!|}^{st}$)

*Actually, all properties listed in the last slide are satisfied over **algebraically closed sets**. Furthermore both $\mathop{\smile\!\!\!\!\!|}^{w}$ and $\mathop{\smile\!\!\!\!\!|}^{st}$ satisfies **Finite Character**, **Extension** and **Transitivity**. $\mathop{\smile\!\!\!\!\!|}^{w}$ satisfies **Local Character** and $\mathop{\smile\!\!\!\!\!|}^{st}$ doesn't. $\mathop{\smile\!\!\!\!\!|}^{st}$ is stationnary over algebraically closed sets.*

Local Character. For all A countable, and B, there exists countable $B_0 \subseteq B$ such that

$$A \mathop{\smile\!\!\!\!\!|}_{B_0} B.$$

For $(K, G) \models ACF_pG$, consider the quotient map

$$\pi : K \to K/G.$$

For $(K, G) \models ACF_pG$, consider the quotient map

$$\pi : K \to K/G.$$

By $(K, K/G)$ we mean the two sorted structure with one sort for the field K,

For $(K, G) \models ACF_pG$, consider the quotient map

$$\pi : K \to K/G.$$

By $(K, K/G)$ we mean the two sorted structure with one sort for the field K, one sort for the $\mathbb{F}_p$-vector space K/G,

For $(K, G) \models ACF_pG$, consider the quotient map

$$\pi : K \to K/G.$$

By $(K, K/G)$ we mean the two sorted structure with one sort for the field K, one sort for the $\mathbb{F}_p$-vector space K/G, and the quotient map $\pi : K \to K/G$.

For $(K, G) \models ACF_pG$, consider the quotient map

$$\pi : K \to K/G.$$

By $(K, K/G)$ we mean the two sorted structure with one sort for the field K, one sort for the $\mathbb{F}_p$-vector space K/G, and the quotient map $\pi : K \to K/G$. This structure is interdefinable with (K, G) hence $NSOP_1$, and Kim-forking can be described.

For $(K, G) \models ACF_pG$, consider the quotient map

$$\pi : K \to K/G.$$

By $(K, K/G)$ we mean the two sorted structure with one sort for the field K, one sort for the $\mathbb{F}_p$-vector space K/G, and the quotient map $\pi : K \to K/G$. This structure is interdefinable with (K, G) hence $NSOP_1$, and Kim-forking can be described. It also satisfies **3-amalgamation** over algebraically closed sets.

For $(K, G) \models ACF_pG$, consider the quotient map

$$\pi : K \to K/G.$$

By $(K, K/G)$ we mean the two sorted structure with one sort for the field K, one sort for the $\mathbb{F}_p$-vector space K/G, and the quotient map $\pi : K \to K/G$. This structure is interdefinable with (K, G) hence $NSOP_1$, and Kim-forking can be described. It also satisfies **3-amalgamation** over algebraically closed sets.

Theorem

$(K, K/G)$ has weak elimination of imaginaries.

Base Monotonicity and Forking

$$A \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{wmon}_{C} B :\iff \forall D \subseteq \overline{BC}\ A \mathop{\smile\!\!\!\!\!\!\;|}\limits^{w}_{CD} B.$$

Base Monotonicity and Forking

$$A \mathop{\smile\!\!\!\!\!\!|}\nolimits^{wmon}_{C} B :\iff \forall D \subseteq \overline{BC}\ A \mathop{\smile\!\!\!\!\!\!|}\nolimits^{w}_{CD} B.$$

$\mathop{\smile\!\!\!\!\!\!|}\nolimits^{f}$ / $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{d}$ / $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}$ = forking/dividing/thorn-forking independence relation.

$$A \mathop{\smile\!\!\!\!\!\!|}_{C}^{wmon} B :\iff \forall D \subseteq \overline{BC}\ A \mathop{\smile\!\!\!\!\!\!|}_{CD}^{w} B.$$

$\mathop{\smile\!\!\!\!\!\!|}^{f}$ / $\mathop{\smile\!\!\!\!\!\!|}^{d}$ / $\mathop{\smile\!\!\!\!\!\!|}^{þ}$ = forking/dividing/thorn-forking independence relation.

Proposition

1. *$\mathop{\smile\!\!\!\!\!\!|}^{wmon}$ doesn't satisfy Local Character.*

$$A \mathop{\smile\!\!\!\!\!\!\;|}^{wmon}_{C} B :\iff \forall D \subseteq \overline{BC}\ A \mathop{\smile\!\!\!\!\!\!\;|}^{w}_{CD} B.$$

$\mathop{\smile\!\!\!\!\!\!\;|}^{f}$ / $\mathop{\smile\!\!\!\!\!\!\;|}^{d}$ / $\mathop{\smile\!\!\!\!\!\!\;|}^{þ}$ = forking/dividing/thorn-forking independence relation.

Proposition

1. $\mathop{\smile\!\!\!\!\!\!\;|}^{wmon}$ *doesn't satisfy Local Character.*
2. ACF_pG *is not rosy* ($\mathop{\smile\!\!\!\!\!\!\;|}^{þ} \to \mathop{\smile\!\!\!\!\!\!\;|}^{wmon}$).

Base Monotonicity and Forking

$$A \mathop{\smile\!\!\!\!\!\!|}\nolimits^{wmon}_{C} B :\iff \forall D \subseteq \overline{BC}\ A \mathop{\smile\!\!\!\!\!\!|}\nolimits^{w}_{CD} B.$$

$\mathop{\smile\!\!\!\!\!\!|}\nolimits^{f}$ / $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{d}$ / $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}$ = forking/dividing/thorn-forking independence relation.

Proposition

1. *$\mathop{\smile\!\!\!\!\!\!|}\nolimits^{wmon}$ doesn't satisfy Local Character.*
2. *ACF_pG is not rosy ($\mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ} \to \mathop{\smile\!\!\!\!\!\!|}\nolimits^{wmon}$).*
3. *Let A, B, C, D be algebraically closed, A, B, D containing C, $B \subseteq D$.*

Base Monotonicity and Forking

$$A \mathop{\smile\!\!\!\!\!\!|}_{C}^{wmon} B :\iff \forall D \subseteq \overline{BC}\ A \mathop{\smile\!\!\!\!\!\!|}_{CD}^{w} B.$$

$\mathop{\smile\!\!\!\!\!\!|}^{f}$ / $\mathop{\smile\!\!\!\!\!\!|}^{d}$ / $\mathop{\smile\!\!\!\!\!\!|}^{þ}$ = forking/dividing/thorn-forking independence relation.

Proposition

1. *$\mathop{\smile\!\!\!\!\!\!|}^{wmon}$ doesn't satisfy Local Character.*
2. *ACF_pG is not rosy ($\mathop{\smile\!\!\!\!\!\!|}^{þ} \to \mathop{\smile\!\!\!\!\!\!|}^{wmon}$).*
3. *Let A, B, C, D be algebraically closed, A, B, D containing C, $B \subseteq D$.*

$$\text{if } A \mathop{\smile\!\!\!\!\!\!|}_{C}^{wmon} B \text{ and } A \mathop{\smile\!\!\!\!\!\!|}_{B}^{st} D \text{ then } A \mathop{\smile\!\!\!\!\!\!|}_{C}^{wmon} D.$$

$$A \mathop{\smile\!\!\!\!\!\!\!|}_{C}^{wmon} B :\iff \forall D \subseteq \overline{BC}\ A \mathop{\smile\!\!\!\!\!\!\!|}_{CD}^{w} B.$$

$\mathop{\smile\!\!\!\!\!\!\!|}^{f}$ / $\mathop{\smile\!\!\!\!\!\!\!|}^{d}$ / $\mathop{\smile\!\!\!\!\!\!\!|}^{þ}$ = forking/dividing/thorn-forking independence relation.

Proposition

1. $\mathop{\smile\!\!\!\!\!\!\!|}^{wmon}$ *doesn't satisfy Local Character.*
2. ACF_pG *is not rosy* ($\mathop{\smile\!\!\!\!\!\!\!|}^{þ} \to \mathop{\smile\!\!\!\!\!\!\!|}^{wmon}$).
3. *Let* A, B, C, D *be algebraically closed,* A, B, D *containing* C, $B \subseteq D$.

$$\text{if } A \mathop{\smile\!\!\!\!\!\!\!|}_{C}^{wmon} B \text{ and } A \mathop{\smile\!\!\!\!\!\!\!|}_{B}^{st} D \text{ then } A \mathop{\smile\!\!\!\!\!\!\!|}_{C}^{wmon} D.$$

4. $\mathop{\smile\!\!\!\!\!\!\!|}^{wmon} = \mathop{\smile\!\!\!\!\!\!\!|}^{f} = \mathop{\smile\!\!\!\!\!\!\!|}^{d} = \mathop{\smile\!\!\!\!\!\!\!|}^{þ}$.

General setting

Let $\mathscr{L}_0 \subseteq \mathscr{L}$, T a complete $\mathscr{L}$-theory and $T_0 = T \upharpoonright \mathscr{L}_0$.

General setting

Let $\mathscr{L}_0 \subseteq \mathscr{L}$, T a complete $\mathscr{L}$-theory and $T_0 = T \restriction \mathscr{L}_0$. Let $\mathscr{L}^S = \mathscr{L} \cup \{S\}$ a new unary predicate

General setting

Let $\mathscr{L}_0 \subseteq \mathscr{L}$, T a complete $\mathscr{L}$-theory and $T_0 = T \upharpoonright \mathscr{L}_0$. Let $\mathscr{L}^S = \mathscr{L} \cup \{S\}$ a new unary predicate and let T^S be the $\mathscr{L}^S$ theory whose models are models $\mathscr{M}$ of T in which $S(\mathscr{M})$ is an $\mathscr{L}_0$-submodel of T_0 of $\mathscr{M}$.

Let $\mathscr{L}_0 \subseteq \mathscr{L}$, T a complete $\mathscr{L}$-theory and $T_0 = T \upharpoonright \mathscr{L}_0$. Let $\mathscr{L}^S = \mathscr{L} \cup \{S\}$ a new unary predicate and let T^S be the $\mathscr{L}^S$ theory whose models are models $\mathscr{M}$ of T in which $S(\mathscr{M})$ is an $\mathscr{L}_0$-submodel of T_0 of $\mathscr{M}$.

General setting

1. T model-complete and T_0 has quantifier elimination in $\mathscr{L}_0$

General setting

Let $\mathscr{L}_0 \subseteq \mathscr{L}$, T a complete $\mathscr{L}$-theory and $T_0 = T \restriction \mathscr{L}_0$. Let $\mathscr{L}^S = \mathscr{L} \cup \{S\}$ a new unary predicate and let T^S be the $\mathscr{L}^S$ theory whose models are models $\mathscr{M}$ of T in which $S(\mathscr{M})$ is an $\mathscr{L}_0$-submodel of T_0 of $\mathscr{M}$.

General setting

1. T model-complete and T_0 has quantifier elimination in $\mathscr{L}_0$
2. T_0 geometric and modular

General setting

Let $\mathscr{L}_0 \subseteq \mathscr{L}$, T a complete $\mathscr{L}$-theory and $T_0 = T \upharpoonright \mathscr{L}_0$. Let $\mathscr{L}^S = \mathscr{L} \cup \{S\}$ a new unary predicate and let T^S be the $\mathscr{L}^S$ theory whose models are models $\mathscr{M}$ of T in which $S(\mathscr{M})$ is an $\mathscr{L}_0$-submodel of T_0 of $\mathscr{M}$.

General setting

1. T model-complete and T_0 has quantifier elimination in $\mathscr{L}_0$
2. T_0 geometric and modular
3. for every $\mathscr{L}$-formula $\phi(x, y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for all model $\mathscr{M}$ of T and $b \in \mathscr{M}$

General setting

Let $\mathscr{L}_0 \subseteq \mathscr{L}$, T a complete $\mathscr{L}$-theory and $T_0 = T \restriction \mathscr{L}_0$. Let $\mathscr{L}^S = \mathscr{L} \cup \{S\}$ a new unary predicate and let T^S be the $\mathscr{L}^S$ theory whose models are models $\mathscr{M}$ of T in which $S(\mathscr{M})$ is an $\mathscr{L}_0$-submodel of T_0 of $\mathscr{M}$.

General setting

1. T model-complete and T_0 has quantifier elimination in $\mathscr{L}_0$
2. T_0 geometric and modular
3. for every $\mathscr{L}$-formula $\phi(x, y)$ there exists an $\mathscr{L}$-formula $\theta_\phi(y)$ such that for all model $\mathscr{M}$ of T and $b \in \mathscr{M}$

$$\mathscr{M} \models \theta_\phi(b) \iff \text{there exists } \mathscr{N} \succ \mathscr{M} \text{ and tuple } a \text{ from } \mathscr{N} \text{ such that } a \text{ is independent over } \mathscr{M} \text{ (for the pregeometry } acl_0\text{) and } \mathscr{N} \models \phi(a, b)$$

Then T_S admits a model companion, call it TS.

Example

- ***Generic $\mathbb{F}_{p^n}$-subvector space.*** *For every model-complete theory of $\mathbb{F}_{p^n}$-vector space that eliminate $\exists^\infty$, one can add one (many) generic $\mathbb{F}_{p^n}$-subvector space. For instance $ACF_p, Psf_p, DCF_p, ACFA_p, PAC_p$....*

Example

- ***Generic $\mathbb{F}_{p^n}$-subvector space.*** *For every model-complete theory of $\mathbb{F}_{p^n}$-vector space that eliminate $\exists^\infty$, one can add one (many) generic $\mathbb{F}_{p^n}$-subvector space. For instance ACF_p, Psf_p, DCF_p, $ACFA_p$, PAC_p....*
- ***Multiplicative subgroup.*** *Generic multiplicative subgroup of an algebraically closed field of characteristic $p \geq 0$ ($p > 0$ can add generic additive and multiplicative subgroup).*

$NSOP_1$-conservative expansion

If T is $NSOP_1$ and Kim-forking satisfies : for A, B, C algebraically closed and Kim-independent over some model $\mathscr{M}$ then

$$acl_0(acl_T(AC), acl_T(BC)) \cap acl_T(AB) = acl_0(A, B)$$

If TS exists, then TS is $NSOP_1$. The following are equivalent:

1. TS is not simple
2. T is not simple or there exist algebraically closed sets A, B, C, D such that A, B, D contain C and $A \mathop{\smile\!\!\!\!\!\!|}^K_C BD$, and such that

$$acl_0(A, acl_T(BD)) \cup acl_T(AD) \neq acl_0(acl_T(AD), acl_T(BD)).$$

General setting

$NSOP_1$-conservative expansion

If T is $NSOP_1$ and Kim-forking satisfies : for A, B, C algebraically closed and Kim-independent over some model $\mathscr{M}$ then

$$acl_0(acl_T(AC), acl_T(BC)) \cap acl_T(AB) = acl_0(A, B)$$

If TS exists, then TS is $NSOP_1$. The following are equivalent:

1. TS is not simple
2. T is not simple or there exist algebraically closed sets A, B, C, D such that A, B, D contain C and $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{C} BD$, and such that

$$acl_0(A, acl_T(BD)) \cup acl_T(AD) \neq acl_0(acl_T(AD), acl_T(BD)).$$

Example

All examples above are $NSOP_1$.

The end

Thanks ;)